Synchronized automata

Marie-Pierre Béal, Dominique Perrin

We present recent insights in automata theory related to synchronizing sequences. Imagine a map with roads which are colored in such a way that a fixed sequence of colors, called a homing sequence or a synchronizing sequence, leads the traveler to a fixed place whatever be the starting point. Such a coloring of the roads is called synchronized.

We first consider Cerny's Conjecture which says that the minimal length of a synchronizing sequence in an n-state synchronized colored (directed) graph is at most (n-1)^2. We present a proof of this conjecture in the particular case of aperiodic colored graphs which is due to Trahtman, and a recent result from Carpi and D'Alessandro for locally strongly transitive colored graphs. We then present Trahtman's Road Coloring Theorem for finding a synchronized coloring. The Road Coloring Theorem states that every aperiodic directed graph with constant out-degree has a synchronized coloring. Finally, we show the importance of the existence of synchronizing sequences, or synchronizing patterns, in the domain of symbolic dynamics.

Topological substitutions

Nicolas Bedaride

We define 2-dimensional topological substitutions. A tiling of the Euclidean plane, or of the hyperbolic plane, is substitutive if the underlying 2-complex can be obtained by iteration of a 2-dimensional topological substitution. We give examples of substitutive tilings of the euclidean plane or the hyperbolic plane.

Text Redundancies

Maxime Crochemore

The talks discuss several questions related to redundancies in texts, regarded as sequences of symbols: repetitions, powers, runs, repeats, etc.
Among the questions treated were the avoidability of long repetitions, bounds on the number of runs in words and efficient algorithms to compute all these redundancies.

CALCULATING THE GARSIA ENTROPY IN LINEAR NUMERATION SYSTEMS

MARCIA EDSON

First consider the sequence-based numeration systems given by the linear recurrence

$$G_{n+2} = aG_{n+1} + bG_n \text{ for } n \geq 0,$$
$$G_0 = 1, G_1 = a + 1 \text{ where } a, b \in \mathbb{N}, a \geq b.$$

The most well known of these is the Fibonacci numeration system, obtained when $a = b = 1$.

The above recurrence is such that the dominating root $\beta_{(a,b)}$ of its characteristic equation satisfying

$$\beta_{(a,b)}^2 = a\beta_{(a,b)} + b$$

is a Pisot number. We simply write β in place of $\beta_{(a,b)}$ unless there is a chance for ambiguity.

We consider sums of the form $\sum_{n=1}^{N} a_n \beta^{-n}$ where $a_n \in A = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$. Let $A_N = \{x : x = \sum_{n=1}^{N} a_n \beta^{-n}\}$, and define a measure $\mu_N = (a+1)^{-N} \sum_{x \in A_N} r(x)\delta_x$, where $r(x)$ is the number of representations of x of length N in base β and δ_x denotes the unit point mass at x. Then these measures converge weakly to a measure μ_β. Jessen and Wintner in 1938 show that any convergent infinite convolution is either purely singular or absolutely continuous. In particular, we have that the measures μ_β are either purely singular or absolutely continuous.

In 1939, Erdős proved that for $\beta = \frac{1+\sqrt{5}}{2}$, μ_β is purely singular. Garsia, in 1963, in order to study the measures μ_β further, introduced the idea of the Garsia entropy which is defined as

$$H(A_n) = -\sum_{x \in A_n} p(x) \ln p(x)$$

where $p(x) = \frac{r(x)}{(a+1)^n}$ is the weight assigned to x by μ_n. Then set $H_\beta = \lim_{N \to \infty} \frac{H(A_N)}{N \ln \beta}$. Garsia proved for general β (not just β satisfying the above linear recurrence) that if $H_\beta < 1$, then μ_β is purely singular. Additionally, he showed that $H_\beta < 1$ for any Pisot number β. Though Garsia proved significant results involving H_β and μ_β, he did not give numerical values for H_β.

Then, in 1991, Alexander and Zagier considered the case $a = b = 1$, so that $\beta = \frac{1+\sqrt{5}}{2}$. Usually the problem of computing entropies is quite difficult but through a graph-theoretical argument, Alexander and Zagier give an explicit value for H_β, where $\beta = \frac{1+\sqrt{5}}{2}$. They make use of the Fibonacci graph, which can be built from the Euclidean tree. The Euclidean tree begins with one node at level 0 labeled with the pair $(1,1)$ and one node at level 1 labeled with the pair $(2,1)$. Then the nodes at level n are defined inductively as follows. Given a node at level n labeled (a,b), there are two edges (left and right) to nodes at level $n+1$ labeled $(a+b,a)$ and $(a+b,b)$, respectively. Therefore this tree corresponds to the *subtractive Euclidean algorithm*, the Euclidean algorithm without division.

For any pair of relatively prime integers (k,i), we define the *length* $e(k,i)$ of the pair (k,i) to be the number of steps in the subtractive Euclidean algorithm applied to the pair k and i. In other words, $e(i,i)=0$ and $e(i+k,i)=e(i+k,k)=e(i,k)+1$.

Grabner, Kirschenhofer, and Tichy, in 2002, give an explicit value for H_β in the case β is the dominating characteristic root of the m-bonacci recurrence which satisfies

$$\beta^m = \beta^{m-1} + \cdots + \beta + 1,$$

extending the results of Alexander and Zagier. The graph-theoretic approach taken by Alexander and Zagier becomes significantly more complicated in this case. Therefore, they abandon this approach in favor of one using generating functions and the method of Guibas and Odlyzko for counting strings with forbidden subwords. A generalization of these results can be found in the 2002 doctoral dissertation of M. Lamberger, a student of Grabner, so that the case $a=b$ is completed. Therefore, when we discuss the Garsia entropy, we assume that $a>b$, and note that the counting is necessarily more complicated in the case where $a>b$, due to the number of forbidden subwords.

In the situation of the general a and b we will discuss here, a graph-theoretic approach would lead to a non-planar graph. Therefore, to simplify the arguments, we shall use combinatorics on words. The main result is as follows.

Theorem 1. *Let* $\kappa_n = \sum_{\substack{0<i<k \\ \gcd(k,i)=1 \\ e(k,i)=n}} k \ln k$, *where* $e(k,i)$ *is the length of the pair* (k,i) *as defined above.*

Furthermore, let $\mathcal{T}(x) = \ln(a+1) - \widehat{M}(x) \sum_{N=1}^{\infty} \kappa_N x^{2N}$,

where

$$\widehat{M}(x) = \frac{(a-b+1)(1-x)\gamma(x)(1-3x^2)^2}{(a+1)(1+x)^3(1-(3+2a-2b)x^2)^2},$$

and

$$\gamma(x) = a + 2ax - (2+3a+2a^2-2b-2ab)x^2 + (2+4a+2a^2-6b-6ab+4b)x^3.$$

Then

$$H_{\beta(a,b)} = \frac{1}{\ln \beta_{(a,b)}} \mathcal{T}\left(\frac{1}{a+1}\right).$$

If time allows, we will see how to explicitly produce values for $H_{\beta(a,b)}$.

Symbolic dynamics, multidimensional subshifts, computability and arithmetic

Mike Hochman

In this course we will discuss recent advances in the classification of the dynamics and certain invariants of combinatorial dynamical systems: by this we mean multidimensional shifts of finite type (Wang tiling systems), sofic shifts, and cellular automata. We will discuss combinatorial invariants, primarily entropy, and the classification of the numbers arising in this way. Then we will discuss the class of effective dynamical systems, which is the natural context within which to study combinatorial dynamical systems, the properties of the category, and invariants related to computability theory. Finally we will discuss the classification of (sub)dynamics of combinatorial dynamical systems.

The lectures will be organized as follows.

Lecture 1: Introduction to combinatorial dynamical systems, combinatorial invariants, and the arithmetic hierarchy of real numbers.

Lecture 2: Proof of the classification of entropies of shifts of finite type and cellular automata.

Lecture 3: Overview of topological dynamics, effective dynamical systems, computability degrees, basics results in the effective category.

Lecture 4: Classification of subdynamics and other results on combinatorial and effective dynamical systems.

Fast arithmetical algorithms in Moebius number systems

P.Kůrka

A Moebius number systems is given by a finite system of Moebius transformations and represents real numbers by symbolic sequences of these transformations. A modular Moebius number system consists of transformations with integer coefficients and unit determinant (continued fractions can be conceived as a modular system). We show that in modular Moebius number system, the computation of any Moebius transformation can be performed by a finite transducer, so that it has linear time complexity. There exist also redundant Moebius number systems (which cannot be modular), in which the computation of any Moebius transformation has linear time complexity with probability one.

Non-normal numbers: The interplay of symbolic and topological dynamical systems

Manfred G. Madritsch*
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On the one hand in a recent paper Olsen [4] considered the extremely non-normal numbers $\mathbb{E}$ of the unit interval. These are real numbers $x \in [0,1)$ having any possible probability vector as accumulation point for the frequency of all digit blocks. He was able to show the following

Theorem ([4, Theorem 1]). *The set $\mathbb{E}$ is residual for $q \geq 2$, i.e. $[0,1) \setminus \mathbb{E}$ is of the first category. In particular, the set $\mathbb{E}$ is of the second category.*

On the other hand Albeverio, Pratsiovytyi and Torbin [1] considered the essentially non-normal numbers $\mathbb{L}$ of the unit interval. These are real number $x \in [0,1)$ having no asymptotic frequency of all digits in their nonterminating q-adic expansion. Their result reads as follows.

Theorem ([1, Theorem 1]). *The Hausdorff dimension of $\mathbb{L}$ is 1.*

The main goals of this talk are to extend these results to fibered systems and to separate the parts of their proof concerning the symbolic structure from those concerning the topological one. In our definitions we mainly follow Cornfeld, Fomin and Sinaĭ [2] (for the topological dynamical systems) and Schweiger [5] (for the fibred system).

Let X be a compact metric space and $T : X \to X$ be a continuous map, then we call the pair (X,T) a topological dynamical system. Moreover, we call the pair (X,T) a fibered system if the following three conditions hold:

1. There is a finite or countable set $\mathcal{D}$ (which is called the digit set).
2. There is a map $d : B \to \mathcal{D}$. Then the sets
$$B_i := d^{-1}(i) = \{x \in B : d(x) = i\}$$
form a partition of B into intervals such that $B_i \cap B_j = \emptyset$ for $i \neq j$.
3. The restriction of T to any B_i, which is denoted by T_i, is an injective map having continuous partial derivatives.

Examples of such dynamical systems are the following:

- *The q-ary numbers.* Let $q \geq 2$ be an integer, then for $i \in \mathcal{D} = \{0, \dots, q-1\}$
$$B = [0,1), \quad B_k = \left[iq^{-1}, (i+1)q^{-1}\right) \quad T_k x = qx - \lfloor qx \rfloor = qx - i.$$
- *The continued fraction expansion.* For $i \geq 1$ let
$$B = [0,1), \quad B_k = \left[\frac{1}{i-1}, \frac{1}{i}\right), \quad T_k x = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} - i.$$

Now we want to extend the topological dynamical system (X,T) to a measure-theoretic one. Therefore let $\mathfrak{B}$ be a σ-algebra of X and μ be a probability measure defined on $\mathfrak{B}$. If $T : X \to X$ is measure preserving, *i.e.* $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathfrak{B}$, then we call the quadruple

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$(X, T, \mathfrak{B}, \mu)$ a measure-theoretic dynamical system. Moreover we call it ergodic if T is ergodic, *i.e.* $\mu(T^{-1}A\Delta A) = 0$ implies that $\mu(A) = 1$ or $\mu(A) = 0$.

We fix a block $\mathbf{b} = b_1 \dots b_k \in \mathcal{D}^k$ and set for $N \geq 1$

$$\Pi(x, \mathbf{b}, N) := \left|\left\{0 \leq i < N : T^i x \in B_{b_1, b_2, \dots, b_k}\right\}\right| .$$

for the frequency of the block of "digits" $\mathbf{b}$ under the first n digits. Furthermore we define by

$$\mathbf{\Pi}_k(x, N) := (\Pi(x, \mathbf{b}, N))_{\mathbf{b} \in \mathcal{D}^k}$$

the vector of frequencies $\Pi(x, \mathbf{b}, N)$ of all blocks $\mathbf{b} \in \mathcal{D}^k$ of length k.

A number $x \in B$ is called k-T-normal if we have

$$\lim_{N \to \infty} \mathbf{\Pi}_k(x, N) = (\mu(B_{b_1, \dots, b_k}))_{\mathbf{b} \in \mathcal{D}^k} ,$$

where μ is the unique ergodic measure. Furthermore we call $x \in X$ T-normal if it is k-T-normal for all $k \geq 1$. Obviously this yields the definition of normal numbers in the q-ary case if we take T to be the transformation of our first example.

An application of Birkhoff's ergodic theorem yields for ergodic T that almost every number $x \in X$ is T-normal (*cf.* Chapter 3.1.2 of [3]). Thus we have that the normal numbers are a large set in the measure theoretic sense.

With this tools in hand we can define extremely non-normal numbers $\mathbb{E}$ and essentially non-normal numbers $\mathbb{L}$ by

$$\mathbb{E} = \bigcap_k \{x \in [0, 1) : \text{each } \mathbf{p} \in \mathbf{S}_k \text{ is an accumulation point of the sequence } (\Pi_k(x, n))_n\} ,$$

$$\mathbb{L} = \left\{x \in [0, 1) : \lim_{N \to \infty} \Pi(x, i, N) \text{ does not exist for all } i \in \mathcal{D}\right\} ,$$

where $\mathbf{S}_k$ denotes the simplex of shift invariant probability vectors in $\mathbb{R}^{|\mathcal{D}|^k}$.

Now we consider the two parts, namely the construction of expansion and the topological structure of (X, T), separately.

Since the definition of $\mathbb{E}$ depends only on the digital expansion we want to generate expansions having the desired properties. In particular, first we have to show that for each block length k every possible probability vector occurs as accumulation point. Secondly we have to show a way of extending this construction to one involving blocks of length $k + 1$.

In the second part we will consider the topological structure. In particular, we are looking for examples for pairs (X, T) such that $\mathbb{E}$ is residual.

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Kadanoff Sand Pile Model

Kevin Perrot

Sand pile models are dynamical systems describing the evolution from N stacked grains to a stable configuration. It uses local rules to depict grain moves and iterate it until reaching a fixed configuration from which no rule can be applied. The main interest of sand piles relies in their Self Organized Criticality (SOC), the property that a small perturbation --- adding some sand grains --- on a fixed configuration has unbounded consequences on the system, involving an arbitrary number of grain fall. Physicists L. Kadanoff et al. inspire KSPM, a model presenting a sharp SOC behavior, extending the well known Sand Pile Model. In KSPM(D), we start from a pile of N stacked grains and apply the rule: D-1 grains can fall from column i onto the D-1 adjacent columns to the right if the difference of height between columns i and i+1 is greater or equal to D. Toward the study of fixed points (stable configurations on which no rule can be applied) obtained from N stacked grains, we propose an iterative study of KSPM evolution consisting in the repeated addition of one single grain on a heap of sand, triggering an avalanche at each iteration. We develop a formal background for the study of avalanches, resumed in a finite state word transducer, and explain how this transducer may be used to predict the form of fixed points. Further precise developments provide a plain formula for fixed points of KSPM(3), showing the emergence of a wavy shape.

Repetitions in Words

Narad Rampersad

We present a selection of topics concerning the avoidance of repetitions in words. We describe some of the main results concerning overlap-free binary words and we give an Fife-like characterization of these words. We also give an overview of the work leading to the resolution of Dejean's Conjecture. We give a short introduction to the use of the probabilistic method in combinatorics on words. We present a result of Carpi on the avoidance of repetitions in arithmetic progressions. We introduce the concept of Abelian repetitions and we illustrate a typical method for showing that a morphism generates a word avoiding Abelian repetitions. We explain the notion of "pattern" and illustrate a way of using generating functions to show the avoidability of certain patterns. Finally we describe some algorithmic results concerning automatic sequences.

Linearly recursive sequences and Dynkin diagrams

Christophe Reutenauer,

Motivated by a construction in the theory of cluster algebras (Fomin and Zelevinsky), one associates to each acyclic directed graph a family of sequences of natural integers, one for each vertex; this construction is called a frieze; these sequences are given by nonlinear recursions (with division), and the fact that they are integers is a consequence of the Laurent phenomenon of Fomin and Zelevinsky. If the sequences satisfy a linear recursion with constant coefficients, then the graph must be a Dynkin diagram or an extended Dynkin diagram, with an acyclic orientation. The converse also holds: the sequences of the frieze associated to an oriented Dynkin or Euclidean diagram satisfy linear recursions, and are even N-rational. One uses in the proof objects called SL_2-tilings of the plane, which are fillings of the discrete plane such that each adjacent 2 by 2 minor is equal to 1. These objects, which application in the theory of cluster algebras, are interesting for themselves. Some problems, conjectures and exercises are given.

MORPHIC WORDS GOVERNING THE BOUNDARIES OF CELLULAR AUTOMATA

CHARLES D. BRUMMITT AND ERIC ROWLAND

Cellular automata are simple machines consisting of cells that update in parallel at discrete time steps. A *one-dimensional cellular automaton* consists of

- an alphabet Σ of size k,
- a positive integer d,
- a function $i : \mathbb{Z} \to \Sigma$, and
- a function $f : \Sigma^d \to \Sigma$.

The function i is called the *initial condition*, and the function f is called the *rule*. We think of the initial condition as an infinite row of discrete cells, each assigned one of k colors. To evolve the cellular automaton, we update each cell according to f, a function of d cells in its vicinity on the previous step. The evolution of a one-dimensional cellular automaton can be visualized two-dimensionally by displaying each row below its predecessor.

We are interested in the two boundaries between the foreground and background regions of a cellular automaton. For a given automaton, with an initial condition in which all but finitely many cells are in a single background state, let $\ell(t)$ be the length of the foreground region on step t. Since information has a maximum propagation speed in a cellular automaton, we have the bound $\ell(t) \leq (d-1)t+\ell(0)$.

Each row in a cellular automaton depends only on the previous row, so the difference sequence $\{\ell(t+1) - \ell(t)\}_{t\geq 0}$ is relevant. This sequence gives the number of cells the automaton grows or shrinks by at each step. Thinking of the difference sequence as an infinite word on the set of integers, we call this word the *boundary word* of a cellular automaton.

The purpose of this talk is to show that many properties of an automaton are reflected in its boundary word. These observations comprise is a new connection between cellular automata and combinatorics on words, and this relationship has number theoretic consequences for the integer sequence $\ell(t)$.

For many automata, the boundary word is eventually periodic, and for these automata $\ell(t)$ has the form

$$\ell(t) = \begin{cases} at + c_0 & \text{if } t \equiv 0 \mod m \\ at + c_1 & \text{if } t \equiv 1 \mod m \\ \quad\vdots & \qquad\vdots \\ at + c_{m-1} & \text{if } t \equiv m-1 \mod m \end{cases} \tag{1}$$

for t sufficiently large, where $a \in \mathbb{Q}$ is the average growth rate.

A systematic inventory of the boundaries of all $2^{16} = 65536$ cellular automaton rules with $k = 2$ colors and depending on $d = 4$ cells, begun from simple initial

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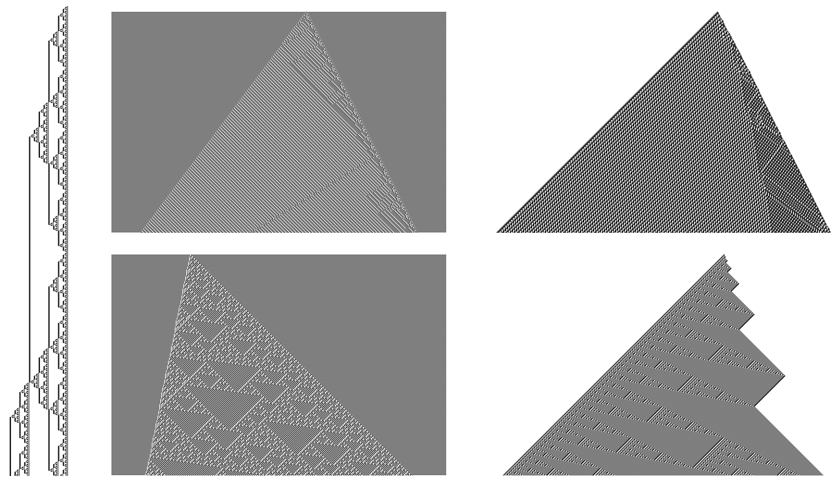

conditions, reveals several types of automata with nonperiodic boundary words. At least four distinct classes of these automata, however, have boundary words that are morphic. We briefly describe each type. In each case, the salient feature of the boundary is captured in a structural property of the boundary word.

Whereas most cellular automata grow linearly on average, the automaton on the left in the figure above grows like $\sqrt{t}$. The boundary word of this automaton is

$$\mathbf{w} = 2210221\bar{1}1\bar{1}1\bar{1}1022102210221\bar{1}1\bar{1}1\bar{1}1\bar{1} \cdots,$$

a word on the alphabet $\{-1, 0, 1, 2\}$, where $\bar{1} = -1$. If φ and ψ are the morphisms

$$\varphi = \{A \to ABC,\ B \to DAB,\ C \to CECE,\ D \to CECD,\ E \to CECE\}$$
$$\psi = \{A \to 2,\ B \to 2,\ C \to 1,\ D \to 0,\ E \to \bar{1}\},$$

then $\mathbf{w} = \psi(\varphi^\omega(A))$. The square-root growth can be derived from these morphisms. Moreover, had the morphism φ been k-uniform, $\ell(t)$ would have been a k-regular sequence. However, φ is not uniform, and indeed it appears that $\ell(t)$ is not k-regular for any small value of k. Whereas terms of k-regular sequences can be computed quickly, we conjecture that terms of $\ell(t)$ cannot be computed as quickly.

Some automata have boundary words that are nearly periodic but that are perturbed occasionally by particles that oscillate in the interior of the automaton. Two are shown in the first row of the figure. The morphisms that generate these words can also be given explicitly.

Finally, there are automata with average linear growth but with nontrivial fractal boundaries. Two are shown in the bottom row of the figure. These automata comprise two classes — those for which the limiting growth rate exists (left), and those for which it does not (right). Existence of the limiting growth rate depends on the existence of letter frequencies in the boundary word.

Of course, every eventually periodic word is also morphic, so we argue that sufficiently simple cellular automaton boundaries are characterized by morphic words, in a way that remains to be made precise.

Entropy minimality of ${\mathbb Z}^d$ shifts of finite type (joint work with Samuel Lightwood)

Michael Schraudner,

It is well-known that an irreducible $\mathbb Z$ shift of finite type X is always entropy minimal, i.e. that every proper subsystem of X has strictly smaller entropy. While this result is very useful and has a lot of applications in the theory of one-dimensional subshifts, it does not extend to the class of ${\mathbb Z}^d$ subshifts for $d>1$. In the multidimensional setting only very strong uniform mixing conditions (e.g.\ UFP) guarantee entropy minimality, whereas non-entropy minimal still uniformly mixing (block or corner gluing) examples exist. In this talk we will show some of these non-entropy minimal examples called wire shifts, analyze the mechanism behind this phenomenon and we will give a necessary and sufficient condition characterizing entropy minimality of general ${\mathbb Z}^d$ shifts of finite type.

Minimal digit sets for parallel addition in non-standard numeration systems

Christiane Frougny, Edita Pelantová, Milena Svobodová[1]

Short abstract for conference CANT 2012, Marseille, France

Considering a (positional) numeration system with base β (which can be integer / real / complex, with modulus $|\beta| > 1$) and a finite alphabet $\mathcal{A}$ of (real / complex) digits, we have to allow a certain level of redundancy for being able to perform the operation of addition in parallel.

Having proved that parallel addition is possible for a large class of complex bases β (in fact any algebraic number β whose all algebraic conjugates have modulus different from 1), we now focus on the question how large the alphabet $\mathcal{A}$ needs to be, in order to allow the addition in parallel. Or, in other words, what is the minimum level of redundancy for the alphabet $\mathcal{A}$ to enable addition in parallel in base β.

We restrict ourselves to the case of alphabets $\mathcal{A}$ of contiguous integer digits containing 0, which already implies that the base β is an algebraic number. For a real positive algebraic number β, the lower bound on the alphabet $\mathcal{A}$ for enabling parallel addition is $\lceil \beta \rceil$. If β is an algebraic integer with minimal polynomial $f(X)$, the lower bound is then equal to $|f(1)|$; and this bound can still be further refined to $|f(1)| + 2$ in case when β is a real positive algebraic integer.

For the bases β being algebraic integers of degree 1 (integers) and of degree 2 (quadratic Pisot units), we prove that these lower bounds are indeed attained in reality. The question of determining the size of minimal alphabet for parallel addition in other numeration systems remains open.

[1] presenting author

TUNING AND PLATEAUX FOR THE ENTROPY OF α-CONTINUED FRACTIONS

CARLO CARMINATI, GIULIO TIOZZO

The family $\{T_\alpha\}_{\alpha\in(0,1]}$ of α-continued fraction transformations is a family of discontinuous interval maps, which generalize the well-known Gauss map. For each $\alpha \in [0,1]$, the map T_α from the interval $[\alpha-1,\alpha]$ to itself is defined as $T_\alpha(0) = 0$ and, for $x \neq 0$,

$$T_\alpha(x) := \frac{1}{|x|} - c_{\alpha,x}$$

where $c_{\alpha,x} = \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$ is a positive integer. These maps have infinitely many branches, but, for $\alpha > 0$, all branches are expansive and so T_α admits an invariant probability measure absolutely continuous with respect to Lebesgue measure. Hence, each T_α has a well-defined metric entropy $h(\alpha)$.

The goal of the paper is to exploit the explicit description of the fractal structure of $\mathcal{E}$ to investigate the self-similarities displayed by the graph of the function $h(\alpha)$. Finally, a complete characterization of the plateaux occurring in this graph is provided, using the Hölder-continuity of the entropy.

Nakada [7], who first investigated the properties of this family of continued fraction algorithms, gave an explicit formula for $h(\alpha)$ for $\frac{1}{2} \leq \alpha \leq 1$, from which it is evident that entropy displays a phase transition phenomenon when the parameter equals the golden mean $g := \frac{\sqrt{5}-1}{2}$ (see also figure 1, left):

$$h(\alpha) = \begin{cases} \frac{\pi^2}{6\log(1+\alpha)} & \text{for } \frac{\sqrt{5}-1}{2} < \alpha \leq 1 \\ \frac{\pi^2}{6\log\frac{\sqrt{5}+1}{2}} & \text{for } \sqrt{2}-1 \leq \alpha \leq \frac{\sqrt{5}-1}{2} \end{cases} \tag{1}$$

Several authors have studied the behaviour of the metric entropy of T_α as a function of the parameter α ([1], [6], [8], [9], [2], [3], [5]); in particular [6] first produced numerical evidence that the entropy is continuous but it displays many more (even if less evident) phase transition points; moreover, they also showed that the entropy is not monotone on the interval $[0,1/2]$. Subsequently, in [8] it was shown that the entropy is monotone over intervals I_r in parameter space for which the orbits of the two endpoints collide after a finite number of steps; the change in monotonicity of entropy is due precisely to the coexistence of intervals with different combinatorics. Maximal components with fixed combinatorics, called *maximal intervals*, are canonically indexed by a set $\mathbb{Q}_E$ of rational numbers, and it is proven that the union of all such intervals has full measure ([2], [3]). The complement of this union, denoted by $\mathcal{E}$, is the set of parameters across which the combinatorics of T_α changes, hence it will be called the *bifurcation set*.

We shall investigate the self-similarities of the graph of entropy by exploiting the self-similarity of $\mathcal{E}$. A common way to study a self-similar object is to define renormalization operators which act on the parameter space of a particular class of dynamical systems. Tuning operators are the inverse of renormalization operators: in [4], by taking as a model the Douady-Hubbard tuning for quadratic maps, tuning operators for continued fractions are defined. In a nutshell, to each rational number r indexing a maximal interval, we associate a *tuning window* W_r and a *tuning map* $\tau_r : [0,g] \to W_r$ which maps parameter space into itself and preserves the bifurcation set $\mathcal{E}$. A tuning window r is called *neutral* if the alternate sum of its partial quotients is zero. Let us define a *plateau* of a real-valued function as a maximal, connected open set where the function is constant.

Theorem 1. *The function h is constant on every neutral tuning window W_r, and every plateau of h is the interior of some neutral tuning window W_r.*

Even more precisely, we will characterize the set of rational numbers r such that the interior of W_r is a plateau. A particular case of the theorem is the following recent result [5]:

$$h(\alpha) = \frac{\pi^2}{6\log(1+g)} \qquad \forall \alpha \in [g^2, g],$$

and (g^2, g) is a plateau (i.e. h is not constant on $[t, g]$ for any $t < g^2$).

On non-neutral tuning windows, instead, entropy is non-constant and h reproduces, on a smaller scale, its behaviour on the whole parameter space $[0, 1]$.

Theorem 2. *If h is increasing on a maximal interval I_r, then the monotonicity of h on the tuning window W_r reproduces the behaviour on the interval $[0, g]$, but with reversed sign: more precisely, if I_p is another maximal interval, then*

1. *h is increasing on $\tau_r(I_p)$ iff it is decreasing on I_p;*
2. *h is decreasing on $\tau_r(I_p)$ iff it is increasing on I_p;*
3. *h is constant on $\tau_r(I_p)$ iff it is constant on I_p.*

If, instead, h is decreasing on I_r, then the monotonicity of I_p and $\tau_r(I_p)$ is the same.

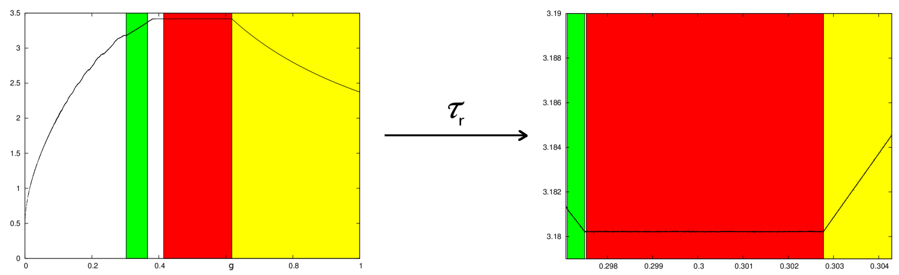

FIGURE 1. An illustration of the theorem is given in the picture: on the left, you see the whole parameter space $[0, 1]$, and the graph of h. The three colored strips correspond to some maximal intervals. On the right, the tuning window $W_{1/3} = [\frac{5-\sqrt{3}}{22}, \frac{\sqrt{3}-1}{2})$ relative to $r = 1/3$. Maximal intervals on the left are mapped via τ_r to maximal intervals of the same color on the right. As prescribed by theorem 2, the monotonicity of h on corresponding intervals is reversed. Note that in the white strips (even if barely visible on the right) there are infinitely many combinatorial types.

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Abstract
Tyler M. White
Topologically Mixing Tiling of $\mathbb{R}^2$ Generated by a
Generalized Substitution

In my talk, I will present a large class of examples of topologically mixing self-similar tilings of the plane. Topologically mixing tiling dynamical systems were investigated by Kenyon, Sadun, and Solomyak in [2]. They studied one-dimensional tiling dynamical system generated by substitions on 2 letters. Given a substitution σ with transition matrix M_σ, they proved that if the lengths of the prototiles are irrationally related, and the eigenvalues λ_1, λ_2 have the property that $\lambda_1 > |\lambda_2| > 1$, then the one-dimensional tiling dynamical system is topologically mixing. They were, however, unable to extend there results beyond an alphabet with 2 letters or one-dimensional tilings.

The examples I have studied were first presented by Kenyon in [1]. He proved that any complex Perron number solving $\lambda^3 - p\lambda^2 + r\lambda + q = 0, p \geq 0, r, q \geq 1$ has a self-similar tiling. Solomyak in [3], studied general self-similar tilings. He was able to prove that any self-similar tiling dynamical system of $\mathbb{R}^d$, $d \in \mathbb{N}$ is never (measure theoretically) strong mixing. Solomyak was also able to prove that any self-similar tiling of $\mathbb{R}^2$ with a complex, non-Pisot similarity is weakly mixing. Soloymak used the construction of Kenyon to provide examples of weakly mixing tiling dynamical systems of the plane. However, the question as to whether any of these examples are topologically mixing tiling dynamical systems of $\mathbb{R}^2$ remained open. In my research, which was suggest by Solomyak, I have used techniques from [2], and [3] to prove that an infinite sub-collection of Kenyon's examples were topologically mixing. These are the first known examples of entropy-zero topologically mixing tiling dynamical systems of the plane.

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