Shape optimization problems and spectral theory
May 28 - June 1, Abstracts of the talks

Juan Pablo Agnelli (Monday 28 - 17:30)
Title: "Shape optimization for tumor location"
Abstract: In non-invasive thermal diagnostics, accurate correlations between the thermal image at skin surface and interior human physiology are desired. In this work an estimation methodology to determine unknown geometrical parameters of an embedded tumor is proposed. We define a functional that represents the mismatch between a measured experimental temperature profile, which may be obtained by infrared thermography on the skin surface, and the solution of an appropriate boundary problem. This functional is related to the geometrical parameters through the solution of the boundary problem, in such a way that finding the minimum of this functional form also means finding the unknown geometrical parameters of the embedded tumor. Sensitivity analysis techniques coupled with the adjoint method were considered to compute the shape derivative of the functional.Then, a non monotone spectral projected gradient method was implemented to solve the optimization problem of finding the optimal geometric parameters.

Jose Arrieta (Tuesday 29 - 8:30)
Title: "Thin domains with oscillatory boundaries"
Abstract: We consider the behavior of the solutions of the Laplace equation with Neumann boundary conditions in a thin domain which presents a highly oscillatory behavior at its boundary. The prototype thin domain we will consider has a locally periodic structure and it is given as $R_\epsilon = \{(x, y) : 0 < x < 1, 0 < y < \epsilon G_\epsilon(x)\}$, where the function G_ϵ is of the type $G_\epsilon(x) = G(x, x/\epsilon^\alpha)$ for some $\alpha > 0$ and with the function G periodic in the second variable. We will analyze the behavior of the solutions as $\epsilon \to 0$ and explore the dependence of the limit operator in terms of the value α.

Rafael D. Benguria (Friday 1 - 8:30)
Title: "A New Estimate on the Two-Dimensional Indirect Coulomb Energy"
Abstract: We prove a new lower bound on the indirect Coulomb energy in two dimensional quantum mechanics in terms of the single particle density of the system. The new universal lower bound is an alternative to the Lieb–Solovej–Yngvason bound with a smaller constant, $C = (4/3)^{3/2}\sqrt{5\pi - 1} \approx 5.90 < C_{LSY} = 192\sqrt{2\pi} \approx 481.27$, which also involves an additive gradient energy term of the single particle density. I will also review the analogous situation in 3-d. In 2-d this is joint work with P. Gallegos and M. Tusek. In 3-d is a joint collaboration with G. Bley and M. Loss.

Guy Bouchitté (Wednesday 30 - 10:30)
Title: "A free boundary problem arising in optimal design of thin torsion rods"
Abstract: We study a variational problem, set on a bounded planar domain D, in which the cost functional depends on the gradient of admissible functions through an integrand which is convex but not strictly convex. We are interested in establishing whether solutions exist whose gradient avoids the region of non-strict convexity. Actually, the answer to this question is related to establishing whether homogenization phenomena occur in optimal design of thin torsion rods. From optimality conditions we are led to a new kind of free boundary problem involving a gradient constraint. We provide some existence results for different geometries of D. Joint work with Ilaria Fragala, Ilaria Lucardesi (Milan) and J.J Alibert (Toulon)

Farid Bozorgnia (Wednesday 30 - 12:00)
Title: "Optimal partitions for first eigenvalues of Laplace operator"
Abstract: This work deals with numerical verification of a conjecture by Caffarelli and Lin which says among all partitions of a given domain, a tiling by regular hexagon (up to boundary) minimizes the sum of first eigenvalues of Laplace operator with Dirichlet boundary condition. To show this numerically, a new idea to approximate the second eigenfunction and second eigenvalue is presented. We use the qualitative properties of the minimization problem from to construct a numerical algorithm to approximate optimal configurations. This problem also has been considered by B. Bourdin, D. Bucur, and E. Oudet.

Friedemann Brock (Thursday 31 - 10:30)
Title: "Weighted isoperimetric in cones inequalities and applications"
Abstract: We study some weighted isoperimetric inequalities relative to cones of $\mathbb{R}^N$. We give some information on the structure of those measures admitting as isoperimetric set the intersection of a cone with the ball centered at the vertex of the cone. For instance, we prove that when the cone is the half-space $\{x_N > 0\}$ and the measure is factorized, this phenomenon can occur only to measures of the form $d\mu = ax_N^k \exp\left(c\,|x|^2\right) dx$ for some $a > 0$, and $k, c \geq 0$. Along with this result, we obtain an isoperimetric estimate for the second Neumann eigenvalue of weighted Laplace-Beltrami operators on the sphere, sharp Hardy type inequalities for functions defined in $\mathbb{R}^N_+$, and a comparison result for degenerate elliptic problems related to these weights. This is joint work with F. Chiacchio and A. Mercaldo (Napoli).

Juan Casado-Diaz (Thursday 31 - 11:15)
Title: "The behavior of a thin flow satisfying the slip condition on a rough wall"
Abstract: We study the asymptotic behavior of the solutions of the Navier-Stokes system in a thin domain satisfying the Navier boundary condition on a periodic rough set of the boundary. The amplitude of the roughness is assumed to be an infinitessimun of the period. We show that three different behavior are possible depending on the relative size of the parameters.

Francesco Chiacchio (Monday 28 - 12:00)
Title: "Isoperimetric estimates for the first Neumann eigenvalue of the Hermite operator"
Abstract: We provide a Szegö-Weinberger type inequality for the first non-trivial Neumann eigenvalue $\mu_1(\Omega)$ for the Hermite operator, where Ω is a smooth and possibly unbounded domain of $\mathbb{R}^N$. Our main result consists in showing that among all sets Ω symmetric about the origin, having prescribed Gaussian measure, $\mu_1(\Omega)$ is maximum if and only if Ω is the euclidean ball centered at the origin.

Julie Clutterbuck (Friday 1 - 10:30)
Title: "Optimal lower bound on the spectral gap"
Abstract: We use heat equation methods to show that the spectral gap for the Schroedinger operator on a convex domain is bounded below by the gap for the one-dimensional problem. This is joint work with Ben Andrews.

Bruno Colbois (Monday 28 - 11:15)
Title: "Upper bound for the spectrum of the Laplacian : a metric approach"
Abstract: I will explain how to get upper bounds for the spectrum of a Riemannian manifold or of a submanifold of the Euclidean spaces thanks to a metric approach. The goal is to obtain estimates not depending on the curvature but on more global invariants.

Andrea Colesanti (Tuesday 29 - 16:00)
Title: "Hadamard's variational formulas for some functionals defined in the class of convex bodies"
Abstract: We review some classical variational formulas for geometric functionals (like the volume) defined in the class of convex bodies (compact convex sets), with respect to perturbations based on the Minkowski addition. Then we describe how these formulas extend to some well-known functionals from the Calculus of Variations, including torsion, eigenvalues of the Laplace operator, and various notions of capacities. Finally, we consider applications to Minkowski type problems. Part of the results have been obtained in collaboration with: Fimiani; Lutwak, Nystroem, Salani, Xiao, Yang and Zhang.

Yves Colin de Verdiere (Wednesday 30 - 11:15)
Title: "The magnetic Morse index formula of Gregory Berkolaiko"
Abstract: Let $H = -\Delta + V$ be a Schrödinger operator on a finite graph or on a closed manifold. Assume that the n-th eigenvalue λ_n is simple and that ϕ_n is an associated eigenfunction. Let B be a small magnetic field and H_B an associated Schrödinger operator. Then the n-th eigenvalue of H_B is a smooth function Λ_n of B whose $B = 0$ is a critical point. In the discrete case, GB founded recently a nice formula for the Morse index of Λ_n at $B = 0$.

I plan to present a simple proof of GB's result and some extension to the case of closed 2-manifolds (joint work with GB).

Gisella Croce (Monday 28 - 16:45)
Title: "An isoperimetric inequality for a nonlinear eigenvalue problem"
Abstract: We present an isoperimetric inequality for a nonlinear generalization of the first twisted Dirichlet eigenvalue. Let $\lambda^{p,q}(\Omega)$ be the set functional defined by

$$\lambda^{p,q}(\Omega) = \inf \left\{ \frac{|\nabla v|_{L^p(\Omega)}}{|v|_{L^q(\Omega)}}, v \in W_0^{1,p}(\Omega) \setminus \{0\}, \int_\Omega |v|^{q-2} v \, dx = 0 \right\}.$$

Under suitable conditions on p and q that ensure the existence of a minimizer function, we investigate the validity of an isoperimetric inequality of the Rayleigh-Faber-Krahn type. More precisely, using an alternative approach, we extend a result proved by P. Freitas and A. Henrot in the case $p = q = 2$, proving that $\lambda^{p,q}(\Omega)$ is minimized by the union of two equal balls, among sets of given volume.

This is a joint work with A. Henrot and G. Pisante.

Marc Dambrine (Thursday 31 - 18:25)
Title: "A Faber-Krahn type inequality for the second eigenvalue of the Ventcel-Steklov operator"
Abstract: In $\mathbb{R}^d$, we consider the eigenvalue problem

$$\Delta u = 0 \text{ in } \Omega \text{ and } \alpha \Delta_\Gamma u + \partial_n u = \lambda u,$$

where volume and surface diffusion are coupled. We are interested in maximizing the second eigenvalue with respect to the domain under a fixed volume constraint. We conjecture the the ball is the optimal design. We applied the coordinates method to obtain a simple (domain dependent) upper bound of the sum of the inverses of $\lambda_1,\ \lambda_d$ and check that the ball is a critical shape. One of the difficulties is the multiplicity of the eigenvalue for the disk.

Ilaria Fragala (Tuesday 29 - 16:45)
Title: "On geometric inequalities for log-concave functions"
Abstract: On the class of log-concave functions on $\mathbb{R}^n$, endowed with a suitable algebraic structure, we study the first variation of the total mass functional, which corresponds to the volume of convex bodies when restricted to the subclass of characteristic functions. We prove some integral representation formulae for such first variation, which lead to define in a natural way the notion of area measure for a log-concave function. In the same framework, we obtain a functional counterpart of Minkowski first inequality for convex bodies; as corollaries, we derive a functional form of the isoperimetric inequality, and a family of logarithmic-type Sobolev inequalities with respect to log-concave probability measures. Finally, we propose a suitable functional version of the classical Minkowski problem for convex bodies, and prove some partial results towards its solution. (joint work with Andrea Colesanti, Dorin Bucur...)

Pedro Freitas (Monday 28 - 8:30)
Title: "Optimization of higher eigenvalues of the Laplace operator"
Abstract: We present some new results on the optimization of higher eigenvalues of the Laplace operator with different boundary conditions. These are a combination of analytic and numerical results focusing on the question of the structure (if any...) behind optimizers, and their behaviour as we move into the high frequencies. Examples of our results are a (rigorous) proof that the asymptotic behaviour of Robin optimizers does not correspond to the Weyl asymptotics and numerical evidence that there are Dirichlet optimizers without any symmetry. This is joint work with Pedro Antunes and James Kennedy.

Nicola Fusco (Thursday 31 - 8:30)
Title: "On the stability of Almgren's isoperimetric inequality"
Abstract: In 1986 F. Almgren proved a deep and beautiful version of the classical isoperimetric inequality for the higher co-dimensional case. We shall discuss a recent result obtained in collaboration with V.Boegelein and F.Duzaar that extends to this more general inequality the stability estimates known in the classical case.

Alexandre Girouard (Tuesday 29 - 12:00)
Title: "Local and global stability of the Dirichlet spectrum"
Abstract: We study the Dirichlet eigenvalues of an inclusion of bounded domains $A \subset B$. Assuming a local spectral stability assumption on A, but nothing on B, we show that the difference of the Dirichlet eigenvalues of A and B is controlled from above in term of the fundamental tone of their difference. Sufficient conditions for the local stability of A to hold will also be discussed. This is joint work with Bruno Colbois and Mette Iversen.

Evans Harrell (Friday 1 - 9:15)
Title "Some semiclassically sharp inequalities for spectra of Schrdinger operators on surfaces and hypersurfaces"
Abstract: I will discuss sum rule identities and related techniques and their uses to prove sharp inequalities for Schrdinger operators on immersed surfaces and manifolds.

Bernard Helffer (Wednesday 30 - 8:30)
Title: "Magnetic characterization of minimal partitions"
Abstract: Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition of Ω by k open sets ω_j, we can consider the quantity $\max_j \lambda(\omega_j)$ where $\lambda(\omega_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in ω_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(\omega_j)$, a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

In this talk, we consider the two-dimensional case and discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases and then give a "magnetic" characterization of these mini-

mal partitions. This work has started in collaboration with T. Hoffmann-Ostenhof (with a preliminary work with M. and T. Hoffmann-Ostenhof and M. Owen) and has been continued with coauthors : V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Vial ... C. Lena

Thomas Hoffmann-Ostenhof (Wednesday 30 - 9:15)
Title: "Spectral minimal partitions: Some recent explicit results for the cyllinder , the torus, etc"
Abstract: We give some explicit examples for non-nodal spectral minimal partitions. In particular we give results for the sufficiently thin cylinders with Neumann boundary conditions and the torus etc. Additional problems and observations are dicussed. This is joint work with Bernard Helffer and partly with S. Terracini and V. Bonnaillie Noel.

Mette Iversen (Thursday 31 - 18:00)
Title: "Minimising convex combinations of the first three eigenvalues"
Abstract: The talk will consider for which values of $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$, minimisers for the variational problem

$$\inf\{\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^2,\ |\Omega| \leq 1\}$$

are connected. Here $\lambda_k(\Omega)$ is the k'th eigenvalue of the Dirichlet Laplacian. This is work with Dario Mazzoleni from Pavia.

David Krejcirik (Thursday 31 - 16:00)
Title: "The Cheeger constant of curved strips"
Abstract: We study the Cheeger constant and Cheeger set for domains obtained as tubular neighbourhoods of curves in the plane. If the reference curve is complete and finite (a "curved annulus"), then the strip itself is a Cheeger set and the Cheeger constant equals the inverse of the half-width of the strip. The latter holds true for unbounded strips as well, but there is no Cheeger set. Finally, for strips about non-complete finite curves, we derive lower and upper bounds to the Cheeger set, which become sharp for infinite curves. The results are illustrated by numerical computations for circular sectors. This is joint work with Aldo Pratelli.

Jimmy Lamboley (Tuesday 29 - 18:00)
Title : "Shape optimization under convexity constraint"
Abstract : We consider the following general shape optimization problem:

$$J(K_0) = \min\{J(K), K \text{ convex} \subset \mathbb{R}^d\},$$

where J is a shape functional. Many open problems, from Functional Analysis, Convex geometry or PDE can be formulated in this setting; the most famous ones are probably the Mahler conjecture and the Polyà-Szegö conjecture. They concern the research of a minimizer, with $J(K) = |K||K^*|$ the product of the volumes of K and of its dual body K^* for the first one, and with $J(K) = Cap(K)^2/P(K)$ the ratio between the electrostatic capacity and the surface area, for the second one. We focus on the way to analyze the convexity constraint on the shapes, using methods from Calculus of Variations, and to deduce some informations on optimal shapes. In dimension 2, we show a large class of functionals J leading to polygonal optimal shapes. In higher dimension, we give a similar weaker result (which applies to both conjecture).

Rick Laugesen (Monday 28 - 10:30)
Title "Sharp isoperimetric bounds on spectral functionals of starlike domains"
Abstract: We extend our Method of Rotations and Tight Frames to treat convex and starlike domains. We prove that the disk extremizes eigenvalue sums and products among such domains, for the Dirichlet (or Neumann) Laplacian, and extremizes more general functionals such as zeta functions and heat traces too. The geometric normalization involves both radial and angular quantities. Possible extensions include the magnetic Laplacian and biLaplacian.

Antoine Lemenant (Tuesday 29 - 18:50)
Titre : "Convex minimizers of the average distance"
Abstract : In this talk we will present the recent work with E. Mainini about convex sets that minimize the average distance functional. The second order optimality condition has been computed, assuming that the boundary of the minimizer is $C^{1,1}$ smooth. This allows us to exclude smooth boundary points of positive curvature for the problem with volume constraint. Some results related to the original average distance minimization problem (among 1-dimensional connected sets introduced by Buttazzo, Oudet and Stepanov 2002) will be also discussed.

Nikolai Nadirashvili (Thursday 31 - 9:15)
Title: "The quantitative isoperimetric inequality: new results and extensions"
Abstract: In the last few years there has been an increasing interest in the study of the stability of the isoperimetric and other related geometric and functional inequalities. I will give an overview of the problem and present two new results. The first one deals with a stronger form of the quantitative isoperimetric inequality for sets of finite perimeter and the second one deals with the stability of the Almgren's isoperimetric inequality for n-dimensional manifolds in R^{n+k}.

Aldo Pratelli (Monday 28 - 11:15)
Title: "A general existence result for minimizers of spectral problems"
Abstract: The problem of finding sets of unit measure which minimize functionals of the Dirichlet eigenvalues is very old and interesting, and there have been several partial results. In particular, Buttazzo and Dal Maso proved, in 1993, that a minimizing set exists for every increasing and l.s.c. functional of the first k eigenvalues if the ambient space is a given bounded subset of $\mathbb{R}^N$. We present a new argument showing that the result is still true if the ambient space is the whole $\mathbb{R}^N$ (joint work with D. Mazzoleni).

Yannick Privat (Friday 1 - 12:00)
Title : "Optimal design problems for conservative equations"
Abstract : We consider a conservative evolution equation on a given domain Ω of $\mathbb{R}^n$. The purpose of this talk is to investigate some natural shape optimization problem arising in the context of mathematics, physics or engineering. Given an initial state, one may observe on a measurable subset ω of Ω with given measure the solution of the equation, or control it (by HUM) or stabilize it (by a linear damping) to rest, with a control supported on ω. In the three cases, we focus on the question to know how to determine the best possible domain ω over all subsets of Ω of fixed measure (say $L|\Omega|$ with $0 < L < 1$) ensuring either the best observation, or the smallest possible norm of control, or the best rate of convergence for the stabilization. These questions are first investigated with fixed initial data. We then provide relevant criterions that do not depend on the initial conditions and analyze the related shape optimization problems. In particular, we comment on the regularity of the optimal domain, which can be regular or of fractal type according to the problem under consideration. One of these problems consists of the optimization (with respect to the domain ω) of observability

constants. Finally, we provide approximation procedures in order to compute numerically the best domain. In particular in dimension one efficient algorithms can be developed by using an interpretation of the problem in terms of optimal control. This is a work in collaboration with E. Trelat (Univ. Paris 6, France) and E. Zuazua (BCAM Bilbao, Spain).

Paolo Salani (Thursday 31 - 16:45)
Title: "An overdetermined problem with non constant boundary condition"
Abstract: Joint work with Chiara Bianchini and Antoine Henrot. We consider the solution u of the torsional rigidity problem in a domain Ω (i.e. $\Delta u = -1$ in Ω and $u = 0$ on $\partial\Omega$) and we add an overdetermination by requiring the norm of the gradient of u coincides with a given function g on $\partial\Omega$, that is $|\nabla u(x)| = g(x)$ for $x \in \partial\Omega$. Then we ask whether such a domain Ω exists (together with the related function u), solving the overdetermined problem. We consider the case when g is a homogeneous function and, under suitable assumptions, we prove existence, uniqueness, regularity and some geometric properties of the solution.

Pier Alberto Sicbaldi (Thursday 31 - 12:00)
Title: "Geometry and topology of some overdetermined elliptic problems"
Abstract: Let Ω be a 2-dimensional domain where there exists a positive solution u of the elliptic problem

$$\Delta u + f(u) = 0$$

with 0 Dirichlet boundary condition and constant Neumann data at the boundary, where f is a Lipshitz function. What information can we obtain on the domain Ω ? A conjecture of Berestycki-Caffarelli-Nirenberg (1997) says that if the complement of Ω is connected then Ω is either a ball or a half-plane. In this talk we will see some strong geometric and topological conditions that Ω must satisfy, and we will prove the Berestycki-Caffarelli-Nirenberg conjecture for a quite large class of functions f. This is a joint work with Antonio Ros.

Jan Sokolowski (Tuesday 29 - 10:30)
Title: "Topological derivatives in shape optimization"
Abstract: We show that topological derivatives of shape functionals can be considered as extensions of classical shape gradients. The examples of applications are presented.

Reference:

A.A. Novotny, J. Sokolowski Topological derivatives in shape optimization, Springer, to appear.

Susanna Terracini (Tuesday 29 - 9:15)

Title: "Entire solutions for competition-diffusion systems and optimal partitions"

Abstract: We study the qualitative properties of a limiting elliptic system arising in phase separation for multiple states Bose-Einstein condensates:

$$\begin{cases} \Delta u = uvr, \\ \Delta v = vu^2, \\ u, v > 0 \quad \text{in } \mathbb{R}^N. \end{cases}$$

We first prove that stable solutions in $\mathbb{R}2$ with linear growth must be one-dimensional. Then we construct entire solutions with polynomial growth $|x|^d$ for any positive integer $d \geq 1$. The construction is also extended to multi-component elliptic systems. We also show the connection between the existence/nonexistence of entire solutions and the qualitative properties of the optimal partitions.

Dan Tiba (Monday 28 - 16:00)

Title: "Finite Element Discretization in Shape Optimization Problems"

Abstract: For optimal design problems, defined in domains of class C and in arbitrary space dimension, governed by elliptic equations with boundary conditions of Neumann or mixed type, we introduce the corresponding fully discretized problems (i.e. we discretize the state equation, the cost functional and the constraints) and we prove convergence results. The discretization method is of fixed domain type, in the sense that it is given in the domain that contains all the admissible open sets. A similar study is performed in the case of the stationary Navier-Stokes equation with Dirichlet boundary conditions, in the absence of the uniqueness property and under Lipschitz assumptions for the unknown domains.

Michiel van den Berg (Friday 1 - 11:15)

Title: "Heat equation and torsion function"

Abstract: We obtain L^p estimates for the torsion function using heat equation methods. For example we show that the torsion function on an open set in Euclidean space with Dirichlet boundary conditions is in L^∞ if and

only if the bottom of the spectrum of the Dirichlet Laplacian is bounded away from 0.

Bozidar Velichkov (Tuesday 29 - 18:25)
Title: "Shape optimization problems with internal constraint"
Abstract: We consider shape optimization problems of the form

$$\min\{J(\Omega), \Omega \in \mathcal{A}\},$$

where J is a suitable cost functional, depending on the spectrum of the Dirichlet laplacian on $\mathbb{R}^d$ and the admissible class $\mathcal{A}$ is defined through both measure and internal constraint:

$$A = \{\Omega : D \subset \Omega \subset \mathbb{R}^d, \, \Omega \text{ quasi-open }, \, |\Omega| \leq m\},$$

where D is a fixed quasi-open set of finite measure, possibly unbounded.

In spite of its simplicity, even for cost functionals like $J(\Omega) = \lambda_1(\Omega)$, the existence proof is rather involved and it is based on a concentration-compactness principle for quasi- open sets which was introduced in [1] as a natural extension of the classical result due to Lions. For this functional, together with the existence of a solution, we prove some global properties for the optimal set: it has to lie in finite distance to D (in particular the optimal set is bounded, provided D is bounded), it has finite perimeter outside $\bar{D}$, it is an open set as soon as its measure is strictly greater than the measure of the (quasi-connected) D. At the end, we will discuss as well the existence question for $J(\Omega) = \lambda_k(\Omega)$. This work is in collaboration with Dorin Bucur and Giuseppe Buttazzo.
References
- D. Bucur: Uniform concentration-compactness for Sobolev spaces on variable domains. Journal of Differential Equations, 162 (2000), 427-450.
- D. Bucur, G. Buttazzo, B. Velichkov: Shape optimization problems with internal constraint. in preparation, 2011.

Alfred Wagner (Tuesday 29 - 11:15)
Title: "The Second Domain Variation for some Optimal Shape Problems"
Abstract: We present some recent results on the second domain variation. After introducing some theoretical background we give explicit examples. The domain functionals are either related to Dirichlet or Robin boundary conditions. Also some fourth order problems are considered. Finally we present a couple of open problems.