Dispersive shock wave: title and abstract

Christophe Besse: "Artificial boundary conditions for Schrödinger equations: Part I"

We consider here the derivation of artificial boundary conditions (ABCs) to solve numerically Schrödinger equations on a bounded computational domain with a fictitious, non-physical boundary. After a general survey of various existing methods, we present the construction of ABCs based on pseudodifferential techniques. In this first part of the presentation we address the one dimensional case with a variable potential $V(x,t)$. We propose efficient numerical schemes associated to fractional operators involved in the expression of the ABCs.

Rémi Carles: "An asymptotic preserving scheme based on a new formulation for NLS in the semiclassical limit."

We consider the semiclassical limit for the nonlinear Schrodinger equation. We introduce a phase/amplitude representation given by a system similar to the hydrodynamical formulation, whose novelty consists in including some asymptotically vanishing viscosity. This system overcomes the technical drawback of Madelung transform, which breaks down in the presence of vacuum. We prove that the system is always locally well-posed in a class of Sobolev spaces, and globally well-posed for a fixed positive Planck constant in the one-dimensional case. We propose a second order numerical scheme which is asymptotic preserving. Before singularities appear in the limiting Euler equation, we recover the quadratic physical observables as well as the wave function with mesh size and time step independent of the Planck constant. This approach is also well suited to the linear Schrodinger equation. This is a joint work with Christophe Besse and Florian Mehats.

Frédéric Chardard: "On the Stability of Travelling Waves Arising in the Kawahara Equation"

The Kawahara equation is a weakly nonlinear model for capillarity-gravity water waves which admits solitary-wave type solutions. For each solitary wave, there exists a family of periodic waves which is asymptotic to the solitary wave when the period tend to infinity. In this talk we study the stability of these travelling waves.

Florent Chazel: "Mathematical and numerical modelling of dispersive water waves".

In this talk we introduce two models for the propagation of dispersive water waves : a Green-Naghdi-type model and a generalized Boussinesq-type model. While the Green-Naghdi model is able to handle weakly dispersive and degenerates into the shallow water equations in the surf zone, the Boussinesq-type model allows for considering highly dispersive waves but fails to properly describe highly nonlinear waves. The numerical strategies are presented and discussed, and the two models are compared on some experimental test cases.

David Chiron: "On the (KdV)/(KP-I) and the (gKdV)/(gKP-I) asymptotic regime for the Nonlinear Schrödinger equation"

We consider the Nonlinear Schrödinger equation with the non standard condition that the wave function Ψ has modulus one at infinity. For small amplitude and long wavelength perturbations of the trivial constant solution $\Psi = 1$, the dynamics can be approximated by the (KdV) or the (KP-I) equation. For some particular nonlinearities, we may obtain an asymptotic regime governed by the (gKdV)/(gKP-I) equation. We shall give some results on these regimes. The same approximation formally holds for the travelling waves, and we shall give a numerical example in two space dimension.

Gennady El: "Whitham modulation theory and dispersive shock waves"

In mid-1960s G.B. Whitham developed an asymptotic theory to treat the problems involving the generation and propagation of slowly modulated, fully nonlinear wavetrains in dispersive media. Locally, these wavetrains are described by periodic or quasi-periodic solutions of nonlinear dispersive PDEs. The Whitham equations then govern slow variations of the wavetrain parameters, such as amplitude, wavenumber, mean etc., on the spatio-temporal scale much larger than the medium characteristic scales (the typical wavelength and period of the periodic solution).

One of the most important applied aspects of the Whitham modulation theory is a mathematical description of *dispersive shock waves* (DSWs) — coherent unsteady nonlinear wave structures providing dispersive regularisation of the wave breaking singularities in conservative media. The fundamental role of DSWs in such media is similar to that of viscous shocks in classical gas and fluid dynamics. At the same time, DSWs are sharply distinct from their well-studied dissipative counterpart both in terms of physical significance and mathematical description. Physical manifestations of DSWs include undular bores on shallow water, nonlinear diffraction patterns in laser optics, and blast waves in Bose-Einstein condensates.

In this mini-course I shall describe the Whitham method in several formulations applicable to integrable and non-integrable nonlinear dispersive PDEs. The problem of the formation and evolution of a dispersive shock wave will be first presented in the framework of the Korteweg - de Vries and defocusing nonlinear Schrödinger equations and then extended to a general dispersive-hydrodynamic context by constructing a system of DSW closure conditions playing the role analogous to the classical shock conditions in viscous fluids. We then consider some applications of the developed theory to shallow water waves, nonlinear optics and superfluid dynamics.

Sergey Gavrilyuk: "Non-classical description of the classical hydraulic jump."

The classical hydraulic jump is a natural phenomenon appearing in open fluid flows and characterizing by an abrupt transition from a supercritical flow to a subcritical one. Defining the Froude number as F = U/ gh, where U is the flow velocity, h is the water depth, and g is the gravity acceleration, one can express the supercritical-subcritical transition in terms of the Froude number as $F_1 > 1$ and $F_2 < 1$. Here the subscripts 1 and 2 correspond to the upstream and downstream flow variables, respectively. The classical shallow water model (Saint-Venant model) fails to explain this phenomena. It is not able to predict the principal characteristics of the hydraulic jump : the form, the length and even the sequent depth ratio. This is a reason why hydraulicians Óhave long ago come to regard the various phenomena of rapidly varied flow as a number of isolated cases each requiring its own specific empirical treatment (Chow 1959). The aim of this work is to propose a mathematical model able to calculate gradually varied flows and, at the same time, some rapidly varied flows such as hydraulic jumps. The hydraulic jump features depend on the upstream Froude number (see, for example, Binnie and Orkney 1954, Chow 1959). When the Froude number F_1 is smaller than about 1.26, the hydraulic jump is undular, i.e. the free surface is oscillating in the downstream part. The analytical and numerical study of unsteady undular bores was recently performed by El et al. (2006, 2010) and Lemetayer et al. (2010). They used the Green-Naghdi to study this phenomena. For the Froude number larger about 1.75 the wholly turbulent jump is formed with a monotonic structure of the free surface. We derive a conservative hyperbolic two-parameters model of shear shallow water flows to study the classical turbulent hydraulic jump (Richard, Gavrilyuk, 2012). The parameters of the model, which are the wall enstrophy and the roller dissipation coefficient, are determined from measurements of the roller length and the deviation from the Bélanger equation of the sequent depth ratio. Stationary solutions to the model describe with a good accuracy the free surface profile of the hydraulic jump. The model is also capable to predict the oscillations of the jump toe. We show that if the upstream Froude number is larger than about 1.5, the jump toe oscillates with a particular frequency, while for a Froude number smaller than 1.5 the solution becomes stationary. In particular, we show that for a given flow discharge, the oscillation frequency is a decreasing function of the Froude number. This is joint work with G.L. Richard.

Philippe Gravejat: "Stability for the solitons of the one-dimensional Gross-Pitaevskii equation."

We present two results in collaboration with F. Béthuel and D. Smets concerning the orbital stability of multi-solitons and the asymptotic stability of single solitons for the one-dimensional Gross-Pitaevskii equation.

Mariana Haragus: "Transverse dynamics of gravity-capillary periodic water waves"

The gravity-capillary water-wave problem concerns the irrotational flow of a perfect fluid in a domain bounded below by a rigid bottom and above by a free surface under the influence of gravity and surface tension. In the case of large surface tension the system has a family of traveling two-dimensional periodic waves for which the free surface has a periodic profile in the direction of propagation and is homogeneous in the transverse direction. We show that these periodic waves are linearly unstable under spatially inhomogeneous perturbations which are periodic in the direction transverse to propagation. As a consequence, the periodic waves undergo a dimension-breaking bifurcation generating a family of spatially three-dimensional solutions which are periodic in both the direction of propagation and the transverse direction.

Johannes Höwing: "Stability of Solitary Waves in generalized Korteweg-deVries, Boussinesq, and Euler-Korteweg equations."

We establish stability of solitary waves for the generalized KdV and for the generalized Boussinesq equation under the assumption that the nonlinearity p satisfies $p'' > 0$ and $p''' \leq 0$. In particular, the Boussinesq equation with $p(v) = kv^{-\gamma}$ with $\gamma \geq 1$ and $k > 0$ describes the flow of an inviscid isothermal ideal (barotropic) fluid with capillarity. Under the sole assumption that p is strictly convex, we still can conclude stability of small-amplitude solitary waves in Euler-Korteweg equations,

such as the generalized Gross-Pitaevskii equation.
At last, we will present a result on stability of solitary waves in an extended Boussinesq equation with quadratic-cubic nonlinearity which was recently proposed as an alternative to the FitzHugh-Nagumo equation in the modeling of pulse propagation in nerves.
Joint work with H. Freistühler.

A.M. Kamchatnov: Effects of instabilities and perturbations in the dispersive shock waves theory.

Dispersive shock wave is a nonlinear oscillatory wave structure connecting two flows with different parameters in a dispersive medium. Theory of dispersive shock waves is well developed for one-dimensional systems described by completely integrable evolution equations. In Whitham approach, dispersive shock wave is normally represented as a modulated nonlinear periodic wave within a finite range of the space coordinate and at one its edge this wave degenerates into linear wave and at the other edge to solitons. Evolution of this structure as a whole is governed by the Whitham modulation equations. In this talk, we shall discuss generalization of this scheme to taking into account of small perturbations in the evolution equations and consider effects of several dimensions. We show that the well-known "snake" instability of multi-dimensional solitons and, hence, of dispersive shock waves, can be effectively stabilized by a fast enough flow along solitons so that unstable modes are convected by flow away from the region of shock generation in experimentally feasible situations. Besides multi-dimensional effects, small perturbations in the evolution equations can play the role comparable with the effects of small modulations at asymptotically large time of evolution. The theory is illustrated by applications to dispersive shock waves in polariton physics.

Christian Klein: "Numerical treatment of nonlinear dispersive partial differential equations."

We present an overview on numerical methods for dispersive PDEs. For the spatial discretization of the PDEs, we use spectral methods, i.e. discrete Fourier transforms and expansions in terms of orthogonal polynomials. We discuss how important analytic information on the solutions can be obtained from the spectral expansions. Boundary conditions are enforced with tau-methods. The spatial discretization leads for dispersive PDEs typically to large systems of stiff ODEs. We discuss stability issues in the time integration and present adapted integration schemes to deal with stiffness. Concrete examples of numerical studies of dispersive PDEs are presented.

Pauline Klein: "Artificial boundary conditions for Schrödinger equations: Part II"

This talk is the continuation of the talk of Christophe Besse. We consider now generalizations to nonlinear Schrödinger equations and to the two dimensional case.

L.-M. Rodrigues: "Slow modulations of periodic waves in Hamiltonian PDEs, with application to capillary fluids."

In a joint work with Pascal Noble and Sylvie Benzoni-Gavage, we provide in an abstract framework for Hamiltonian PDEs the expected link between the weak hyperbolicity of a slow modulation averaged system — the well-known Whitham system — and the spectral stability of periodic traveling waves to side-band perturbations.

Our abstract framework is general enough to recover a wealth of already known results (for instance for the KdV equation) but most importantly to us general enough to include the Euler-Korteweg system (EK), a system involved among other things in the description of the evolution of capillary flows. For the EK system, we've carried out numerical investigations of spectral stability through the verification of the obtained necessary condition.

Kristel Roidot: "Numerical Study of Asymptotic behavior and Blow-up phenomena in NLS equations."

Rapid oscillations in solutions are observed in dispersive PDEs without dissipation where solutions of the corresponding PDEs without dispersion present shocks. To solve numerically these oscillations, the use of efficient methods without using artificial numerical dissipation is necessary, in particular in the study of PDEs in some dimensions. As studied PDEs in this context are typically stiff, efficient integration in time is the main problem. We focus in this talk on NLS equations, mainly the Davey Stewartson II equations (DS II) (a two-space dimensions generalization of the cubic NLS equation). In semiclassical limit, the DS II equations show a similar behavior as the nonlinear Schrödinger equation (NLS) in (1+1) dimensions, i.e. which is characterized by the appearance of dispersive shocks and the possibility of blowup in the solutions. So far there are no analytic predictions about solutions of DS in this regime, and it is unclear whether there will be blow-up in this case. Numerical studies are supposed to provide more insight into these questions. After discussing efficient time integration of these equations, we present few numerical results about the asymptotic behavior and blow-up phenomena in their solutions.

Frédéric Rousset: "Multi-solitons solutions of the water-waves system."

I will describe the construction of semi-global solutions of the full water waves system with surface tension that behave like a sum of decoupling solitary waves for large times.

Jean-Claude Saut: "Weakly dispersive perturbations of quasilinear hyperbolic equations".

Many relevant physical models are weakly dispersive perturbations of quasilinear hyperbolic equations or systems. This leads to questions of spaces of resolution, existence time, blow-up, existence of solitary waves, etc,.. We will review the (few) available results and the (many) open questions.