Mixed sequences

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For expositional ease, we will consider a very particular case.

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Remark: Any $\vec{p}$-saturated sequence is the union of $\vec{q}$-saturated sequences for $\vec{q} \subset \vec{p}$.

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Two $\vec{p}$-sequences (x_k) and (y_k) are *equimixed* if there is a bijection $\varepsilon : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$(x_k)_{k \in A} \text{ contains a } p_i\text{-subsequence}$$
$$\Leftrightarrow$$
$$(y_{\varepsilon(k)})_{k \in A} \text{ contains a } p_i\text{-sequence.}$$

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$\log(N(n)) \sim 9^n \cdot slow(n)$

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$J(n)$ is the number of $2 \times n$ matrices with entries $1, -1, 0$ such that

1. the lower row is nonzero
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The machinery

Three steps

1. Descriptive set theory
2. Ramsey theory
3. finite combinatorics

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STEP 2. After Step 1, we need to control what happens in subsets of T other than $[i]$-chains. A new partition theorem for trees, *finer* than Milliken, is needed.

STEP 3. The output of the previous procedure are finite combinatorial objects that rule the behavior of $\vec{p}$-sequences.

Construction of pathological Gâteaux-differentiable functions

Robert Deville (joint work with M. Ivanov and S. Lajara)

Definition 1 *X, Y Banach spaces. $F : X \to Y$ has the jump property if F is Gâteaux differentiable at every point of X and if $\exists \alpha > 0$ such that*

$$\|F'(x) - F'(y)\|_{\mathcal{L}(X,Y)} \geq \alpha \qquad \text{whenever } x, y \in X \text{ and } x \neq y.$$

We say that (X, Y) has the jump property if there is a Lipschitz, bounded function $F : X \to Y$ with the jump property.

Deville-Hajek :

- If (X, Y) has property $(*)$, then $\mathcal{L}(X, Y)$ is nonseparable.

- $(X, \mathbb{R})$ never has the jump property.

- $(\ell^1, \mathbb{R}^2)$ has the jump property.

- if $1 \leq p, q < +\infty$, (ℓ^p, ℓ^q) has the jump property if and only if $p \leq q$.

Theorem 2 *Let X and Y be Banach spaces. Suppose that there exist a total, bounded, biorthogonal system $\{e_n, e_n^*\}_n \subset X \times X^*$ and an unconditional basic sequence $\{f_n\}_n \subset Y$ such that, for each $h \in X$, the series $\left(\sum_{n=1}^{\infty} e_n^*(h)(f_{2n-1} + f_{2n})\right)$ converges. Then (X, Y) has the jump property.*

Let us notice, that under the above assumptions, if

$$T(h) = \sum_{n=1}^{\infty} e_n^*(h) f_{2n-1} \ and \ S(h) = \sum_{n=1}^{\infty} e_n^*(h) f_{2n}$$

then, according to the uniform boundedness principle, T and S are bounded linear operators from X into Y.

Lemma 3 *Let X and Y be Banach spaces, and let $\{e_n^*\}_n \subset X^*$ and $\{f_n\}_n \subset Y$ be sequences satsfying the hypothesis of the theorem. Then, for every $a = \{a_n\}_n \in \ell^\infty$, if*

$$T_a(h) = \sum_{n=1}^{\infty} a_n e_n^*(h) f_{2n-1} \text{ and } S_a(h) = \sum_{n=1}^{\infty} a_n e_n^*(h) f_{2n}$$

Then $T_a, S_a \in \mathcal{L}(X, Y)$, and $\exists C > c > 0$ such that

$$c\|a\|_\infty \leq \|T_a\| \leq C\|a\|_\infty \text{ and } c\|a\|_\infty \leq \|S_a\| \leq C\|a\|_\infty \text{ whenever } a \in \ell^\infty.$$

In particular, $\mathcal{L}(X, Y) \supset \ell^\infty$.

Lemma 4 *For each* $p = (a, b) \in \mathbb{R}^2$ *with* $a < b$ *and each* $\varepsilon > 0$ *there exists a* $\mathcal{C}^\infty$ *function* $\varphi = \varphi_{p,\varepsilon} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *such that:*

(i) $\|\varphi(s,t)\| \leq \varepsilon$ *for all* $(s,t) \in \mathbb{R}^2$*,*

(ii) $\left\|\frac{\partial \varphi}{\partial s}(s,t)\right\| \leq \varepsilon$ *for all* $(s,t) \in \mathbb{R}^2$*,*

(iii) $\left\|\frac{\partial \varphi}{\partial t}(s,t)\right\| \leq 1$ *for all* $(s,t) \in \mathbb{R}^2$*, and*

(iv) $\varphi(s,t) = 0$ *whenever* $s < a$*,*

(v) $\left\|\frac{\partial \varphi}{\partial t}(s,t)\right\| = 1$ *whenever* $s \geq b$*.*

(Here, $\|\cdot\|$ *denotes an arbitrary norm on* $\mathbb{R}^2$*.)*

Proof when $\|\cdot\|$ is the Euclidean norm. If $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathcal{C}^\infty$ such that

- $0 \leq \beta(s) \leq 1$

- $\beta(s) = 0$ for $s \leq a$ and $\beta(s) = 1$ if $s \geq b$,

then the function

$$\varphi(s,t) = \frac{\beta(s)}{n}(\sin(nt), \cos(nt))$$

satisfies the required properties whenever $n > \varepsilon^{-1}$.

Lemma 5 *Let $\{F_k\}_k$ be a sequence of mappings between the Banach spaces X and Y such that :*

1. *The series $\sum\limits_{k=1}^{\infty} F_k$ converges pointwise to a function $F : X \longrightarrow Y$.*

2. *For all $h \in X$, the series $\sum\limits_{k=1}^{\infty} F_k'(x)(h)$ converges uniformly with respect to $x \in X$.*

Then F is Gâteaux differentiable and Lipschitz on all of X, and, for every $x, h \in X$, we have

$$F'(x)(h) = \sum_{k=1}^{\infty} F_k'(x)(h).$$

Step 1: Construction of F. Let us write $\mathbb{P} = \{(a,b) \in \mathbb{Q}^2 : a < b\}$ and let $k \mapsto (n_k, p_k)$ be a bijection from $\mathbb{N}$ into $\mathbb{N} \times \mathbb{P}$ such that $n_k \neq k$ for all $k \in \mathbb{N}$. Assume $\sum \varepsilon_k \ll 1$ and set $\varphi_k = \varphi_{p_k, \varepsilon_k}$. Define

$$s_k(x) = \left(e^*_{n_k}(x), e^*_k(x)\right),\ x \in X,$$

$$i_k(s,t) = t f_{2k-1} + s f_{2k},\ (s,t) \in \mathbb{R}^2$$

and

$$F_k = i_k \circ \varphi_k \circ s_k.$$

$F_k \in \mathcal{C}^\infty(X,Y)$, and $\|F_k(x)\| \leq \varepsilon_k$ for all $x \in X$. Since the series $\sum\limits_{k=1}^{\infty} \varepsilon_k$ converges, the formula

$$F(x) = \sum_{k=1}^{\infty} F_k(x)$$

defines a continuous, bounded function from X into Y.

Step 2. If $h \in X$, then the series $\sum_{k=1}^{\infty} F_k'(x)(h)$ converges uniformly with respect to x.

Hence, F is Gâteaux differentiable and Lipschitz on all of X, and

$$F'(x)(h) = \sum_{k=1}^{\infty} F_k'(x)(h) \textit{ whenever } x, h \in X.$$

Step 3: F has the jump property. Fix $x, y \in X$, $x \neq y$. Pick $n \in \mathbb{N}$ such that $e_n^*(x) \neq e_n^*(y)$. We can assume that $e_n^*(x) < e_n^*(y)$. Now, let $k \in \mathbb{N}$ such that $n_k = n$ and $e_n^*(x) < a_k < b_k < e_n^*(y)$. We get $F_k'(x)(e_k) = 0$ and $\|F_k'(y)(e_k)\| = 1$, and thus,

$$\|F_k'(x)(e_k) - F_k'(y)(e_k)\| = 1.$$

Moreover, $\|F_m'(x)(e_k)\| \leq \varepsilon_m$ and $\|F_m'(y)(e_k)\| \leq \varepsilon_m$ for all $m \neq k$. Therefore,

$$\|F'(x) - F'(y)\| \geq \|F'(x)(e_k) - F'(y)(e_k)\| \geq 1 - 2\sum \varepsilon_m > 0.$$

Example 6 *(Bayart) If X is a separable Banach space, then (X, c_0) has the jump property.*

Proof : There is a Markushevich basis $\{e_n, e_n^*\}_n \subset X \times X^*$, such that $\|e_n^*\| = 1$ for all $n \in \mathbb{N}$. In particular, for every $h \in X$ we have $\lim_n e_n^*(h) = 0$, that is, $\{e_n^*(h)\}_n \in c_0$, and if we denote by $\{f_n\}_n$ the unit vector basis of c_0, then

$$\left\| \sum_{n=1}^{\infty} e_n^*(h) f_{2n-1} \right\| \leq \|h\| \ \textit{and} \ \left\| \sum_{n=1}^{\infty} e_n^*(h) f_{2n} \right\| \leq \|h\|.$$

Since $\{f_n\}_n$ is unconditional, according to the theorem, (X, c_0) has the jump property.

Corollary 7 *Let X be a Banach space with an Schauder basis $\{e_n\}_n$, Y be a Banach space and $U \in \mathcal{L}(X,Y)$ such that $\{U(e_n)\}_n$ is a subsymmetric basic sequence in Y. Then (X,Y) has the jump property.*

In particular, if X has a subsymmetric basis, then (X,X) has the jump property.

Example 8 *If $q \geq p \geq 1$, then (ℓ^p, ℓ^q) has the jump property.*

More generally, let M and N be two Orlicz functions such that $N(t) \leq k_1 M(k_2 t)$, for some constants k_1, k_2 and all t in a neighbourhood of zero. If h_M and h_N are the corresponging Orlicz sequence spaces, then (h_M, h_N) has the jump property.

Proof : if $q \geq p \geq 1$, then $U : \ell^p \to \ell^q$ is continuous.

Corollary 9 *Let X be a Banach space with an Schauder basis $\{e_n\}_n$, Y be a Banach space and $U \in \mathcal{L}(X,Y)$ such that :*
$\{U(e_n)\}_n$ is an unconditional basic sequence in Y, and
Y is isomorphic to $Y \oplus Y$.
Then (X,Y) has the jump property.

Example 10 *If $p \geq q \geq 1$ and $p \neq 1$, then $(L^p([0,1]), L^q([0,1]))$ has the jump property.*

Proof : If $p \geq q > 1$, the inclusion $U : L^p([0,1]) \longrightarrow L^q([0,1])$ is continuous if $p \geq q$, and the Haar system is an unconditional basis of $L^q([0,1])$.

Remark 11 $\left(L^2([0,1]), L^p([0,1])\right)$ *also has the jump property for every $p \geq 1$. (It is not clear whether there exist couples (p,q) such that $(L^p([0,1]), L^q([0,1]))$ fails to have the jump property. The question is open in particular whenever $p = q = 1$.*

Corollary 12 *If X is a Banach space with an unconditional basis, and X is isomorphic to $X \oplus X$, then (X, X) has the jump property.*

Example 13 *If $\mathcal{T}$ is the Tsirelson's space, then $(\mathcal{T}, \mathcal{T})$ has the jump property. Notice that Tsirelson's space does not have any subsymmetric Schauder basis.*

Example 14 *If J is the James space, then (J, J) has the jump property. Indeed, $\left(J, \ell^2\right)$ has the jump property. Since ℓ^2 is isomorphic to a subspace of J, (J, J) also enjoys this property. Notice that J does not have any unconditional Schauder basis.*

Example 15 *The space X constructed by Argyros and Kaydon such that every $T \in \mathcal{L}(X)$ is of the form $\lambda I + K$, is a separable infinite dimensional Banach space such that $\mathcal{L}(X)$ is separable. Hence (X, X) fails the jump property.*

The set of functions satisfying the jump property.

Let $G(X,Y)$ be the space of bounded and Lipschitz functions from X to Y which are Gâteaux-differentiable at each point of X, endowed with its natural norm $\|f\| := \sup\{f(x);\, x \in X\} + \sup\{\|f'(x)\|;\, x \in X\}$.

Denote $G_*(X,Y) := \{f \in G(X,Y);\, f$ has the jump property$\}$.

Bayart : if X is a separable Banach space, then $G_*(X,c_0)$ is spaceable, i. e. $G_*(X,c_0) \cup \{0\}$ contains a closed infinite dimensional subspace of $G(X,c_0)$.

Proposition 16 *Let X and Y be Banach spaces, and $\{e_n\}_n$ be a Schauder basis of X such that $\{U(e_n)\}_n$ is a subsymmetric basic sequence in Y. Then $G_*(X,Y)$ is lineable, i. e. $G_*(X,Y) \cup \{0\}$ contains an infinite dimensional subspace of $G(X,Y)$.*

Problem : Is $G_*(X,Y)$ spaceable?

Isometric representation and maximal norms

Valentin Ferenczi, University of São Paulo

CIRM, August 29th, 2012

Terminology[1]

The results presented here are joint work with Christian Rosendal, from the University of Illinois at Chicago.

In this talk all Banach spaces are separable, infinite dimensional, and, for expositional ease, assumed to be complex.

[1] The author acknowledges the support of FAPESP, processes 2008/11471-6, 2010/05182-1, 2010/17493-1

Contents

1. Introduction: Mazur's rotation problem, transitive and maximal norms
2. Main result: Isometry groups on HI spaces
3. Main result: Renorming theory and ergodic decompositions
4. Open questions: about ℓ_p, ℓ_2...

Contents

Introduction: Mazur's rotations problem

Definition

- $\mathrm{Isom(X)}$ *is the group of linear surjective isometries on a Banach space* X.
- *The group* $\mathrm{Isom(X)}$ *acts* ***transitively*** *on the unit sphere* S_X *of* X *if for all* x, y *in* S_X*, there exists* T *in* $\mathrm{Isom}(X)$ *so that* $Tx = y$.

It is clear that $\mathrm{Isom(H)}$ acts transitively on any Hilbert space H. Conversely if $\mathrm{Isom(X)}$ acts transitively on a Banach space X, must it be isomorphic? isometric to a Hilbert space?

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Introduction: Mazur's rotations problem

Conversely if $\mathrm{Isom}(X)$ acts transitively on a Banach space X, must it be isomorphic? isometric to a Hilbert space?

Answer

- *if* $\dim X < +\infty$*: YES to both*
- *if* $\dim X = +\infty$ *é separável: ???*
- *if* $\dim X = +\infty$ *é não separável: NO to both*

Idea of the case $\dim X < +\infty$: $X = \mathbb{R}^n$ with inner product $< .,. >$ such that $\|x_0\| = \sqrt{< x_0, x_0 >}$ for some x_0, and assume $\mathrm{Isom}(X, \|.\|)$ acts transitively. Define

$$[x, y] = \int_{T \in \mathrm{Isom}(X, \|.\|)} < Tx, Ty > dT,$$

This a new inner product for which the T still are isometries, and $\|x\| = \sqrt{[x, x]}$, since holds for x_0 and by transitivity.

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This a new inner product for which the T still are isometries, and $\|x\| = \sqrt{[x, x]}$, since holds for x_0 and by transitivity.

So we have the next unsolved problem which appears in Banach's book "Théorie des opérateurs linéaires".

Problem (Mazur's rotations problem, 1932)

If X is separable and $\mathrm{Isom}(X)$ acts transitively on S_X, must X be isomorphic? isometric to the Hilbert space?

It is not even known whether X isomorphic to ℓ_2 and $\mathrm{Isom}(X)$ acts transitively implies that X is isometric to ℓ_2.

Mazur's rotations problem: isometric versus bounded representations of groups

If G is a group of isometries on $(X, \|.\|)$ and $|||.|||$ is another norm on X, then G becomes a bounded group of isomorphisms on $(X, |||.|||)$. Conversely:

Observation

If G is a bounded group of isomorphisms on $(X, \|.\|)$, then the formula $|||x||| = sup_{g \in G} \|gx\|$ defines a G-invariant equivalent norm on X.

So there is no difference between bounded representations or isometric representations of groups on a Banach space, up to equivalent renorming.

However if $\|.\|$ was a Hilbert norm on X, the new norm $|||.|||$ has no reason to be a Hilbert norm.

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However if $\|.\|$ was a Hilbert norm on X, the new norm $|||.|||$ has no reason to be a Hilbert norm.

Observation: Mazur and Dixmier

Fact

Let $\|.\|$ be an equivalent norm on ℓ_2 such that $G = \mathrm{Isom}(\ell_2, \|.\|)$ acts transitively on $(\ell_2, \|.\|)$. Assume

- *G is unitarizable: exists $A \in GL(\ell_2)$ such that AgA^{-1} is unitary for all $g \in G$.*

Then

- *$\|.\|$ is a Hilbert norm.*

Proof: we may assume $\|Ax_0\|_2 = \|x_0\| = 1$ for some fixed x_0. Then we define $< x, y >:=< Ax, Ay >_2$ and claim that $< x, x >= \|x\|^2$ for all x. Indeed this holds for $x = x_0$. For any other x such that $\|x\| = 1$ let $g \in G$ be such that $x_0 = gx$, then

$$\sqrt{< x, x >} = \|Ax\|_2 = \|AgA^{-1}Ax\|_2 = \|Agx\|_2 = \|Ax_0\|_2 = 1.$$

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Observation: Mazur and Dixmier

So one part of Mazur's problem is related to the question of which bounded representations on the Hilbert space are unitarizable, i.e. which bounded subgroups of $GL(\ell_2)$ are unitarizable.

Theorem (Day-Dixmier, 1950)

Any bounded representation of an amenable group on the Hilbert space is unitarizable.

By Ehrenpreis and Mautner (1955) this does not extend to all (countable) groups.

Question (Dixmier's unitarizability problem)

If all bounded representations of G on ℓ_2 are unitarisable, is G amenable?

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General objective of renorming theory: replace the norm by a better one (i.e. an equivalent one with more properties).

In general, one tends to look for an equivalent norm which make the unit ball

- smoother, more differentiable, more convex, more rotund...
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Introduction: transitive and maximal norms

In 1964, Pełczyński and Rolewicz looked at Mazur's rotations problem and defined properties of a given norm $\|.\|$ which may be interpreted as saying that $\|.\|$ induces many isometries. In what follows $\mathcal{O}_{\|.\|}(x)$ represents the orbit of the point x of X, under the action of the group $\mathrm{Isom}(X, \|.\|)$, i.e. $\mathcal{O}_{\|.\|}(x) = \{Tx, T \in \mathrm{Isom}(X, \|.\|)\}$.

Definition

Let X be a Banach space and $\|.\|$ an equivalent norm on X. Then $\|.\|$ is

(i) *transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x) = S_X$.*

(ii) *quasi transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x)$ is dense in S_X.*

(iii) *maximal if there exists no equivalent norm $|||.|||$ on X such that $\mathrm{Isom}(X, \|.\|) \subseteq \mathrm{Isom}(X, |||.|||)$ with proper inclusion.*

Of course $(i) \Rightarrow (ii)$, and also $(ii) \Rightarrow (iii)$ (Rolewicz).

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Examples of (i): ℓ_2, of (ii): $L_p(0,1)$, of (iii): ℓ_p.

Note that (iii) means that $\mathrm{Isom}(X, \|.\|)$ is a maximal bounded subgroup of $GL(X)$.

Indeed if G were a bigger bounded subgroup then $|||x||| = \sup_{g \in G} \|gx\|$ would define an equivalent norm for which $\mathrm{Isom}(X, \|.\|) \subseteq \mathrm{Isom}(X, |||.|||)$ with proper inclusion.

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Classical problems of renorming theory in Banach spaces have the following form: given a Banach space X, does X admit an equivalent norm with property (i),(ii) or (iii)? Three more accessible problems stand out.

Introduction: transitive and maximal norms

Question (Wood, 1982)

Does every Banach space admit an equivalent maximal norm?

Question (Deville-Godefroy-Zizler, 1993)

Does every superreflexive Banach space admit an equivalent quasi-transitive or even transitive norm?

Question

Let $1 < p < +\infty$. *Does* $L_p([0,1])$ *admit an equivalent transitive norm? Does* ℓ_p *admit an equivalent quasi transitive norm?*

The main result

Theorem (F. - Rosendal, 2011)

There exists a separable uniformly convex Banach space X without an equivalent maximal norm. Equivalently $GL(X)$ does not have a maximal bounded subgroup.
In particular, X is a counterexample to the questions of Wood and Deville-Godefroy-Zizler.

Our main result is actually about ”small” subgroups of isometries on any separable reflexive space.

Contents

The main result: HI spaces

By Gowers and Maurey, every operator on a complex HI space is of the form $\lambda Id + S$, λ scalar, S strictly singular. For isometries we have:

Theorem (Rabiger-Ricker, 1998)

Any isometry on a complex HI space is of the form $\lambda Id + K$, K compact.

But actually this is much more general:

Lemma

a) if an isometry on any space X has the form $\lambda Id + S$, then S is compact (actually belongs to $\overline{\mathcal{F}(X)}$).
b) if moreover X does not contain unconditional basic sequences, then S has finite range.

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b) if moreover X does not contain unconditional basic sequences, then S has finite range.

Proof

a) if $T = Id + S$ is an isometry then its image $\tilde{T}$ in the Banach algebra $\mathcal{L}(X)/\overline{\mathcal{F}(X)}$ generates a bounded group and has spectrum $\{1\}$, therefore by Gelfand's theorem $\tilde{T}$ is a multiple of the unit, i.e. $T = Id + K$, $K \in \overline{\mathcal{F}(X)}$.

b) By Fredholm theory the spectrum is a finite sequence or an infinite sequence of eigenvalues tending to 1 together with the value 1. In the former case, use Gelfand's theorem to see that $T - Id$ has finite range. In the latter case, pick a sequence $e^{i\theta_n}$ of eigenvalues tending fast enough to 1, then show that the associated sequence x_n of eigenvectors is unconditional, a contradiction. Indeed for well-chosen k

$$\Big\|\sum_{j=1}^{n} a_j x_j\Big\| = \Big\|\sum_j a_j e^{ik\theta_j} x_j\Big\| \simeq \Big\|\sum_j \pm a_j x_j\Big\|,$$

The main result: HI spaces

So this applies to HI spaces, and we obtain:

Corollary

If X is HI, then every isometry T on X has the form $\lambda Id + F$. It follows that T acts nearly trivially on X:
there exists a decomposition $X = F \oplus H$, with $\dim F < +\infty$, $T(F) = F$, $T_{|H} = \lambda Id_H$ for some λ.

Note that this result holds for any equivalent renorming of X, and is the least we can get from an isomorphic property.

So on the one hand $\mathrm{Isom}(X)$ is closed in $GL(X)$; on the other hand all isometries are of the form $\lambda Id + F$, $F \in \mathcal{F}(X)$. So why do we not get, as limits of those, isometries of the form $\lambda Id + K$, $K \notin \mathcal{F}(X)$?

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The main result: HI spaces

One answer would be that the whole isometry group acts trivially apart on a finite-dimensional subspace, i.e. the decomposition $F \oplus H$ would be the same for all $T \in \mathrm{Isom}(X)$ on a HI space X. Then we would be done! Indeed:

Observation

If $\mathrm{Isom}(X)$ acts nearly trivially on X, i.e. there exists a decomposition

$$X = F \oplus H,$$

isometry invariant, where F is finite dimensional, and every T in $\mathrm{Isom}(X)$ acts as a multiplie of the identity on H, then the norm is not maximal.

Indeed decompose the finite codimensional part $H = F' \oplus H'$ and renorm to increase the "real" action of the isometry group to $F \oplus F'$ instead of F...

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isometry invariant, where F is finite dimensional, and every T in $\mathrm{Isom}(X)$ acts as a multiplie of the identity on H, then the norm is not maximal.

So if $\mathrm{Isom}(X, \|.\|)$ acts nearly trivially for any norm $\|.\|$ on X, then X admits no maximal norm.

The main theorem

Theorem

If X is HI separable reflexive without a Schauder basis, then $\mathrm{Isom}(X)$ acts nearly trivially on X: there exists a decomposition

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isometry invariant, where F is finite dimensional, and every T in $\mathrm{Isom}(X)$ acts as a multiple of the identity on H. It follows that X does not admit a maximal norm.
Moreover there exists such a space, superreflexive.

Much more generally:

Theorem

If X is separable reflexive and G a group of isometries of the form $Id + F$ which is SOT-closed in $GL(X)$, then G acts nearly trivially or X has a complemented subspace with a Schauder basis.

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For the proof, we shall first look at topologies on isometry groups.

Principles of representation theory - topologies

If X is a Banach space then we shall look at the norm topology or the pointwise convergence topology (SOT) on $GL(X)$.

A continuous representation of a topological group G on a Banach space X is an a homomorphism π of G into $GL(X)$ such that for all $x \in X$ the map $g \mapsto g.x$ is continuous.
In other words π is SOT-continuous.

Lemma (classical from descriptive set theory)

If X is separable, then $\mathrm{Isom}(X)$ is a Polish group (i.e. separable completely metrizable) for the SOT.

Indeed $\mathrm{Isom}(X)$ is SOT-closed in $GL(X)$ (but not in $L(X)$).

Lemma

If X is HI and has separable dual, then $\mathrm{Isom}(X)$ is norm separable, therefore the norm and pointwise convergence topologies coincide on $\mathrm{Isom}(X)$.

Proof: $\mathrm{Isom}(X)$ is a Polish group both for the norm and SOT, so use a theorem of Pettis.

A more general version of this is:

Lemma

If X is a Banach space with separable dual, and G is a SOT-closed subgroup of isometries of the form $Id + F$, then the norm and pointwise convergence topologies coincide on G.

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The main result: HI spaces

Going back to HI spaces what does it mean that the norm and strong operator topology coincide? Two cases come to mind:

1. the action is limited to a finite-dimensional space,
2. the isometry group is ”essentially” discrete in the SOT-topology.

Theorem

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Going back to HI spaces what does it mean that the norm and strong operator topology coincide? Two cases come to mind:

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Renorming theory: LUR norms

Objective: find an equivalent norm on a space X which is smoother, more differentiable, more convex, more rotund..., and relate the existence of such a norm to isomorphic properties of X.

For example, as we all know, are equivalent for a Banach space:

(i) X admits an equivalent uniformly convex norm,

(ii) X is superreflexive.

Definition

A norm $\|.\|$ is locally uniformly convex (LUR) if

$$\forall x_0 \in S_X \forall \epsilon > 0 \exists \delta > 0 \forall x \in S_X (\|x - x_0\| \geq \epsilon \Rightarrow \|\frac{x + x_0}{2}\| \leq 1 - \delta).$$

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Can we relate isomorphic properties of X to the existence of an equivalent LUR norm? Not really:

Theorem (Kadec, 61)

Every separable space X admits an LUR renorming.

Can we combine both directions of renorming theory, i.e. renorm with a smoother norm without diminishing the isometry group? The following is due to Bader, Furman, Gelander, Monod, 07.

Fact

If $(X, \|.\|)$ is superreflexive then there exists an equivalent uniformly convex norm $|||.|||$ on X such that $\mathrm{Isom}(X, \|.\|) \subset \mathrm{Isom}(X, |||.|||)$.

Indeed let $G = \mathrm{Isom}(X, \|.\|)$. Let $|.|_{uc}$ be an equivalent uniformly convex norm on X. Then define $|||x||| = \sup_{g\in G} |gx|_{uc}$.

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In the LUR case we have a result of G. Lancien.

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If $(X, \|.\|)$ is separable with the RNP then there exists an equivalent LUR norm $|||.|||$ on X such that $\mathrm{Isom}(X, \|.\|) \subset \mathrm{Isom}(X, |||.|||)$.

Can this be improved to all separable spaces X? No:

Example

The space L_1 cannot be renormed with an LUR norm $|||.|||$ such that $\mathrm{Isom}(L_1, \|.\|_{L_1}) \subset \mathrm{Isom}(L_1, |||.|||)$.

Proof: The usual norm on L_1 is quasi transitive. A new LUR norm with as many isometries would also be quasi transitive, so would have to be a multiple of $\|.\|_{L_1}$. But $\|.\|_{L_1}$ is not LUR.

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We may now use this towards our problem.

Theorem (Lancien, 93)

Let X have the RNP and separable dual. Then for every bounded subgroup G of $GL(X)$ there exists a G-invariant equivalent norm on X which is LUR and dual LUR.

Therefore to deduce from isomorphic properties of X that the isometry group G acts nearly trivially, we may assume that the norm is LUR with dual LUR norm.

This will allow us to use (or even reprove) Alaoglu-Birkhoff and Jacobs - de Leeuw - Glicksberg decompositions associated to group representations.

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Alaoglu - Birkhoff theorem (1940) and Jacobs - de Leeuw - Glicksberg theorems (1960s) from ergodic theory relate, for reflexive X:

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(a) *(Alaoglu - Birkhoff type decomposition)*

$$X = H_G \oplus (H_{G^*})^{\perp},$$

where $H_G = \{x \in X : Gx = \{x\}\}$, $H_{G^*} = \{\phi \in X^* : G\phi = \{\phi\}\}$, *and moreover* $H_{G^*}^{\perp} = \{x \in X : 0 \in \overline{conv}(Gx)\}$.

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- Existing proofs were about isometric representations on X reflexive, strictly convex with dual strictly convex norm. Lancien's result and our observation show that in the separable case, we may remove the strict convexity hypotheses and work with bounded representations.

- Then we may even assume that the norm is LUR with dual LUR norm and give an alternate proof of these decompositions. The point is that the duality map taking a point to its support functional will be an homeomorphism....

- However these authors also obtain results for bounded semigroups of operators and we don't.

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Conclusion of the proof of the main theorem

Theorem

Let X be separable, reflexive Banach space and G be a group of isometries on X of the form $Id + F$. Then

a) *G acts nearly trivially or X has a complemented subspace with a FDD.*

b) *if G is SOT-closed in $GL(X)$ then G acts nearly trivially or X has a complemented subspace with a Schauder basis.*

Proof of a): for each finitely generated subgroup $L =< T_1, \ldots, T_n >$ of G, the Alaoglu-Birkhoff decomposition is

$$X = H_{L^*}^{\perp} \oplus H_L = F_L \oplus H_L,$$

where $H_L = \cap_{i=1}^n ker(Id - T_i)$ is finite-codimensional and F_L finite dimensional.
So either G acts nearly trivially, or we may find a sequence $T_n \in G$ such that $F_{<T_1,\ldots,T_n>}$ is a strictly increasing sequence of finite-dimensional subspaces giving a FDD.

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Proof of b) (sketch): By Jacobs - de Leeuw - Glicksberg

$$X = K_{G^*}^{\perp} \oplus K_G.$$

First case: if $K_{G^*}^{\perp}$ is finite dimensional then it is trivial. Therefore G is almost periodic, i.e. all orbits are relatively compact, i.e. G is SOT-compact. Since both topology coincide, G is norm compact, and this implies that G acts nearly trivially (Shiga 55).

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Proof of b) (sketch): Second case: If $Y = K_{G^*}^{\perp}$ is infinite dimensional then using the fact that no non-trivial G-orbit on Y is relatively compact and some group theory, find U and a sequence $T_n \in G$ such that the Alaoglu-Birkhoff decompositions associated to $T_n^{-1}UT_n$ are sufficiently "disjoint", build from this a complemented subspace with an FDD with uniform dimension $\mathrm{codim} H_U$, then refine to a Schauder basis.

Conclusion

Theorem

There exists a separable, uniformly convex, HI space X, without a Schauder basis . Therefore there exists a separable, uniformly convex space X for which the isometry group acts nearly trivially for any equivalent renorming. In particular X does not have an equivalent maximal norm.

Contents

Open problems

Question

Show that L_p does not admit an equivalent transitive norm, for $p \neq 2$.
Show that ℓ_p does not admit an equivalent quasi transitive norm, for $1 < p \neq 2$.

It also remains open whether the isometry group must always act nearly trivially on any (reflexive) HI space. Or on the contrary:

Question

Find a SOT-closed group of isometries of the form $Id + F$ which is infinite discrete on a HI space? on the Hilbert space? on some separable Banach space?
Find whether such a group may act quasi transitively?

F. Cabello-Sánchez, *Regards sur le problème des rotations de Mazur*, Extracta Math. 12 (1997), 97–116.

V. Ferenczi and C. Rosendal, *On isometry groups and maximal symmetry*, preprint 2011,
arXiV math.FA.1106.5020, or http://www.math.uic.edu/ rosendal.

R. Fleming and J. Jamison, *Isometries on Banach spaces. Vol. 2. Vector-valued function spaces*, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 138. Chapman and Hall/CRC, Boca Raton, FL, 2008.

G. Pisier, *Are unitarizable groups amenable?*, Infinite groups: geometric, combinatorial and dynamical aspects, 323362, Progr. Math., 248, Birkhauser, Basel, 2005.

Radon Nikodým compacta

Piotr Koszmider

Polish Academy of Sciences

Definition (Namioka, Reynov, Stegall; 1979-87)

A compact space is said to be a Radon-Nikodým compact (or RNC for short) if and only if it is homeomorphic to a weak* compact subset of a dual Banach space with the Radon-Nikodým property.

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Theorem (Namioka, 87)

K is an RNC iff K is compact and there is a lower semi-continuous (l.s.c.) metric on K^2 which fragments K.

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Theorem (Orihuela, Schachermayer, Valdivia, 1991)

If a compact Hausdorff space is RNC and Corson compact, then it is Eberlein compact.

Question (Namioka, 87)

Are continuous images of RNC spaces RNCs?

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Theorem (A. Avilés, P.K.)

There is a continuous image of RNC which is not RNC.

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- *The class of subspaces of Asplund generated Banach spaces is associated with the class of continuous images of RN compacta, (Fabian)*

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Theorem (P.K., 2004)

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Theorem (A. Avilés, P. K)

Let L be a continuous image of a RN-compactum which is not RN-compactum. The space $C(L)$ is not isomorphic to any $C(K)$ where K is totally disconnected and L is sequentially compact, has many nontrival projections and the hyperplanes of $C(L)$ are isomorphic to the entire space.

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Differentiability of functions inside small subsets of infinite-dimensional spaces

Olga Maleva

University of Birmingham

Geometry of Banach Spaces, Luminy CIRM

27 August 2012

Background

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X^* separable $\Rightarrow$ X has "small" universal differentiability subsets.

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$\exists\ g : \mathbb{R} \to \mathbb{R}$ Lipschitz, not differentiable everywhere on S,
thus $f := g \circ P : X \to \mathbb{R}$ is Lipschitz and
$\forall x \in E$, directional derivative $f'(x, e)$ does not exist
$\Rightarrow \forall x \in E$, f is not differentiable at x.

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We construct a sequence (x_k, e_k), $x_k \in E$ and $\|e_k\| = 1$ such that $f'(x_k, e_k)$ exists and is "almost maximal" among $f'(x, e)$ when $x \in E$, $\|x - x_k\|$ is small and e is arbitrary direction.

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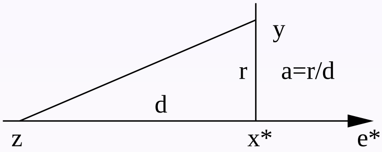
We then prove f is differentiable at x^* and $f'(x^*)(u) = f'(x^*, e^*)\Phi_{e^*}(u)$, where Φ_{e^*} is the Fréchet derivative of the norm $\|\cdot\|$ at e^*.

Finding a point of differentiability



$$M = f'(x^*, e^*) \geq 0$$

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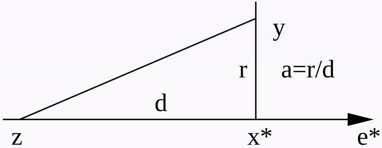

$f(y) > f(x^*) + \varepsilon r$
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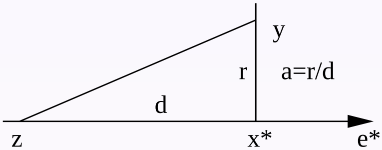

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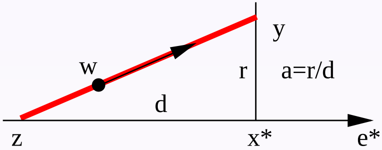
$$\frac{f(y) - f(z)}{\|y - z\|} \geq \frac{Md + \varepsilon r}{\sqrt{d^2 + r^2}} = \frac{M + \varepsilon a}{\sqrt{1 + a^2}} > M + \varepsilon a + O(a^2) > M + \tau$$

Finding a point of differentiability



$\frac{f(y)-f(z)}{\|y-z\|} > f'(x^*, e^*) + \tau$, $\tau > 0$ is fixed

Therefore there exists $w \in [y, z]$ such that $f'(w, \frac{y-z}{\|y-z\|}) > f'(x^*, e^*) + \tau$

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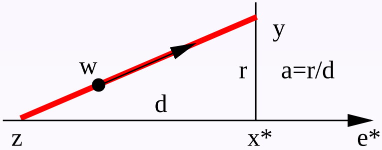

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If $[y, z] \subseteq E$, we get a contradiction.

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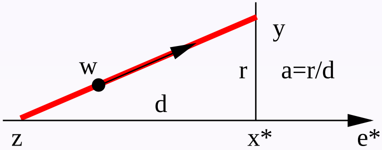

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Thus $f'(x^*, e^{*\perp}) = 0$.

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    - Need to "dynamically" define G_n on each step in order to control the Minkowski dimension.

Equivalent definitions of a u.p.u. sets

Theorem. G. Alberti, M. Csörnyei, D. Preiss (2010): $S \subseteq \mathbb{R}^n$ The following two conditions are equivalent:

1. There exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\forall x \in S$ and $\forall \|e\| = 1$ the directional derivative $f'(x, e)$ does not exist

Geometric measure theory

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Each uniformly purely unrectifiable set is **purely unrectifiable**:
its intersection with any smooth curve has 1-dimensional measure 0.

Geometric measure theory: Open question

Does there exist a purely unrectifiable set which is NOT uniformly purely unrectifiable?

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  - If $x_n \in E_n \setminus E_{n+1}$ then how to find $x_{n+1} \in E_{n+1}$ close to x_{n+1}?

THANK YOU!

Quasi-minimality and tightness in Banach spaces

A. Manoussakis

Department of Sciences
Technical University of Crete

joint work with S.A. Argyros and A. Pelczar-Barwacz.

Luminy, August, 2012

W.T. Gowers started a classification program for Banach spaces The goal of the program is to identify disjoint classes of Banach spaces which are

- inevitable, i.e. any Banach space contains an infinite dimensional subspace in one of those classes,
- hereditary, i.e. if a space belongs to a given class, then all of its closed infinite dimensional subspaces as well,
- belonging to one class gives a lot information about the structure of the space and what operators can be defined on it.

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The starting point was W.T. Gowers and B. Maurey example of a Hereditarily Indecomposable (H.I). Banach space.

Definition

A Banach space X is said to be H.I. if no subspace of X admits a non-trivial projection.

Theorem (Gowers 1st dichotomy)

Every Banach space contains either an HI subspace or a subspace with an unconditional basis.

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The H.I property$\Leftrightarrow$ $\mathrm{dist}(S_Y, S_Z) = 0$ for all Y, Z inf. dimensional subspaces of X implies *quasi-minimality*.

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X is quasi-minimal if every two inf. dimensional subspaces Y, Z of X have further subspaces Y_1, Z_1 resp. which are isomorphic.

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Definition (H.P. Rosenthal)

A Banach space X is said to be minimal if every infinite dimensional subspace of X has further subspace isomorphic to X.

ℓ_p-spaces are minimal, Schlumprecht's space is minimal space not containing any ℓ_p.

A quasi-minimal space which does not contain a minimal subspace is called strictly quasi-minimal .
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Theorem (Gowers classification)

Every Banach X space has a subspace Y with one of the following properties

- Y *is H.I space*
- Y *has unconditional basis and no two subspaces with disjoint support are isomorphic*
- Y *has unconditional basis and is quasi-minimal*
    1. *is strongly quasi-minimal*
    2. *is minimal*

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A dichotomy with respect the minimality was asked.

Definition (V. Ferenczi-C. Rosendal (2007))

A space X with a Schauder basis $(e_n)_n$ is tight if for every subspace Y of X there exists a successive sequence

$$I_0 < I_1 < I_2 < \dots$$

of subsets of $\mathbb{N}$ such that for every infinite subset A of $\mathbb{N}$, $Y \not\hookrightarrow [e_n, n \notin \cup_{i \in A} I_i]$.

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Example: Tsirelsons space T, Gowers space G_u with unconditional basis solving the hyperplane problem.

V.Ferenczi- C.Rosendal distinguish three forms of tightness

- *Tight with constants*: For every subspace Y the sequence of subsets $I_0 < I_1 < \dots$ witnessing the tightness of Y maybe chosen so that
$$Y \not\hookrightarrow_K [e_n : n \notin I_K] \ \forall K.$$
- *Tight by support*: For every block subspace $Y = [y_k]$ the sequence of subsets $I_0 < I_1 < \dots$ witnessing the tightness of Y maybe chosen so that
$$I_k = \mathrm{supp}(y_k).$$
- *Tight by range*: For every block subspace $Y = [y_k]$ the sequence of subsets $I_0 < I_1 < \dots$ witnessing the tightness of Y maybe chosen so that
$$I_k = \mathrm{range}(y_k).$$

Theorem ((FR)-classification 2007)

Any infinite dimensional Banach space contains a subspace from one of the following classes:

1. *HI, tight by range (Gowers HI space with asymptotic unconditional basis),*
2. *HI, tight, sequentially minimal (?)*
   (X is sequentially minimal if it is quasiminimal end every block subspace has subsequentiall minimal subspace)
3. *tight by support (the space G_u),*
4. *unconditional basis, tight by range, quasi-minimal (?),*
5. *unconditional basis, tight, sequentially minimal,(T-space),*
6. *unconditional basis, minimal (ℓ_p, c_0, T^*, S)*

The proof that the above examples of spaces are in classes 1,3,5,6 and for more examples in classes 1,3,5,6, V. Ferenczi-C. Rosendal, Banach spaces without minimal subspaces(Examples), JFA 2009, (Annales de l'Institut Fourier, (2012))

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Example of a space in class 2? i.e. an HI, tight by range and sequentially minimal space?

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Example of a space in class (4)?

We shall present an example of space in class (4). i.e. a space X with unconditional basis which is
a) quasi-minimal
b) tight by range $\Leftrightarrow$ no two block subspaces with disjoint ranges being comparable

Basic Notation

Given a symmetric subset $W \subset c_{00}(\mathbb{N})$, which contains the uvb $(e_n)_n$ we define

$$\|x\|_W = \sup\{< x, f >: f \in W\}$$

Given $x, y \in c_{00}(\mathbb{N})$ we say that

$$x < y \quad \text{if} \quad \max \operatorname{supp}(x) < \min \operatorname{supp}(y).$$

Let $\theta \in (0,1)$ and $n \in \mathbb{N}$. We say that a subset $W \subset c_{00}(\mathbb{N})$ is closed under $(\theta, \mathcal{A}_n)$-operation if whenever

$$f_1 < \cdots < f_n \text{ in } W \text{ then } \theta(f_1 + \cdots + f_n) \in W.$$

We consider two sequences of natural numbers
$(n_j)_j \nearrow \infty, (m_j)_j \nearrow \infty$.
We say that $f \in c_{00}(\mathbb{N})$ has weight m_j, $w(f) = m_j$, if f is the result of an $(m_j^{-1}, \mathcal{A}_{n_j})_j$ operation.

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Definition

A finite sequence $(f_1, \ldots, f_{n_{2j+1}})$ of $c_{00}(\mathbb{N})$ is said to be special sequence if every f_i is the result if an $(m_{2j_i}^{-1}, \mathcal{A}_{n_{2j_i}})$-operation, j_1 sufficiently large and for every $i > 1$, j_i is uniquely determined by the sequence $(|f_1|, |f_2|, \ldots, |f_{i-1}|)$

A variation of Gowers example G_u

Let W_1 be the smallest subset of $c_{00}(\mathbb{N})$ satisfying

(i) W_1 contains $(e_n)_n$

(ii) For every $f \in W_1$ and $g \in c_{00}$ with $|g| = |f|$, then $g \in W_1$

(iii) It is closed in the projections on the subsets of $\mathbb{N}$

(iv) It is closed in the even operations $(\frac{1}{m_{2j}}, \mathcal{A}_{n_{2j}})$

(v) It is closed in the odd operations $(\frac{1}{m_{2j+1}}, \mathcal{A}_{n_{2j+1}})$ on special sequences $f_1, f_2, \ldots, f_{n_{2j+1}}$

Let $\mathcal{X}_1 = \overline{(c_{00}(N), \|\cdot\|_{W_1})}$.

Properties of the space $\mathcal{X}_1$

The space $\mathcal{X}_1$ is

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(W.T. Gowers- B.Maurey, Math. Annalen 97) Every bounded operator T in $\mathcal{X}_1$ is of the form $T = D + S$ where D is a diagonal and S is strictly singular.

Variations of the norming set W_1

The projection on subsets is essential to prove the tightness by support.
Since we are looking for quasi-minimality we substitute this property with the projections on intervals of $\mathbb{N}$. i.e. we take W_2 to have the same properties as W_1 except that is closed in the projections on the intervals of $\mathbb{N}$.

Let W_2 be the smallest subset of $c_{00}(\mathbb{N})$ satisfying

(i) W_2 contains $(e_n)_n$

(ii) For every $f \in W_2$ and $g \in c_{00}$ with $|g| = |f|$, then $g \in W_2$

(iii) It is closed in the projections on the intervals of $\mathbb{N}$ (this only difference with the definition of W_1)

(iv) It is closed in the even operations $(\frac{1}{m_{2j}}, \mathcal{A}_{n_{2j}})$

(v) It is closed in the odd operations $(\frac{1}{m_{2j+1}}, \mathcal{A}_{n_{2j+1}})$ on special sequences $f_1, f_2, \ldots, f_{n_{2j+1}}$

Let $\mathcal{X}_2 = \overline{(c_{00}(N), \|\cdot\|_{W_2})}$.
The space $\mathcal{X}_2$ is reflexive with a 1-unconditional basis and quasi-minimal.

If in the set W_2 we add the condition to be closed in the rational convex combinations then we get a space X_3 with unconditional basis which is tight by support and hence not quasi-minimal.
In order to get tightness by range we need an additional property for the set W_2.

The G-operation

Definition

A subset $F = \{n_1 < n_2, \ldots, n_k\}$ of $\mathbb{N}$ is Schreier admissible, $\mathcal{S}$-admissible, if

$$k \leq n_1 < n_2 < \cdots < n_k.$$

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Definition (The G-operation)

Given $f \in c_{00}$ and $F = \{n_1, \dots, n_{2k}\}$ $\mathcal{S}$-admissible let

$$G_F(f) = \frac{1}{2}\sum_{i=1}^{k} E_{2i-1}f \quad \text{where} \quad E_{2i-1} = [n_{2i-1}, n_{2i})$$

The norming set a type-(4) space

Let W_4 be the smallest subset of $c_{00}(\mathbb{N})$ such that

(i) W_4 contains $(e_n)_n$

(ii) For every $f \in W_4$ and $g \in c_{00}$ with $|g| = |f|$, then $g \in W_4$

(iii) It is closed in the projections on the intervals of $\mathbb{N}$

(iv) It is closed in the even operations $(\frac{1}{m_{2j}}, \mathcal{A}_{n_{2j}})$

(v) It is closed in the odd operations $(\frac{1}{m_{2j+1}}, \mathcal{A}_{n_{2j+1}})$ on special sequences $f_1, f_2, \ldots, f_{n_{2j+1}}$

(vi) It is closed in the G-operation.

The set W_4 is the set W_2 with the additional property to be closed in the G-operation.

Let $\mathcal{X}_4$ be the completion of $(c_{00}, \|\cdot\|_{W_4})$.

Theorem

The space $\mathcal{X}_4$ is reflexive with unconditional basis, quasi-minimal and tight by range.

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Sketch of the tightness. Let (x_i) be normalized block sequence. We show that there exists no bounded operator T such that supp $T(x_i) \cap \text{range}(x_i) = \emptyset$ and T can be extended to an isomorphism from $[(x_i)]$ to X. This will prove that $\mathcal{X}_{(4)}$ is tight by range.

Let T be an operator as above and assume without loss of generality that $\|T\| \leq 1$. By the reflexivity of the space and passing to a subsequence we may assume that $(T(x_i))_i$ is a block sequence and moreover

$$\text{range}(x_i + Tx_i) < \text{range}(x_{i+1} + Tx_{i+1}) \ \forall i \in \mathbb{N}.$$

Lemma

Let $(x_k)_k$ be a normalized block sequence. Then for every $n \in \mathbb{N}$ there exists $l(n) \in \mathbb{N}$ such that for every finite subsequence $(x_n)_{n\in F}$ of (x_k) with $\#F \geq l(n)$ there exists a block sequence $y_1 < y_2 < \cdots < y_n$ of $(x_n)_{n\in F}$ such that $\|y_i\| \leq 2$ and

$$\|\frac{1}{n}(y_1 + y_2 + \cdots + y_n)\| = \|\sum_{n\in F} a_n x_n\| > 1$$

Observe that taking $l(n) \leq F$ we have that $(x_n)_{n\in F}$ is $\mathcal{S}$-admissible sequence i.e. $(\min \operatorname{supp}(x_i))_{i\in F}$ is $\mathcal{S}$-admissible.

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$$G_f = \tfrac{1}{2}(\operatorname{range}(x_{n_1})f + \operatorname{range}(x_{n_3})f + \cdots + \operatorname{range}(x_{n_{\#F-1}})f),$$

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$$G_f = \frac{1}{2}(\operatorname{range}(x_{n_1})f + \operatorname{range}(x_{n_3})f + \cdots + \operatorname{range}(x_{n_{\#F-1}})f),$$

It holds $G_f(x) \geq 1/4$ and $\operatorname{supp} G_f \cap supp T x_i = \emptyset$ for every i

We use now the machinery which has been developed for mixed Tsirelson spaces with 'conditional structure', i.e. we construct for $j \in \mathbb{N}$ a double block sequence $(u_i, f_i)_{i=1}^{n_{2j+1}}$ with the following main properties

1. $(f_1, \ldots, f_{n_{2j+1}})$ is a special sequence
2. $\|u_i\| = 1$ and $f_i(u_i) \geq 1/4$, $\text{range}(f_i) = \text{range}(u_i)$ $\forall i$
3. $\text{supp}\, f_i \cap \text{supp}\, Tx_j = \emptyset$ for all i, j.
4. for every $f \in W_4$ with $w(f) \neq w(f_i)$ it holds $|f(u_i)| \leq \max\{w(f), m_{2j_i}^{-2}\}(\approx 0)$.

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The functional $\frac{1}{m_{2j+1}}(f_1 + \dots + f_{n_{2j+1}})$ is the result of an $(m_{2j+1}^{-1}, \mathcal{A}_{n_{2j+1}})$-operation and hence belongs to W_4. It holds

$$\|\frac{1}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} u_i\| \geq \frac{1}{m_{2j+1}} f_i(\frac{m_{2j+1}}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} u_i) \geq 1/4m_{2j+1}$$

$$\text{while } \|\frac{1}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} Tu_i\| \leq \frac{1}{m_{2j+1}^2} \Rightarrow T \text{ no-isomorphism}$$

(V.Ferenczi-C.Rosendal 5th dichotomy). Every Banach space X contains a subspace Y with basis that is
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In the spaces $G_u, \mathcal{X}_1$ every bounded operator is of the form $T = D + S$, D diagonal and S strictly singular.
Question: Does the same holds for the operators on $\mathcal{X}_4$?

The quasi-minimality

The space $\mathcal{X}_4$ is
tight by range$\Leftrightarrow$ no two block subspaces with disjoint range be comparable
In order to get quasi-minimality we have to use block sequences with no disjoint range and to some extension to have the same range.

The basic idea is the following:
Step 1. Given two block subspace Y, Z of $\mathcal{X}_4$ we choose for j,

A) $y_1 < z_1 < y_2 < z_2 < \cdots < y_{n_{2j}/2} < z_{n_{2j}/2}$ $y_i \in S_Y, z_i \in S_Z$,
B) $y_i^*, z_i^* \in W$ such that $y_i^*(y_i) = 1 = z_i^*(z_i)$
C) $\mathrm{range}(y_i^*) = \mathrm{range}(y_i)$, $\mathrm{range}(z_i^*) = \mathrm{range}(z_i)$

and for $u = y + z = \dfrac{m_{2j}}{n_{2j}}(y_1 + z_1 + \cdots + y_{n_{2j}/2} + z_{n_{2j}/2})$ and

$$f = \tfrac{1}{m_{2j}}(y_1^* + z_1^* + \cdots + y_{n_{2j}/2}^* + z_{n_{2j}/2}^*) \qquad \text{it holds}$$

(I) $\|u\| = 1 (= f(u))$

(II) $\forall x^* \in W_4$:
$w(x^*) \neq m_{2j} \Rightarrow |x^*(u)| \leq \max\{w(x^*), m_{2j}^{-2}\} (\approx 0)$.

Note that $f(y) = 1/2 = f(z)$.

STEP 2. We construct vectors $y \in Y, z \in Z$ such that

a) they have almost the same range

b) they are normed by the same (unique) functional.

Using Step 1 we choose a double sequence $(u_i, f_i)_{i=1}^{n_{2j+1}}$ with the following properties

1. for every i, (u_i, f_i) satisfies $(I), (II)$ constructed as in step 1. i.e.
   - $u_i = y_i + z_i = \frac{m_{2j_1}}{n_{2j_i}} \sum_{k=1}^{n_{2j_i}/2} (y_{ik} + z_{i,k})$
   - $f_i = \frac{1}{m_{2j_i}} \sum_k (y^*_{i,k} + z^*_{i,k})$,
   - $f_i(y_i) = 1/2 = f_i(z_i)$
2. $(f_1 \ldots, f_{n_{2j+1}})$ is a special sequence

Let $y = \frac{m_{2j+1}}{n_{2j+1}} \sum_i y_i$, $z = \frac{m_{2j+1}}{n_{2j+1}} \sum_i z_i$ and $f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i$

Note that range$(y) \approx$ range(z) and $f(y) = \frac{1}{2} = f(z)$.

Property (2) in the choice of the double sequence (u_i, f_i) i.e. for $f \in W_4$ with $w(f) \neq w(f_i)$ it holds

$$|f(u_i)| \leq \max\{w(f), m_{2j_i}^{-2}\}(\approx 0).$$

as well as the definition of the special sequences,

the weight of f_i is uniquely determined by $(|f_1|, \ldots, |f_{i-1}|)$,

implies that if a functional g gives good estimation on y then g must equal to f or at least be " an initial segment" of f. So if a functional "norms" y also "norms" also z.

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So if a functional "norms" y also "norms" also z.
Taking sequences $(y_i) \in Y$ and $(z_i) \in Z$ where for every i, y_i, z_i are as y and z above we prove that $(y)_i \sim (z_i)_i$.

Thank you

A Banach space with rich spreading model structure

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(joint work with Spiros A. Argyros)

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Introduction

In this lecture we will present an example of a Banach space $\mathfrak{X}_{\text{usm}}$, the main property of which, is that every block subspace of $\mathfrak{X}_{\text{usm}}$ contains a spreading model universal block sequence $\{z_k\}_k$.

The definition of the space $\mathfrak{X}_{\text{usm}}$ is very similar to the corresponding one of the space $\mathfrak{X}_{ISP}$, as the method used to construct the norming set W is saturation under constraints and the unconditional frame is Tsirelson space.

The key difference between the two constructions, is the way special functionals are defined.

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The Problem

- In 2005 G. Androulakis, E. Odell, Th. Schlumprecht, N. Tomczak-Jaegermann, pose the following question. Does there exist a Banach space X such that for every subspace Y of X and $1 \leqslant p < \infty$, ℓ_p is a spreading model of Y?

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Theorem (S. A. Argyros, P. M.)

There exists a reflexive space $\mathfrak{X}_{\text{usm}}$ with a Schauder basis $\{e_n\}_n$ satisfying the following properties.

(i) The space $\mathfrak{X}_{\text{usm}}$ is hereditarily indecomposable.

(ii) There exists a uniform constant C satisfying the following. Every block sequence contains a further block sequence $\{x_k\}_k$, such that every 1-subsymmetric sequence is C-equivalent to the spreading model generated by a subsequence of $\{x_k\}_k$.

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Further properties of the space $\mathfrak{X}_{\text{usm}}$

- The space $\mathfrak{X}_{\text{usm}}$ does not contain a subspace which is tight by range, in a strong sense. More precisely, every Y block subspace of $\mathfrak{X}_{\text{usm}}$ contains a seminormalized block sequence $\{x_n\}_n$ satisfying the following. There exists an onto isomorphism $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ such that $T(x_{2n-1}) = x_{2n}$ for $n \in \mathbb{N}$.

A similar result to the above was previously proven by V. Ferenczi and Th. Schlumprecht. In particular, they constructed a Gowers-Maurey type space, not containing a subspace which is tight by range. (*Subsequential minimality in Gowers and Maurey spaces*, to appear in Proc. London Math. Soc.)

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  This fact can actually be deduced from the previous property and V. Ferenczi - Ch. Rosendal dichotomy (*Banach spaces without minimal subspaces*, J. Funct. Anal. (2009)). In the present case however it is possible to identify the sequence $\{x_k\}_k$, witnessing the sequential minimality of the space.

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Spreading model universal sequences

- Let $\{u_k\}_k$ denote the unconditional universal basic sequence of Pełczyński.
- Then, for any $\varepsilon > 0$, every 1-subsymmetric sequence, is $1 + \varepsilon$-equivalent to some subsequence of $\{u_k\}_k$.
- Assume that you have a sequence $\{x_k\}_k$ in some Banach space, satisfying the following.
- There exists a constant $C > 0$, such that for any $F \in \mathcal{S}$, $\{x_k\}_{k \in F}$ is C-equivalent to $\{u_k\}_{k \in F}$.
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- Then, for any $\varepsilon > 0$, every 1-subsymmetric sequence, is $1+\varepsilon$-equivalent to some subsequence of $\{u_k\}_k$.
- Assume that you have a sequence $\{x_k\}_k$ in some Banach space, satisfying the following.
- There exists a constant $C > 0$, such that for any $F \in \mathcal{S}$, $\{x_k\}_{k\in F}$ is C-equivalent to $\{u_k\}_{k\in F}$.
- Then, every 1-subsymmetric sequence, is C-equivalent the spreading model generated by some subsequence of $\{x_k\}_k$.
- The goal is to construct a space, where such a sequence can be found in any further subspace.

The norming set W_{usm}

- The construction of the set W_{usm} uses as an unconditional frame the set $W_{(1/2^n, \mathcal{S}_n)_n}$ and it is quite similar to the norming set of the space $\mathfrak{X}_{\text{ISP}}$.
- The $(\frac{1}{2^n}, \mathcal{S}_n, \alpha)$ and $(\frac{1}{2^n}, \mathcal{S}_n, \beta)$ operations are also used here.
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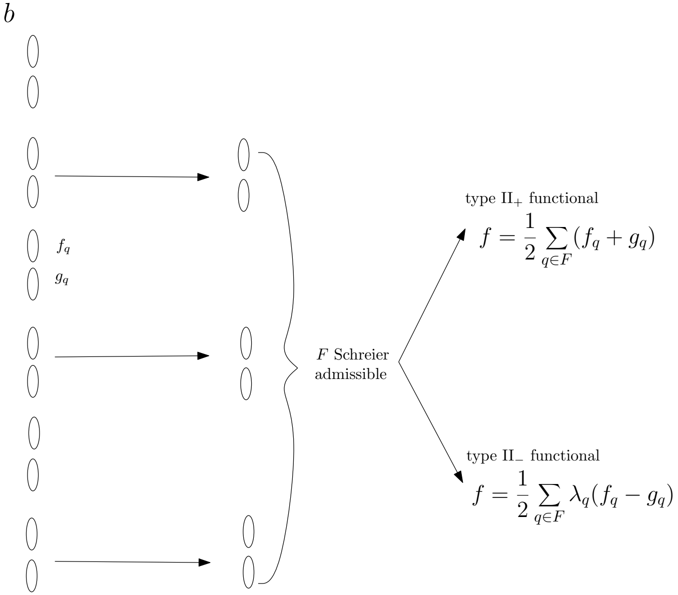
The norming set W_{usm}

In every HI construction the special functionals form an infinite branching tree, such that the weight of each functional appearing in the tree, uniquely determines all its predecessors. In our case the tree structure is more involved, as every branch of the tree defines two kinds of special functionals.

The norming set W_{usm}



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The norm on the space $\mathfrak{X}_{\text{usm}}$ is induced by a norming set W_{usm} which is the minimal set satisfying the following properties.

(Type I_α functionals) The set W_{ISP} is closed in the $(\frac{1}{2^n}, \mathcal{S}_n, \alpha)$ operations.

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- (special sequences) A sequence of pairs of functionals of type I_α $\{f_q, g_q\}_q$, is called a special sequence if
- $$f_1 < g_1 < f_2 < g_2 < \cdots < f_q < g_q < \cdots$$
- $$w(f_q) = w(g_q) \quad \text{for} \quad q = 1, \ldots, d$$
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(Type II$_-$ functionals) The set W_{usm} includes all $E\phi$, with E an interval of the naturals and

$$\phi = \frac{1}{2}\sum_{q\in F}\lambda_q(f_q - g_q)$$

with $\{f_q, g_q\}_q$ a special sequence of type I$_\alpha$ functionals, $F \subset \mathbb{N}$ with $\#F \leqslant \min F$ and $\{\lambda_q\}_{q\in F} \subset \mathbb{Q}$ such that $\|\sum_{q\in F}\lambda_q u_q^*\|_u \leqslant 1$, where $\{u_k^*\}_k$ denotes the biorthogonals of the unconditional basis of Pełczyński.

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c_0 spreading models in $\mathfrak{X}_{\text{usm}}$

- As in the case of the space $\mathfrak{X}_{\text{ISP}}$,the critical ingredient is sequences generating c_0 spreading models, which are achieved through the fact that the norming set W_{usm} is saturated under constraints.
- Here, the α and β indices are also used to identify the spreading models admitted by a block sequence. However, unlike in the previous case, they are not enough to fully describe them.

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$$\lim_k \sup \left\{ |f_q(x_k)| \vee |g_q(x_k)| : q \in \mathbb{N} \right\} = 0$$

Then there exists a subsequence $\{x_{k_n}\}_n$ of $\{x_k\}_k$ generating a c_0 spreading model.

- The type II_+ functionals play a crucial in role identifying block sequences satisfying property (ii) of the above proposition, in every block subspace.

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Spreading models

- In every block subspace, there exist seminormalized block sequences $x_1 < y_1 < x_2 < y_2 < \cdots < x_q < y_q < \cdots$ and $\{f_q, g_q\}_q$ special sequences such that $\{x_q, f_q\}, \{y_q, g_q\}$ are exact pairs.
- Then the sequence $\{x_q - y_q\}_q$ is spreading model universal, i.e. every 1-subsymmetric sequence is 146-equivalent to the spreading model generated by a subsequence of $\{x_q - y_q\}_q$.
- Moreover, the sequences $\{x_q\}_q, \{y_q\}_q$ generate ℓ_1 spreading models and both admit as biorthogonal sequence, the sequence $\{f_q + g_q\}_q$, which generates a c_0 spreading model in the dual.

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The space $\mathfrak{X}_{\text{usm}}$ is not tight by range

Proposition (Ferenczi - Schlumprecht)

Let $\{x_n\}_n, \{y_n\}_n$ be block sequences in a Banach space with a basis. If the maps $x_n \to x_n - y_n$ and $y_n \to x_n - y_n$ extend to bounded linear operators, then the sequences $\{x_n\}_n$ and $\{y_n\}_n$ are equivalent.

- We choose sequences $\{x_q\}_q, \{y_q\}_q$ as above and any L infinite subset of the natural numbers such that $\{x_q - y_q\}_{q \in L}$ generates a spreading model not equivalent to ℓ_1. Since $\{x_q\}_q$ and $\{y_q\}_q$ generate ℓ_1 spreading models the assumptions of the above proposition are satisfied and therefore they are equivalent.
- Moreover, since $\{f_q + g_q\}_q$ generates a c_0 spreading model in the dual and they are biorthogonal to both $\{x_q\}_q$ and $\{y_q\}_q$, the exists an isomorphism $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ such that $Tx_q = y_q$ for $q \in \mathbb{N}$.

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The space $\mathfrak{X}_{\text{usm}}$ is sequentially minimal

- In every block subspace there exists a block sequence $\{x_q\}_q$ and $\{f_q\}_q$ a sequences of type I_α functionals in W_{usm}, such that $\{x_q, f_q\}$ are exact pairs and for every natural number n, there exist infinitely many q with $w(f_q) = n$.

Theorem

Let $\{x_q\}_q$ be a sequence as above. Then in every block subspace Y of $\mathfrak{X}_{\text{usm}}$, there exists a sequence $\{y_k\}_k$, a strictly increasing sequence of natural numbers $\{q_k\}_k$ and $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an onto isomorphism, such that $Tx_{q_k} = y_k$ for $k \in \mathbb{N}$.

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Let $1 < q < \infty$ and q' be its conjugate. Then the following holds.

- If $\{x_m^*\}_m$ is a block sequence in $\mathfrak{X}_{\text{usm}}^*$ and $\{x_k\}_k$ is a block sequence in $\mathfrak{X}_{\text{usm}}$ satisfying the following,

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then the map $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ with $Tx = \sum_{k=1}^{\infty} x_k^*(x)x_k$ is bounded, non compact and strictly singular.

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Let Y be a block subspace of $\mathfrak{X}_{\text{usm}}$. Then there exists $S : Y \to Y$ a strictly singular operator which is not polynomially compact.

- The construction of S goes as follows.
- We choose a strictly increasing sequence of real numbers $\{p_n\}_n, p_1 > 2$ and appropriate block sequences $\{x_k^n\}_n$, such that $\{x_k^n\}_n$ generates an ℓ_{p_n} spreading model.
- Using the previous proposition, construct operators S_n such that $S_n x_k^n = x_k^{n+1}$ for all k, n, whereas $S_n x_k^l = 0$ for all $n \neq l$.
- Then $S = \sum_{n=1}^{\infty} \frac{1}{2^n} S_n$ is the desired operator.

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Thank you!

Fréchet Differentiability of Lipschitz Functions and Porous Sets in Banach Spaces

Based on a recent book with the same title by
Joram Lindenstrauss, David Preiss and Jaroslav Tišer,
(Annals of Mathematics Studies 179)

with background results mostly from
Yoav Benyamini and Joram Lindenstrauss,
Geometric Nonlinear Functional Analysis
(AMS Colloquium Publications, volume 48)

Luminy, August 2012

Key open problem and (optimistic) conjectures

Key problem. Is it true that any three real-valued Lipschitz functions on an (infinite dimensional separable real) Hilbert space have a common point of Fréchet differentiability?

Conjecture I. Any countable collection of real-valued Lipschitz functions on a Banach space with separable dual has a common point of Fréchet differentiability.

Conjecture II. On a Banach space with separable dual, if we have a countable collection f_i of Lipschitz, RNP-valued functions, there is a point x such that for each i,

- f_i is Gâteaux differentiable at x, and
- f_i is Fréchet differentiable at x provided that the set of Gâteaux derivatives of f_i is norm separable.

Basic notions

Let $f : X \to Y$, where X, Y are separable real Banach spaces.

Lipschitz: $\|f(y) - f(x)\| \leq C\|y - x\|$ for all $x, y \in X$. The least such C is the Lipschitz constant of f, denoted $\mathrm{Lip}(f)$.

Derivative of f at x_0 in the direction of $u \in X$ is

$$f'(x_0; u) := \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

Fréchet derivative of f at $x_0 \in X$ is $f'(x_0) = f'_F(x_0) \in \mathcal{L}(X, Y)$, i.e., a continuous linear map $f'(x_0) : X \to Y$, such that

$$f(x_0 + u) = f(x_0) + f'(x_0)u + o(\|u\|) \text{ as } \|u\| \to 0.$$

Gâteaux derivative of f at x_0 is $f'(x_0) = f'_G(x_0) \in \mathcal{L}(X, Y)$ such that $f'(x_0)(u) = f'(x_0; u)$ for every u.

Porosity

A set $P \subset X$ is said to be porous at x if there are $x_k \to x$, $r_k > \|x_k - x\|$ and $c > 0$ such that $B(x_k, cr_k) \cap P = \emptyset$.

Trivial but important observation. P is porous at x_0 iff

$$x \to \mathrm{dist}(x, P)$$

is not Fréchet differentiable at x_0.

Definition. Porous sets are sets porous at all their points.
Definition. σ-porous sets are countable unions of porous sets.

Note. Conjectures I, II are often stated as asking for a concept of null sets (a proper σ-ideal of Borel subsets of X) such that every Lipschitz $f : X \to \mathbb{R}$ is Fréchet differentiable a.e., etc.

The observation says σ-porous set must be null. For Gâteaux differentiability, a similar role is played by directional porosity, which requires the x_k to be on one line through x.

Exercises

(P.1) For every σ-porous set S construct a Lipschitz $f : X \to \mathbb{R}$ that is Fréchet non-differentiable at every point of S.

(P.2) Can you do the same for σ-directional porosity and Gâteaux non-differentiability?

(P.3) Let $0 < c < 1$. Show every σ-porous set can be written as a union of porous sets P_i such that each P_i is porous with constant c (i.e., for each $x \in P_i$ the definition of porosity of P_i holds with the given c).

(P.4) Let P be porous in direction $x = \sum_{i=1}^{n} \alpha_i x_i$. Show that $P = \bigcup_{i=1}^{n} P_i$ where P_i is porous in direction x_i.

(P.5) If in (P.4), P is porous in direction x with constant c, what are possible porosity constants of the sets P_i?

Lipschitz isomorphism problem.

Problem. If f is a Lipschitz isomorphism of a separable X onto Y, are X and Y (linearly) isomorphic?

Approaches via differentiability.

- If $f'_F(x_0)$ exists at some x_0, it is an isomorphism of X onto Y; but f may be nowhere Fréchet differentiable.
- If $f'_G(x_0)$ exists at some x_0, it is an isomorphism onto a subspace of Y. In some cases, this leads to a positive answer: Hilbert space, other classical spaces, or spaces not isomorphic to any proper subspace.
- If $f'_G(x_0)$ exists at some x_0, and $(y^* \circ f)'_F(x_0)$ exists for a dense set of $y^* \in Y^*$, then $f'_G(x_0)$ is an isomorphism of X onto Y. Hence Conjecture II implies a positive answer for RNP spaces with separable dual.

Lipschitz quotient problem.

Lipschitz quotient $f : X \to Y$ is a map which is Lipschitz and co-Lipschitz (there is $c > 0$ so that $f(B(x,r)) \supset B(f(x), cr)$).

Problem. Under what conditions does the existence of a Lipschitz quotient of X onto Y imply that Y is a (linear) quotient of X?

Comments.

- Separability is not sufficient: if $X \supset \ell_1$, every separable Y is a Lipschitz quotient of X.
- If for some x_0 there is a dense set of $y^* \in Y^*$ such that $(y^* \circ f)'_F(x_0)$ exists, then $y^* \to (y^* \circ f)'_F(x_0)$ is an embedding of Y^* to X^*.
- Conjecture I implies a positive answer for separable reflexive spaces.

RNP and Gâteaux differentiability

Definition. Y has the RNP if every Lipschitz $f : \mathbb{R} \to Y$ is differentiable at least at one point (equivalently a.e.)

Theorems. *If X is separable and Y has the RNP, every Lipschitz $f : X \to Y$ is Gâteaux differentiable a.e.*

Definitions of null sets. A Borel set $N \subset X$ is null if

- (Mankiewicz, 1972) Whenever $x_i \in X$ have dense span, $\sum_i \|x_i\| < \infty$, and $x \in X$, then $\{t \in [0,1]^{\mathbb{N}} : x + \sum_i t_i x_i \in N\}$ is a null set in $[0,1]^{\mathbb{N}}$.
- (Aronszajn, 1976) For every $x_i \in X$ with dense span, $N = \bigcup_i N_i$, N_i Borel and null on every line in direction of x_i.
- (Phelps, 1978) N is null for every non-degenerated Gaussian measure on X.

Theorem (Csörnyei, 1999) *These three notions coincide.*

Haar null sets

Definition (Christensen, 1972) A Borel set $N \subset X$ is Haar null if there is a Borel probability measure μ on X such that $\mu(x+N) = 0$ for every $x \in X$.

Remarks.

- Form a proper σ-ideal.
- Definition makes sense in topological groups.
- Our previous null sets are Haar null.
- Rediscovered by Hunt, Sauer and Yorke (1992) and called "shy" with complements called "prevalent."
- There is an interesting extension defining shy subsets of convex completely metrizable sets (Anderson and Zame, 2001)

Strengthening of Gâteaux differentiability results

Theorem (Zajíček,P) *For any Lipschitz $f : X \to Y$ there is a σ-directionally porous set $S \subset X$ such that f is Gâteaux differentiable at every point of $X \setminus S$ at which it is differentiable in a spanning set of directions.*

Corollary. Given any Lipschitz $f : X \to Y$, there is a Lipschitz $h : X \to \mathbb{R}$ such that f is Gâteaux differentiable at every point of X for which there is a spanning set of directions in which $(f, h) : X \to Y \times \mathbb{R}$ is differentiable.

Note. The published proof adds countably many functions h, but it should be possible to reduce it to just one by methods of Exercise (P.2).

Corollary. Aronszajn's Theorem.

Gâteaux differentiability and behaviour of sets on curves

Definition. A Borel set $N \subset X$ is null on curves close to direction $u \in X \setminus \{0\}$ if there is $\eta > 0$ so that N has measure zero on every curve $\gamma : \mathbb{R} \to X$ such that $\mathrm{Lip}(\gamma(t) - tu) < \eta$.

Denote by $\mathcal{N}$ the σ-ideal generated by such sets.

Theorem (Zajíček, P, 2001) *Every Lipschitz $f : X \to Y$, X separable, Y with RNP, is Gâteaux differentiable $\mathcal{N}$-a.e.*

Possibly spurious improvements and problems.

- This can be improved to Aronszajn-like decomposition, but we do not know if it is a genuine strengthening?
- If we consider only real-valued f, we can replace null by "regularly null:" For every $\varepsilon > 0$ there is open $G \supset N$ so that $\lambda(\gamma^{-1}(N)) < \varepsilon$ for all $\gamma : \mathbb{R} \to X$ with $\mathrm{Lip}(\gamma(t) - tu) < \eta$. Does this hold in general? Is it a genuine strengthening?

Problem. In ℓ_2, do the σ-ideals generated by sets of Gâteaux nondifferentiability of real-valued and ℓ_2-valued functions coincide?

The UPU problem

The simplest version. Let $N \subset \mathbb{R}^2$ be a Borel set such that

$$\lambda\{t; (t, g(t)) \in N\} = 0$$

for every $g : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(g) < 2$.
Is there an open set $G \supset N$ such that

$$\lambda\{t; (t, g(t)) \in G\} < 1$$

for every $g : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(g) < 1$?

Comments

- True for K_σ sets. (Easy exercise.)
- Unknown, and in fact most interesting, for G_δ sets N.

How big can porous sets be on curves?

The real problem. On what X can one find a Lipschitz $f : X \to \mathbb{R}$ (or $f : X \to Y$, Y with RNP) whose set of points of Gâteaux differentiability is σ-porous?

Subproblem. In what X can one find a σ-porous set whose complement belongs to $\mathcal{N}$?

Partial answers

- X^* is separable iff σ-porous sets are null on typical curves.
- (Maleva) If $X \supset \ell_1$, there is a σ-porous set with complement null on all curves.
- In ℓ_p, $p \geq n \geq 2$, it is impossible to find a σ-porous set with complement expressible as a union of n sets, each of which is null on curves close to some direction.
- In c_0-like spaces it is impossible to find a σ-porous set with complement in $\mathcal{N}$.

Fréchet differentiability of Lipschitz functions – history

Literature contains several examples of real-valued Lipschitz functions on Hilbert spaces that are nowhere Fréchet differentiable.

These are also counterexamples to our Key Problem.

However, at the 1979 Conference on Banach Spaces at Kent State University, Robert Phelps gave a careful analysis of these examples and found a (different) mistake in each of them.

Theorem (P, 1990) *Every real-valued Lipschitz function on a Banach space with separable dual is Fréchet differentiable at least at one point.*

In spite of quite a lot of effort, the question whether or not this can be extended to complex-valued or $\mathbb{R}^n$-valued function remains largely open.

A real-valued example (Averbukh & Smolyanov, 1967)

Claim (without proof) A Lipschitz, nowhere Fréchet differentiable function:

$$f(x) = \int_0^\pi \sin(x(t))\,dt, \text{ as a map of } L_2(0,\pi) \text{ to } \mathbb{R}$$

Calculation of Fréchet derivative of f.

$$f(x+u) = f(x) + \int_0^\pi \cos(x(t))u(t)\,dt + \text{error},$$

where error is estimated by

$$\Big|\int_0^\pi (\sin(x(t)+u(t)) - \sin(x(t)) - \cos(x(t))u(t))\,dt\Big| \leq \frac{1}{2}\int_0^\pi |u(t)|^2 dt = \frac{1}{2}\|u\|^2 = o(\|u\|)$$

A vector-valued example (Sova 1966)

Example. A Lipschitz, nowhere Fréchet differentiable map

$$f(x)(t) = \sin(x(t)), \text{ as a map of } L_2(0,\pi) \text{ to itself}$$

Proof for $x = 0$. If $f'(0)$ exists,

$$f'(0)u = \lim_{t\to 0} \frac{\sin(0+tu) - \sin(0)}{t} = u.$$

But, as the measure of a set $S \subset (0,\pi)$ tends to zero,

$$\|f(0+\pi\mathbf{1}_S) - f(0) - f'(0)(\pi\mathbf{1}_S)\| = \|\sin(\pi\mathbf{1}_S) - \sin(0) - \pi\mathbf{1}_S\| = \pi\|\mathbf{1}_S\|$$

is not $o(\pi\|\mathbf{1}_S\|)$.

Almost Fréchet differentiability

Lindenstrauss, P, 1996. *On uniformly smooth spaces, for any Lipschitz $f : X \to \mathbb{R}^n$ and $\varepsilon > 0$ there are $x_0 \in X$ and a continuous linear $T : X \to \mathbb{R}^n$ so that*

$$\limsup_{\|u\|\to 0} \frac{\|f(x_0+u) - f(x_0) - Tu\|}{\|u\|} < \varepsilon$$

Johnson, Lindenstrauss, Schechtman, P, 2002. *The same result holds in asymptotically uniformly smooth spaces.*

Comment. These are the only known Fréchet-type differentiability results that hold even in situations when the so called "mean value estimates" are false.

Smoothness and Asymptotic Smoothness

Definition. Modulus of (uniform) smoothness of X is

$$\rho_X(t) = \sup_{\|x\|=1, \|y\|\le t} \frac{\|x+y\| + \|x-y\|}{2} - 1, \qquad t > 0$$

The space X is said to be uniformly smooth if $\lim_{\tau\to 0} \rho_X(t)/t = 0$.

Definition. Modulus of asymptotic uniform smoothness of X is

$$\bar\rho_X(t) = \sup_{\|x\|=1} \inf_{\dim(X/Y)<\infty} \sup_{\substack{y\in Y\\ \|y\|\le t}} \|x+y\| - 1, \qquad t > 0.$$

The space X is said to be asymptotically uniformly smooth if $\lim_{\tau\to 0} \bar\rho_X(t)/t = 0$.

Differentiability and Asymptotic Smoothness

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Suppose*

$$\bar\rho_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0.$$

Then every Lipschitz $f : X \longrightarrow \mathbb{R}^n$ has points of Fréchet differentiability.

Examples.

- $\bar\rho_{\mathrm{Hilbert}}(t) = \sqrt{1+t^2} - 1 = o(t^2 \log(1/t))$,
- $\bar\rho_{\ell_p}(t) = (1+t^p)^{1/p} - 1 = o(t^n \log^{n-1}(1/t))$ if $p \ge n$.
- $\bar\rho_{c_0}(t) = \max\{t-1, 0\} = o(t^n \log^{n-1}(1/t))$ for every n

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Conjecture II holds for spaces satisfying*

$$\bar\rho_X(t) = o(t^n) \text{ for every } n.$$

One dimensional mean value estimates

With the notable exception of "almost Fréchet differentiability results," the differentiability statements that we prove also include a (multidimensional) mean value estimate.

One dimensional mean value estimate. If $f : X \to \mathbb{R}$ is Lipschitz, then for every $x, u \in X$ and $\varepsilon > 0$, we can shift the segment $[x, x+u]$ slightly so that f is Gâteaux differentiable a.e. on it, and infer that

$$f(x+u) - f(x) \le \varepsilon + \sup\{f'_G(y)(u) : \mathrm{dist}(y, [x, x+u]) < \varepsilon\}$$

If X^* is separable, this holds even with Fréchet derivatives (but not by the same simple argument). In other words, the one dimensional mean value estimate holds for Fréchet derivatives of Lipschitz functions on spaces with separable dual.

Multidimensional mean value estimates

If M is a reasonable n-dimensional surface in X and $f : X \to \mathbb{R}^n$, results such as divergence theorem, Stokes theorem, etc, express (boundary) integrals of f as surface integrals of some linear forms of f'. This leads to inequalities analogous to those in the one dimensional case, and they will again hold with Gâteaux derivatives at points close to M.

Hence, to define what we mean by validity of multidimensional mean value estimates for Fréchet derivatives of $\mathbb{R}^n$-valued functions, we can avoid discussion of surfaces, and require that for every x at which f is Gâteaux differentiable, every $T \in \mathcal{L}(X, \mathbb{R}^n)^*$ and every $r > 0$,

$$\sup\{T(f'_F(y)) : \|y - x\| < r\} \geq T(f'_G(x))$$

Invalid mean value estimates

Definition. A normalized sequence of elements $x_i \in F$ is said to satisfy the upper q estimate if there is a constant A such that for any sequence (c_i) of real numbers and any k,

$$\|\sum_{i=1}^k c_i x_i\| \leq A(\sum_{i=1}^k |c_i|^q)^{1/q}.$$

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Suppose that, for some $q > n/(n-1)$, the dual space X^* contains a normalized sequence satisfying the upper q estimate. Then the multidimensional mean value estimate does not hold for Fréchet derivatives of Lipschitz $\mathbb{R}^n$-valued functions on X.*

Corollary. The multidimensional mean value estimate fails for Fréchet derivatives of Lipschitz $\mathbb{R}^3$-valued functions on Hilbert spaces, and for Fréchet derivatives of Lipschitz $\mathbb{R}^n$-valued functions on ℓ_p when $p < n$.

Large porous sets — again

An important part of our arguments are statements that σ-porous sets are small. (Of course, we also know that for differentiability results to hold this is necessary.)

In situations that we cannot handle, σ-porous sets can be rather large.

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Let X be a separable Banach space, and let $n > 1$. Suppose that, for some $\beta > 1$, the dual space X^* contains a normalized sequence satisfying the upper $t^{n/(n-1)}/\log^{\beta}(1/t)$ estimate. Then X contains a σ-porous subset whose complement meets every n-dimensional Lipschitz surface in a set of n-dimensional Hausdorff measure zero.*

Γ-null sets

Γ_n-null sets: Borel sets $N \subset X$ such that $\lambda_n(\gamma^{-1}(N)) = 0$ for typical (in the sense of Baire category) $\gamma \in C^1([0,1]^n, X)$.

Γ-null sets: Borel sets $N \subset X$ such that $\lambda_\infty(\gamma^{-1}(N)) = 0$ for typical (in the sense of Baire category) $\gamma \in C^1([0,1]^{\mathbb{N}}, X)$, where C^1 means continuous with continuous partial derivatives.

Theorem (Lindenstrauss, P) *Conjecture II holds for spaces in which every porous set is Γ-null.*

Problem. Is it true that every real-valued Lipschitz function X is Fréchet differentiable a.e. with respect to the σ-ideal generated by sets of Gâteaux non-differentiability (of real-valued Lipschitz functions, of RNP-valued Lipschitz functions) and porous sets?

Porosity and Γ_n-nullness

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *A separable Banach space has separable dual iff every porous set in X is Γ_1 null.*

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Every directionally porous set is Γ_1-null and Γ_2-null.*

Theorem (Speight, 2012) *For every $n \geq 3$, every space of dimension $> n$ is a union of a Γ_n-null set and a σ-directionally porous set.*

Theorem (Lindenstrauss, Tišer, P, 2002–2012). *Suppose*

$$\bar{\rho}_X(t) = o(t^n \log^{n-1}(1/t)) \text{ as } t \to 0. \qquad (*)$$

Then every porous set in X is contained in a union of a σ-directionally porous set and a Γ_n-null G_δ set.

It follows that if (*) holds for every n, every porous set is Γ-null.

Use of variational principles

Observation. *Suppose that for the given Lipschitz function f on a Hilbert space H we find a point $x \in H$ and a unit vector u such that $f'(x;u)$ exists and is equal to the Lipschitz constant of f. Then f is Fréchet differentiable at x.*

Improved observation. *Suppose that f is a Lipschitz function and $\Phi : X \times X \to \mathbb{R}$ is a locally uniformly continuous function having continuous derivative with respect to the second variable. Then f is Fréchet differentiable at a point x provided that we may find a vector u such that the function*

$$(y,v) \to f'(y;v) - \Phi(y,v)$$

attains its maximum at (x,u).

One may hope to use a smooth variational principle to find such Φ and (x,u).

A smooth variational principle

Suppose that $h\colon M \to \mathbb{R}$ is lower bounded and lower semicontinuous on a complete metric space (M,d). Suppose further that $F_j\colon M \times M \to [0,\infty]$, $j \geq 0$, are functions lower semicontinuous in the second variable with $F_j(x,x) = 0$ for all $x \in M$ and that $0 < r_j \leq \infty$ are such that $r_j \to 0$ and

$$\inf_{d(x,y)>r_j} F_j(x,y) > 0.$$

If $x_0 \in M$ and $(\varepsilon_j)_{j=0}^\infty$ is any sequences of positive numbers such that

$$h(x_0) < \varepsilon_0 + \inf_{x\in M} h(x) \quad \text{and} \quad \inf_{d(x_0,y)>r_0} F_0(x_0,y) > \varepsilon_0,$$

then one may find a sequence $(x_j)_{j=1}^\infty$ of points in M converging to some $x_\infty \in M$ such that the function

$$H(x) := h(x) + \sum_{j=0}^{\infty} F_j(x_j, x)$$

attains its minimum on M at x_∞.

An idea from descriptive theory

A serious difficulty with the use of smooth variational principles is that the domain of the function $h(x,u) = f'(x;u)$ whose maximum we are seeking (after a perturbation) is not a complete metric space (it is not defined on all of $X \times X$) and, even if it is defined everywhere, it is not continuous (not even semi-continuous).

The domain M of h is a Borel set, so it is a 1-1 projection of a G_δ set G in some $X \times X \times Z$; metrizing G by a complete metric, we can move it to a not topologically equivalent metric on M to make it a complete metric space. Similarly, we can imagine making h continuous.

Obviously, abstract parameters Z would not help, but this led us to seeking "natural" parameters or distance that give the required completeness and continuity, and at the same time control the "differentiability behaviour" of f.

Cone monotone functions

Definition. A function $f : X \to \mathbb{R}$ is cone-monotone if there is a nonempty open cone C in X such that $f(y) \geq f(x)$ whenever $y - x \in C$.

Theorem. Every cone-monotone function defined on a space X with separable dual has a point of Fréchet differentiability outside any given σ-porous set.

An appropriate version of the mean value estimate holds as well: For any $a, b \in X$ for which $b - a$ belongs to C, any σ-porous set P, and any $\varepsilon > 0$ there is $x \in X \setminus P$ with $\operatorname{dist}(x, [a, b]) < \varepsilon$ at which f is Fréchet differentiable and $f'(x; b - a) < f(b) - f(a) + \varepsilon$.

Metric space for Lipschitz $f : X \to \mathbb{R}$

We metrize

$$M = \{(x, u) : \|x - x_0\| < r,\ \|u - u_0\| < r,\ f'(x; u) \text{ exists}\}$$

by

$$d((x, u), (y, v)) = \max\{\|x - y\|, \|u - v\|,\ \varrho(f_{x,u}, f_{y,v})\}$$

where $f_{x,u} = f(x + u) - f(x)$ and

$$\varrho(g, h) = \sup_{t \in (-r,r)\setminus\{0\}} \frac{|g(t) - h(t)|}{|t|}.$$

Perturbations for $f : X \to \mathbb{R}$

$F_i : M \times M \longrightarrow [0, \infty)$,

$$F_i((x, u), (y, v)) = \Phi_i(x, y) + \Psi_i(u, v) + Q_i((x, u), (y, v)) + \Delta_i((x, u), (y, v)),$$

where

$$\Phi_i(x, y) = \lambda_i\|y - x\|, \quad \Psi_i(u, v) = \lambda_i\|v - u\|^2,$$

$$Q_i((x, u), (y, v)) = \sigma_i\big(f'(y; v) - f'(x; u)\big)^2,$$

and

$$\Delta_i((x, u), (y, v)) = \min\{\lambda_i, \max\{0, \varrho(f_{x,u}, f_{y,u}) - s_i\}\}.$$

Notice that the peculiarity in the definition of Δ_i is not a misprint: Δ_i really does not depend on v.

Key lemma

Lemma. Let $h\colon [a, b] \to \mathbb{R}$ be such that $h(b) = h(a) = 0$. Assume that

$$\kappa := \sup_{t\in[a,b]} \overline{D}h(t) < \infty.$$

Then there is a set $S \subset [a, b]$ such that $\lambda(S) \geq \|h\|_\infty/(3\kappa)$ and for every $\xi \in S$,

- h is differentiable at ξ;
- $h'(\xi) \geq \dfrac{\|h\|_\infty}{3(b - a)}$;
- $|h(t) - h(\xi)| \leq 60\sqrt{\kappa h'(\xi)}|t - \xi|$ for every $t \in [a, b]$.

Zippin's Embedding Theorem, and Beyond

Thomas Schlumprecht

Texas A&M University

Luminy, August, 2012

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2. *If X is reflexive (e_i) is shrinking and boundedly complete ($X \equiv \overline{\mathrm{span}(e_j^* : j \in \mathbb{N})}^*$ canonically).*

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[Ryan Causey, 2012] If X^* is separable and $\mathrm{Sz}(X) \le \omega^{\alpha}$, then Z embeds into a space Z such that $\mathrm{Sz}(Z) \le \omega^{\alpha+1}$.

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Skipped means

$$\max \operatorname{supp}_E(x_n) < \min \operatorname{supp}_E(x_{n+1}) - 1.$$

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Starting with a separable space X we want to construct a space Z with an FDD, being as close to X as possible.

Finite Dimensional Decompositions

A sequence of finite dimensional subspaces (F_n) of X is called

Finite Dimensional Decomposition of X (FDD)

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Proposition (Johnson 1971)

If (F_n) is an FDD for a Banach space X then there are finite dimensional spaces (G_n) so that $X \oplus (\oplus G_n)_{\ell_2}$ has a basis.

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General Assumption: (after renorming)

(e_i, e_i^*) is 1-bounded and 1-norming and $\|e_i\| = \|e_i^*\| = 1$, for all $i \in \mathbb{N}$.

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$m_0 = 0$, $m_1 \in \mathbb{N}$ arbitrary, $m_{j+1} = k(\varepsilon_{j+1}, m_j)$, and then

$$E_j = \text{span}(e_i : m_j < i \leq m_{j+1}) \text{ and } E^*_j = \text{span}(e^*_i : m_j < i \leq m_{j+1}).$$

Notation: For $x^* \in X^*$ we let

$\mathrm{supp}_{E^*}(x^*) = \{j \in \mathbb{N} : x^*|_{E_j} \neq 0\}$, and

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- that (F_j) is unconditional, if X has the UTP,
- to be able to turn Z into a reflexive space (still containing X and having an FDD) if X is reflexive.

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Lemma

If X^* is separable then $\mathcal{A}_c = \big\{\{n_1(x^*), \dots n_{l(x^*)}(x^*)\} : x^* \in S_{X^*}\big\}$ is compact in $[\mathbb{N}]^\omega$.

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4. $\left\| \sum_{j=l_1}^{l_2} x_j^* \right\| \leq 5$, *if* $1 \leq l_1 \leq l_2 \leq l$
5. $\{\min rg_{E^*}(x^*) : i = 1, 2 \ldots l\} \in \{A \cup B : A, B \in \mathcal{A}_c\}$

Replacing in proof of [FOS] c–decomposition by overlapping c-decomposition leads to a Bourgain Delbaen space Z containing X with $Z^* = \ell_1$.

In order to assure $\mathrm{Sz}(Z) \leq \omega^{\alpha\omega}$ if $\mathrm{Sz}(X) \leq \omega^{\alpha\omega}$, we need that $CB(\mathcal{A}_c) < \mathrm{Sz}(X)$. We will need to block the E_n and E_n^* again

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Sketch: Choose $n > m$ large enough so that

$$(m+1, n-1) \notin \Big\{ \bigcup_{s=1}^{4} A_i : A_i \in \mathcal{A}_\varepsilon \Big\}.$$

Let $G_j = E_{n_{j-1}+1} + E_{n_{j-1}+2} + \ldots E_{n_j}$, with $n_{j+1} = n(\varepsilon_j, n_j)$.

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$$m_{j+1}(x^*) = \min\Big\{ r > m_j(x^*) : \Big\| \sum_{i=m_j(x^*)+1}^{m_{j+1}(x^*)} dv_i^*(x^*) \Big\| > c \Big\}.$$

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Then $CB(\mathcal{B}_c) < \mathrm{Sz}(X)$, and thus, if $\mathrm{Sz}(Z) = \mathrm{Sz}(X)$, where Z is defined as before, by replacing E_j by G_j.

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So on $\tilde{Z} = c_{oo}(\oplus_{j=1}^{\infty} G_j + G_{j+1})$ use

$$|||(z_j)||| = \sup \Big\{ \Big| \sum_{j\in A} dv_j^*(x^*)(z_j) \Big| : x^* \in B,\ A \subset \mathbb{N} \Big\},$$

$$\Big[\text{instead of } |||(z_j)||| = \sup \Big\{ \Big| \sum_{j=1}^{l} dv_j^*(x^*)(z_j) \Big| : x^* \in B,\ l \in \mathbb{N} \Big\}\Big]$$

Recent advances in the theory of isomorphic polyhedrality

Richard J. Smith[1] (with V. P. Fonf[2], A. J. Pallares[3] and S. Troyanski[3])

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[2] Ben Gurion University of the Negev, Israel

[3] University of Murcia, Spain

Luminy, August 2012

Isomorphically polyhedral Banach spaces

Definition (Klee 60)

A Banach space $(X, \|\cdot\|)$ is *polyhedral* if, given any finite-dimensional subspace $E \subseteq X$, there exist $f_1, \ldots, f_n \in X^*$ such that

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6. If $p \in (1, \infty)$, there exists i.p. $E_p \not\hookrightarrow C(K)$ (K countable), and $E_p \twoheadrightarrow \ell_p$.

Boundaries and property $(*)$

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Constructing boundaries with $(*)$ is the principal tool for obtaining isomorphically polyhedral spaces.

w^*-LRC sets

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We say that a subset $E \subseteq X^*$ is *w^*-locally relatively norm compact* (w^*-LRC) if, given $f \in E$, there exists a w^*-open set U containing f, such that $E \cap U$ is relatively norm compact.

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If $\dim X = \infty$ then S_{X^*} is never σ-w^*-LRC, by Baire Category. However, it is possible for some boundaries to be σ-w^*-LRC.

σ-w^*-LRC boundaries

Theorem

Let $\varepsilon > 0$ and let $(X, \|\cdot\|)$ admit a boundary that is covered by a σ-w^*-LRC and w^*-K_σ set.

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If X admits a boundary covered by a norm K_σ, then X admits a norm as above.

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Proposition

Suppose that $(C(K), \|\cdot\|_\infty)$ admits a boundary that is covered by a σ-w^*-LRC and w^*-K_σ set. Then K is σ-discrete.

Splitting the unit sphere of X

We can extend the previous theorem by considering countable splittings of S_X.

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Let $H_n \subseteq B_{X^*}$, $n \in \mathbb{N}$, be an increasing sequence of relative boundaries, each covered by a σ-w^*-LRC and w^*-K_σ set.

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Moreover, suppose we can write $S_X = \bigcup_{n=1}^{\infty} S_n$, such that

$$b_n \;=\; \inf\{\sup\{f(x) \;:\; f \in H_n\} \;:\; x \in S_n\} \;>\; 0 \quad \text{and} \quad b_n \;\to\; 1.$$

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Then X admits an equivalent (polyhedral) norm with a boundary having $(*)$.

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Proposition

Let $(X, \|\cdot\|)$ be *separable* and isomorphically polyhedral. Then, for every $\varepsilon > 0$, there are countable subsets $H_n \subseteq B_{(X^*,\|\cdot\|)}$, $n \in \mathbb{N}$, having $(*)$, such that

$$b_n = \inf\{\sup\{f(x) : f \in H_n\} : x \in S_X\} > 1 - \varepsilon \quad \text{and} \quad b_n \to 1.$$

Isomorphically polyhedral Orlicz spaces

Corollary (cf FPTS 08)

Let $(x_\gamma, x_\gamma^*)_{\gamma\in\Gamma}$ be a strong M-basis, and suppose that $S_X = \bigcup_{n=1}^\infty S_n$ and

$$b_n \;=\; \inf\left\{\sup\left\{f(x) \;:\; f \in B_{X^*}, \mathrm{card}(\mathrm{supp}(f)) \leq n\right\} \;:\; x \in S_n\right\}$$

behave as above. Then X admits an equivalent (polyhedral) norm as above.

Isomorphically polyhedral Orlicz spaces

Corollary (cf FPTS 08)

Let $(x_\gamma, x_\gamma^*)_{\gamma\in\Gamma}$ be a strong M-basis, and suppose that $S_X = \bigcup_{n=1}^\infty S_n$ and

$$b_n \;=\; \inf\{\sup\{f(x) \;:\; f \in B_{X^*}, \operatorname{card}(\operatorname{supp}(f)) \leq n\} \;:\; x \in S_n\}$$

behave as above. Then X admits an equivalent (polyhedral) norm as above.

Example (Leung 94)

Let Γ be a set, M a non-degenerate Orlicz function and consider $(h_M(\Gamma), \|\cdot\|_M)$. Suppose further that there exists $K > 1$ satisfying

$$\lim_{t\to 0} \frac{M(Kt)}{M(t)} \;=\; \infty.$$

Then h_M is isomorphically polyhedral.