Théorie de Galois géométrique et différentielle
CIRM, 29 mars - 2 avril 2010

**PROGRAMME**

The aim of the conference is to present recent developments in differential and geometric Galois theories, including in particular the following topics: fundamental groups, moduli spaces, Riemann-Hilbert problems, anabelian geometry, unipotent representations, Galois groups of functional equations, reduction and lifting, arithmetic applications.

**PROGRAMME DES EXPOSES:**

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Lundi

- 9h - 9h50: Minhyong Kim: *Diophantine Geometry and Galois Theory*
- 10h - 10h50: Tamas Szamuely: *Around Grothendieck's section conjecture*
- 11h30 - 12h20: Anna Cadoret: *l-adic representations of étale fundamental groups*

- 16h30 - 17h20: Michael Singer: *A Jordan-Hoelder Theorem for Differential Algebraic Groups*
- 17h30 - 18h20: Alexei Ovchinnikov: *Parametrised differential Galois theory with non-closed fields of constants*
- 18h30 - 19h20: Charlotte Hardouin: *Galois arithmetic of q-difference equations*

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Mardi

- 9h - 9h50: Bjorn Poonen: *Néron-Severi groups under specialization*
- 10h - 10h50: Joao Pedro Dos Santos: *Lifting D-modules from positive to zero characteristic*
- 11h30 - 12h20: Heinrich Matzat: *Frobenius modules and Galois representations*

- 16h30 - 17h10: Anne Granier: *A Galois D-groupoid for q-difference systems*
- 17h20 - 18h10: Guy Casale: *Galois theory for difference or differential equations*
- 18h30 - 19h20: Alexandru Buium: *Galois groups arising from arithmetic differential equations*

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Mercredi

- 9h - 9h50: Stefan Wewers: *Group actions on the p-adic disk and the local lifting problem*
- 10h - 10h50: Andrew Obus: *Arithmetic and Geometry of Three-Point Covers*
- 11h30 - 12h20: Irene Bouw: *Towards Hurwitz numbers in positive characteristic*

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Jeudi

- 9h - 9h50: Frits Beukers: *Monodromy of A-hypergeometric functions*
- 10h - 10h50: Julia Hartmann: *Patching and Local-Global Principles for Homogeneous Spaces*
- 11h30 - 12h20: Jean-Marc Couveignes: *Computing mod l Galois representations associated to modular forms*

- 16h - 16h40: Yves André: *Tempered Anabelian Geometry I*
- 16h50 - 17h30: Emmanuel Lepage: *Tempered Anabelian Geometry II*
- 17h50 - 18h30: Andreas Maurischat: *On the infinitesimal inverse problem in ID Galois theory*
- 18h40 - 19h20: Lior Bary Soroker: *On irreducible specializations of polynomials*

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Vendredi

- 9h - 9h50: Marius van der Put: *Families of linear differential equations and the Painlevé equations*
- 10h - 10h50: Jeroen Sijsling: *(1;e)-curves and Lamé equations*
- 11h30 - 12h20: Philip Boalch: *Geometry of moduli spaces of meromorphic connections on curves*

- 14h - 14h50: Francis Brown: *Differential and motivic galois theory of polylogarithms*
- 15h - 15h50: Hiroaki Nakamura: *On an elliptic analogue of Ihara's power series*

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**ABSTRACTS**

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Yves André

Title: *Tempered Anabelian Geometry I*

Abstract: we will give a quick outline of tempered fundamental groups of p-adic curves and their applications.

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Lior Bary Soroker

Title: *On irreducible specializations of polynomials*

Abstract: Hilbert's irreducibility theorem asserts that every irreducible polynomial in two (or more) variables, say f(T,X), with rational coefficients admits irreducible specializations, i.e., rational a for which f(a,X) is irreducible. A stronger property follows -- the existence of group-preserving specializations, i.e., rational a's for which the Galois group of f(a,X) is isomorphic to the Galois group of f(T,X). In this work, we study irreducible specializations, in particular, when group-preserving specialization may not exist. This applies to analogs of the Schinzel's hypothesis H for function fields over certain families of fields, including large finite fields.

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Frits Beukers

Title: *Monodromy of A-hypergeometric functions*

Abstract: The system of A-hypergeometric differential equations is a vast generalisation of the one-variable hypergeometric equation and the Apell-Lauricella and Horn systems. They were introduced at the end of the 1980 by Gel'fand Kapranov and Zelevinski. In this lecture we indicate a method to compute the monodromy group of such systems.

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Philip Boalch

Title: *Geometry of moduli spaces of meromorphic connections on curves*

Abstract: I will survey some results about moduli spaces of meromorphic connections on principal G-bundles over curves, as well as the corresponding spaces of (enriched) monodromy data. Some motivation comes from the approach of Witten and collaborators to geometric Langlands.

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Irene Bouw

Title: *Towards Hurwitz numbers in positive characteristic*
*(joint work with Brian Osserman)*

Abstract: Hurwitz numbers, which count branched covers of the projective line, are a very classical topic, with equivalent formulations in algebraic geometry, group theory, complex geometry, and topology, as well as connections to number theory and string theory. We briefly review the basic definitions and the difficulties of generalizing to positive characteristic, and then present some preliminary progress in this direction in a family of examples of genus 0, with four branch points. Our techniques rely on a combination

of degenerations and stable reduction theory.

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Francis Brown

Title: *Differential and motivic galois theory of polylogarithms*

Abstract : I will begin by reviewing the bar construction and unipotent fundamental groups from the point of view of differential Galois theory, and describe these structures for configuration spaces of points on a curve. In genus 0 one obtains a Hopf algebra of polylogarithms with an action of the symmetric group. The corresponding periods are multiple zeta values, along with their motivic coproduct. If time permits I will discuss the corresponding constructions in genus 1 (joint with A. Levin).

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Alexandru Buium

Title: *Galois groups arising from arithmetic differential equations*

Abstract: Arithmetic differential equations are analogues of differential equations in which derivative operators are replaced by Fermat quotient operators. There are interesting situations when Galois (profinite) groups can be attached to such equations. We will present explicit computations of such Galois groups. We will also present an application of such a computation to the study of arithmetic differential Fourier expansions of arithmetic differential modular forms.

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Anna Cadoret

Title: *$\ell$-adic representations of étale fundamental groups*
*(joint work with Akio Tamagawa)*

Abstract: $\ell$-adic representations of Galois groups are a powerful tool to investigate objects of arithmetico-geometric nature (\textit{e.g.} Grothendieck semi-stable reduction theorem, Weil conjectures \textit{etc.}) Of particular interest are $\ell$-adic representations of the \'etale fundamental group of a scheme $X$ attached to geometric objects; typically, the representations
$\rho_{\ell,i,Y}:\pi_{1}(X)\rightarrow \hbox{\rm GL}(H^{i}(Y_{\overline{\eta}},\mathbb{Q}_{\ell}))$, $i\geq 0$ arising from the action of $\pi_{1}(X)$ on the $\ell$-adic \' etale cohomology groups of the geometric generic fiber of a smooth proper scheme $Y\rightarrow X$. By specialization theory, $\rho_{\ell,i,Y}:\pi_{1}(X)\rightarrow \hbox{\rm GL}(H^{i}(Y_{\overline{\eta}},\mathbb{Q}_{\ell}))$ encodes the family of representations
$\rho_{\ell,i,Y,x}:\Gamma_{k(x)}\rightarrow \hbox{\rm GL}(H^{i}(Y_{\overline{x}},\mathbb{Q}_{\ell}))$, $x\in X$ (where $k(x)$ denotes the residue field at $x $ and $\Gamma_{k(x)}$ its absolute Galois group) hence provides a natural tool to understand how arithmetico-geometric invariants vary in this family.

The main result I am going to discuss in this talk is the following. Let $k$ be a field of characteristic $0$, $X$ a smooth, separated, geometrically connected scheme over $k$ with generic point $\eta$ and $\rho:\pi_{1}(X)\rightarrow \hbox{\rm GL}_{m}(\mathbb{Z}_{\ell})$ an $\ell$-adic representation. Let $G$ denote the image of $\rho$. Any closed point $x$ on $X$ induces a splitting $x:\Gamma_{k(x)}\rightarrow\pi_{1}(X_{k(x)})$ of the canonical restriction epimorphism $\pi_{1}(X_{k(x)})\rightarrow \Gamma_{k(x)}$ so one can define the closed subgroup $G_{x}:=\rho\circ x(\Gamma_{k(x)})\subset G$ (up to inner automorphisms). Given an integer $d\geq 1$, let $X^{\leq d}$ denote the set of closed point $x\in X$ such that $[k(x):k]\leq d$.

Theorem: Let $k$ be a finitely generated field of characteristic $0$. Assume furthermore that $X$ is a curve. Then,
- for any representation $\rho:\pi_{1}(X)\rightarrow \hbox{\rm GL}_{m}(\mathbb{Z}_{\ell})$ and any integer $d\geq 1$, the set $X_{\rho, d,\geq 3}$ of all $x\in X^{\leq d}$ such that $G_{x}$ has codimension $\geq 3$ in $G$ is finite.
- Furthermore, if $\rho=\rho_{\ell,i,Y}:\pi_{1}(X)\rightarrow \hbox{\rm GL}(H^{i}(Y_{\overline{\eta}},

\mathbb{Q}_{\ell}))$, $i\geq 0$ for some smooth proper scheme $Y\rightarrow X$ then the set $X_{\rho, d,\geq 1}$ of all $x\in X^{\leq d}$ such that $G_{x}$ has codimension $\geq 1$ in $G$ is finite and there exists an integer $B_{\rho,d}\geq 1$ such that $[G:G_{x}]\leq B_{\rho,d}$, $x\in X^{\leq d}\smallsetminus X_{\rho,d,\geq 1}$.

I will briefly mention the ingredients involved in the proof and then focus on several applications of this theorem such as uniform open image theorems for abelian varieties parametrized by curves (e.g Serre's uniform open image theorem, uniform boundedness of $\ell$-primary torsion, $1$-dimensional modular tower conjecture etc.) or variation of 'motivized' motivic Galois groups and application to the variation of the rank of motivized cycles under specialization.

In the last part of this talk, I will focus on the $\ell$-adic representations arising from the action of $\pi_{1}(X)$ on the generic $\ell$-adic Tate module $T_{\ell}(A_{\eta})$ of an abelian scheme $A$ over $X$ and discuss developments of our work when the base scheme $X$ is replaced by an algebraic stack, when the base scheme $X$ is a surface or when one considers mod $\ell$ representations ($\ell$ varying).

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Guy Casale

Title: *Galois theory for difference or differential equations*

Abstract : I will give the definition of the Galois pseudgroup (from Malgrange point of view) of a dynamical system and apply it to difference equations seen as a birational transformations and to differential equation seen as a vector fields. After recalling the results obtained by Anne Granier in the difference case, and their differential versions, I will give the proof of the `surjection theorem' proved by Umemura in the differential case and conjectured in the difference case. This theorem is the easiest remainder of a Galois correspondence. It usually means that in a tower of two Galoisian extensions K c L c M , if K c M is Galoisian then the Galois group of K c L is a quotient of the Galois group of K c M. The form of a complete correspondence in nonlinear situations is not known.

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Jean-Marc Couveignes

Title: *Computing mod l Galois representations associated to modular forms (joint work with Edixhoven, de Jong and Merkl)*

Abstract: I will present a polynomial time algorithm to compute the mod l Galois representations associated to modular forms. Our previous algorithm was probabilistic (due to the use of computation in the Jacobian of modular curves over finite fields). The deterministic variant uses computations with complex numbers instead.

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Anne Granier

Title: *A Galois D-groupoid for q-difference systems*

Abstract: In "Le groupoïde de Galois d'un feuilletage" (2001), Bernard Malgrange defines for non linear differential equations an object generalizing the Galois group of linear differential equations. Following this construction, we define a Galois D-groupoid for q-difference systems and we show that the definition is legitimate by proving that, for a class of linear q-difference systems, the linear Galois group can be recovered from the Galois D-groupoid. In relation to the talks by Charlotte Hardouin and Guy Casale, further applications will also be mentioned.

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Charlotte Hardouin

Titre: *Galois arithmetic of $q$-difference equations*

Abstract: In this joint work with Lucia Di Vizio, we firstly prove in the case of a generic $q$-difference equation with coefficients in a finite extension of a given field $k(q,x)$, with $q$ transcendental over $k$, an equivalence between the following facts:

1. the $q$-difference equation is trivial over $k(q,x)$;
2. its specialization at $q =\xi$ has zero curvature for almost all primitive roots of unity $\xi$;
3. for almost all primitive roots of unity $\xi$, its specialization at $q =\xi$ is endowed with an iterated $\xi$-difference structure.

The equivalence between 1. and 2. is an avatar of the Grothendieck conjecture on $p$-curvatures, while the equivalence between 1. and 3. is an analog of a conjecture of Matzat-van der Put for difference equations. This result leads to an arithmetic description of the generic (also called intrinsic) Galois group, both algebraic and differential, attached to the $q$-difference equation. The latter, by specialization, gives an upper bound for the generic Galois group of a differential equation.

Secondly, we show that our results combined with those obtained by Di Vizio and Hendricks, allow us to describe the generic Galois group of a complex $q$-difference equations, with $q\in\C\smallsetminus{0,1}$ by means of the curvatures (but not the same curvatures according to whether $q$ is algebraic or transcendental).

As an application, we relate the Malgrange-Granier groupoid of a $q$-difference equation to the generic differential Galois group.

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Julia Hartmann

Title: *Patching and Local-Global Principles for Homogeneous Spaces*

Abstract: Patching methods are usually used to construct a global object from local data. When the machinery is applied in the converse direction, one obtains local-global principles. This talk explains such principles for homogeneous spaces under certain linear algebraic groups as well as applications to quadratic forms and central simple algebras. We also discuss when these local-global principles fail and how to describe the kernel of the local-global map. (Joint work with David Harbater and Daniel Krashen.)

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Minhyong Kim

Title: *Diophantine Geometry and Galois Theory*

Abstract: In his manuscripts from the 1980's Grothendieck proposed ideas that have been interpreted variously as embedding the theory of schemes into either
- group theory and higher-dimensional generalizations;
- or homotopy theory.
It was suggested, moreover, that such a framework would have profound implications for the study of Diophantine problems. In this talk, we will discuss mostly the little bit of progress made on this last point using some mildly non-abelian motives associated to hyperbolic curves.

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Emmanuel Lepage

Title: *Tempered Anabelian Geometry II*

Abstract: We will discuss what can be recovered of a p-adic hyperbolic curve from its geometric tempered fundamental group. S. Mochizuki proved that one could recover the graph of the stable reduction of the curve from the tempered fundamental group. In fact it can be recovered from a prime-to-p version: vertices correspond to maximal compact subgroups of the prime-to-p tempered fundamental group. In the case of Mumford curves, one can moreover recover a natural metric on the graph. We will also be interested in the maximal compact subgroups of the full tempered fundamental group.

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Heinrich Matzat

Title: *Frobenius modules and Galois representations.*

Abstract: Frobenius modules encode many of the typical phenomena in positive characteristic. They enjoy a difference Galois theory with linear algebraic group schemes as partners similar to differential modules. Here we study adic Frobenius modules in connection with their induced differential equations and Galois representations. This applies among others to ordinary Galois extensions, p-adic differential equations and t-motives.
Applications contain the solution of the inverse Galois problem for p-adic (and t-adic) differential equations with strong Frobenius structure and related results for their monodromy groups.

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Andreas Maurischat

Title: *On the infinitesimal inverse problem in iterative differential Galois theory*

Abstract: Whenever one considers Picard-Vessiot extensions for iterative differential equations in positive characteristic, there occur intermediate iterative differential extensions over which the PV-field is inseparable. In the Galois correspondence, they correspond to nonreduced subgroup schemes of the Galois group scheme. Moreover, the Galois group scheme itself can be nonreduced, and even infinitesimal. In this talk, I will show how one can decide which infinitesimal group schemes occur as ID-Galois group schemes over a given ID-field, and I will give some illustrating examples.

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Hiroaki Nakamura

Title: *On an elliptic analogue of Ihara's power series*

Abstract: After a quick review on Ihara's power series (adelic beta function) arising from the fundamental group of the projective line minus three points, we discuss an elliptic analogue, i.e, certain arithmetic invariants arising from fundamental groups of once punctured elliptic curves.
An explicit formula in terms of Kummer properties of modular units is given to describe these invariants.
In the complex analytic model of universal family of once punctured elliptic curves, the formula turns out to feature the generalized Dedekind-Rademacher functions as a main periodic part of the invariant. For the complete article, see:
http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1691.pdf

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Andrew Obus

Title: *Arithmetic and Geometry of Three-Point Covers*

Abstract: Since the work of Grothendieck, it has been known that branched Galois covers of curves in characteristic p behave much like those in characteristic 0, provided that p does not divide the order of the Galois group G. However, if p | #G, then many new phenomena arise. In particular, the reduction of a smooth G-Galois cover in characteristic zero might not be smooth. In this talk, we examine the case of three-point covers of the projective line. We will discuss results of Raynaud and Wewers concerning the stable reduction of such covers when p exactly divides the order of G, some arithmetic applications to fields of moduli, and generalizations by the speaker to the case where G has a cyclic p-Sylow group of arbitrary order. If time permits, we will mention some

of the difficulties inherent in generalizing to the case where a p-Sylow group of G is non-cyclic.

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Alexei Ovchinnikov

Title: *Parametrised differential Galois theory with non-closed fields of constants (joint work with Henri Gillet and Sergey Gorchinskiy).*

Abstract: A general Galois theory of parameterized differential equations was introduced by Landesman. The special case of parameterized linear differential equations was developed by Cassidy and Singer who discussed various special properties of the parameterized Picard-Vessiot Galois group when the field of constants of the base field is differentially closed. In the language of Tannakian categories, the latter extra condition was used to show the existence of a differential fibre functor that computes solutions of the system. Using geometric techniques we show how to construct a differential fibre functor for a large class of base differential fields not requiring the constants to be differentially closed. In particular, Atiyah extensions are used to show that the fibre functor respects the differential structure. Our argument also provides simpler axioms defining a differential Tannakian category.

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Bjorn Poonen

Title: *Néron-Severi groups under specialization (joint work with Davesh Maulik and Claire Voisin)*

Abstract: We prove that given a smooth proper family X --> B of varieties over an algebraically closed field k of characteristic 0, there exists a closed fiber having the same Picard number as the geometric generic fiber, even if k is countable. In fact, we give two proofs, and they show that the locus on the base where the Picard number jumps is "small" in two different senses. The first proof uses Hodge theory and the actions of geometric monodromy groups and Galois groups to show that the locus is small in a sense related to Hilbert irreducibility. The second proof uses the "p-adic Lefschetz (1,1) theorem" of Berthelot and Ogus to show that in a family of varieties with good reduction at p, the locus is nowhere p-adically dense.

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Marius van der Put

Title: *Families of linear differential equations and the Painlevé equations*

Abstract: First we review recent work of Masa--Hiko Saito and the speaker concerning the construction of the ten ``isomonodromic'' families, giving rise to Painlev\'e equations, and their monodromy spaces. Then we focus on the fine details of some special families related to the Painlev\'e equations I and III (D8).

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Joao Pedro Dos Santos

Title : *Lifting D-modules from positive to zero characteristic*

Abstract: The theme of the talk is liftings, or deformations, of D-modules (D being the ring of differential operators from EGA IV) from positive characteristic to characteristic zero using ideas of Matzat and Berthelot's theory of arithmetic D-modules. More precisely, I will try to address the following question: is it possible to find liftings which "preserve" the differential Galois group?

Then I will explain how to run the machinery of deformation theory (Schlessinger) to analyze the space of all liftings of a given D-module in positive characteristic. (This is reminiscent of the work of Mazur on Galois representations). If time permits, I will then address the question of comparing the deformation theory of a D-module with that of a representation (of a naturally

associated group scheme).

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Jeroen Sijsling

Title: (*1;e)-curves and Lamé equations*

Abstract: A (1;e)-curve is a quotient of the upper half plane that is of genus 1 and ramifies above only one point. The multivalued inverse map back to the upper half plane can be determined using the Picard-Fuchs differential equation that lives on this curve, which is in this case called a Lamé equation. We explore the finite list, due to Takeuchi, of arithmetic (1;e)-curves, i.e. those naturally commensurable to a Shimura curve coming from a quaternion algebra over a totally real field. The emphasis throughout is on explicit calculation of these curves and the resulting Lamé equations.

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Carlos Simpson

Title: *Foliations on moduli spaces of regular connections*

Abstract: We look at a natural family of Lagrangian subspaces of the moduli space of regular connections, in particular in the case of the projective line minus four points corresponding to Painlevé VI. Iwasaki-Inaba-Saito as well as Szabo have given explicit constructions, which we compare with the abstract definition using the nonabelian Hodge filtration. This allows us to see that the subspaces form a foliation in this case, the quotient of which has appeared in work by Arinkin on the Langlands program.

For a related result, see D. Arinkin and R. Fedorov, *An example of Langlands correspondence for irregular rank two connections on P^1 ,* on arXiv:1003.6112.

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Michael Singer

Title: *A Jordan-Hoelder Theorem for Differential Algebraic Groups*
*(joint work with Phyllis Cassidy).*

Abstract: An ordinary differential operator can be factored as a product of irreducible operators and any two such factorizations have the same number of factors and, after a possible permutation, these factors are equivalent in a suitable sense. Examples showing that such a result is not true for partial differential operators have been known for over 100 years.

Solutions of systems of homogeneous linear partial differential equations form a group under addition and are an example of a differential algebraic group. We show that a Jordan-Hoelder type theorem holds for such groups, that is, any such group can be filtered by a finite subnormal series of differential algebraic groups such that successive quotients are "almost simple". Furthermore, any two such series have the same length and, after a possible permutation, successive quotients are "isogenous".

This talk will not assume that the audience is familiar with differential algebraic groups and will expose the necessary background. Special attention will be given to the meaning of this result in the context of systems of linear partial differential equations and many examples will be given.

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Tamas Szamuely

Title: *Around Grothendieck's section conjecture*

Abstract: We survey recent work on this famous conjecture, including contributions by

Harari and the speaker.

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Stefan Wewers

Title: *Group actions on the p-adic disk and the local lifting problem*

Abstract: In the first part of the talk, I will report on joint work with Louis Brewis related to the local lifting problem. The main result yields an obstruction against liftability to characteristic zero of the action of a finite group G on a power series ring k[[z]], where k is a field of characteristic p. It refines the obstruction defined by Bertin. In the second part of the talk I will consider the case where the group G is cyclic of p power order. In this case our obstruction vanishes, which is consistent with a conjecture of Oort that all cyclic actions should lift.

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