Computing up to isomorphism: proof-relevant PERs

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What is this talk about?

A partial equivalence relation (PER) is an homogeneous binary relation that is symmetric and transitive.

PERs are important in semantics of type theory and programming languages, as well as higher-order computability.

To build a PER model, we start with some realizers/programs. Types are interpreted as PERs over realizers. When aRb we think of a and b as implementing identical programs of type R.

In this talk, we will consider proof-relevant PERs, where there are named witnesses to the relatedness of a and b. Motivation: build realizability models of ITT and HoTT.

Realizers

A combinatory algebra (CA) is a set A together with a binary "application" operation:

$$(-) \bullet (=) : A \times A \to A$$

that is functionally complete: for every polynomial $t(x_1, ..., x_n)$ over A, there is some "code" $c \in A$ that "represents" it:

$$c \bullet a_1 \bullet ... \bullet a_n = t(a_1, ..., a_n)$$

Examples: categorical models of the λ-calculus

A model of the untyped λ-calculus is a cartesian closed category $\mathcal{C}$ with a reflexive object $U \in \mathcal{C}$:

$$U^U \xrightarrow{\mathsf{lam}} U \xrightarrow{\mathsf{app}} U^U \quad = \quad \mathsf{id}$$

$$[\![x_1, ..., x_n \vdash \lambda y.t]\!] \coloneqq U^n \xrightarrow{\lambda[\![x_1,...,x_n,y \vdash t]\!]} U^U \xrightarrow{\mathsf{lam}} U$$

$$[\![x_1, ..., x_n \vdash tu]\!] \coloneqq U^n \xrightarrow{\langle \mathsf{app} \circ [\![x_1,...,x_n \vdash t]\!], [\![x_1,...,x_n \vdash u]\!] \rangle} U^U \times U \xrightarrow{\mathsf{eval}} U$$

$\mathcal{C}(1, U)$ is a CA with application:

$$a \bullet b = \mathsf{eval} \circ \langle \mathsf{app} \circ a, b \rangle$$

The category of PERs over the CA A

A morphism $R \to S$ is a function:

$$f : {}^{A}\!/_{R} \to {}^{A}\!/_{S}$$

between subquotients that is "tracked" by some $e \in A$:

$$\begin{tikzcd}
{}^{A}\!/_{R} \arrow[r, "f"] \arrow[d, "\|-\|_R"'] \arrow[dr, phantom, "\leq"] & {}^{A}\!/_{S} \arrow[d, "\|-\|_S"] \\
\mathsf{P}(A) \arrow[r, "\mathsf{P}(e\bullet(-))"'] & \mathsf{P}(A)
\end{tikzcd}$$

where $\|[a]\|_R := \{b \in A \mid bRa\}$.

Subquotients

The subquotient of a set A by a PER R is often described as the quotient by R of the subset $\mathsf{Dom}(R) := \{a \in A \mid aRa\} \subseteq A$.

Alternatively, we may view a PER R on A as an inverse semicategory whose source-target span is jointly injective.

A construction of [DeWolf-Pronk '18] exhibits a right adjoint K to the forgetful functor $U : \mathbf{Gpd} \to \mathbf{ISCat}$.

- Objects of $K(\mathcal{C})$ are pairs $(X, f^{-1}f)$, where $f : X \to X$.
- If $f : X \to Y$ in $\mathcal{C}$, then $f : (X, f^{-1}f) \to (Y, ff^{-1})$ in $K(\mathcal{C})$.
- Composition is inherited from $\mathcal{C}$ and $\mathsf{id}_{(X,e)} := e$.

Applying K to R yields R considered as a groupoid with object set $\mathsf{Dom}(R)$. The coequalizer of the source and target functions of this groupoid is the subquotient ${}^{A}\!/_{R}$.

Proof-relevant PERs

A proof-relevant ER on a groupoid A is a groupoid $\mathbb{R}$ internal to $\mathbf{Gpd}$ (hence a double groupoid), with $\mathbb{R}_0 = A$, whose source-target span is a two-sided discrete fibration (cf. cateads [Bourn-Penon '78, Bourke '11]).

ERs are effective in $\mathbf{Gpd}$, in the sense that an ER is the 2D kernel of its codescent morphism (2D quotient map). The codescent object $Q(\mathbb{R})$ of an ER $\mathbb{R}$ is its horizontal groupoid $\mathbb{R}_\mathsf{h}$.

A proof-relevant PER on A is an inverse semicategory $\mathbb{R}$ in $\mathbf{Gpd}$, with $\mathbb{R}_0 = A$, whose source-target span is a two-sided discrete fibration.

Proof-relevant subquotients

There is an internal / double-categorical version of the right adjoint:

$$U \dashv K : \mathbf{IsCat}(\mathbf{Gpd}) \to \mathbf{Gpd}(\mathbf{Gpd})$$

The horizontal groupoid $K(\mathbb{R})_{\mathsf{h}}$ of the double groupoid $K(\mathbb{R})$ is $K(\mathbb{R}_h)$.

K respects the source-target span being a two-sided discrete fibration, so the adjunction restricts:

$$U \dashv K : \mathbf{ISGrat} \to \mathbf{Grat}$$

Thus we define the subquotient of a PER $\mathbb{R}$ to be the codesecent object $Q(K(\mathbb{R}))$ of $K(\mathbb{R})$.

2D combinatory algebras

A 2D combinatory algebra (2CA) consists of a groupoid A and an "application" functor:

$$(-) \bullet (=) : A \times A \to A$$

such that for every polynomial $t(x_1, ..., x_n)$ over A_0 there exists $c \in A_0$ and a natural isomorphism:

$$\gamma_{a_1,...,a_n} : c \bullet a_1 \bullet ... \bullet a_n \to t(a_1, ..., a_n)$$

2D models of the λ-calculus

A 2D model of the untyped λ-calculus is a cartesian closed bicategory $\mathfrak{C}$ with a pseudoreflexive object $U \in \mathfrak{C}$.

$$U^U \xrightarrow{\mathsf{lam}} U \xrightarrow{\mathsf{app}} U^U \quad \cong \quad \mathsf{id}$$

Examples:

- generalised species of structures [Fiore-Gambino-Hyland-Winskel '08],
- profunctorial Scott semantics [Galal '20],
- categorified relational (distributors-induced) model [Olimpieri '21],
- categorified graph model [Kerinec-Manzonetto-Olimpieri '23].

Similar to before, $\mathsf{Core}(\mathfrak{C}(1, U))$ is a 2CA.

The category of proof-relevant PERs over the 2CA A

A morphism $\mathbb{R} \to \mathbb{S}$ is a functor:

$$F : \; Q(K(\mathbb{R})) \; \to \; Q(K(\mathbb{S}))$$

that is "tracked" up to natural isomorphism:

$$\begin{tikzcd}
Q(K(\mathbb{R})) \arrow[r, "F"] \arrow[d, "\|-\|_R"'] \arrow[dr, phantom, "\Nearrow \varepsilon"] & Q(K(\mathbb{S})) \arrow[d, "\|-\|_S"] \\
\mathsf{Psh}(A) \arrow[r, "\mathsf{Psh}(e\bullet(-))"'] & \mathsf{Psh}(A)
\end{tikzcd}$$

where $\big\|(a, \rho^{-1}\rho)\big\|_R(b) := \mathsf{Hom}_{\mathbb{R}_\mathsf{h}}(b, a)$ and $\mathsf{Psh}(G) \dashv (-) \circ G$.

Outlook

Is the (2,1)-category of proof-relevant PERs a complete small category in the (2,1)-category of groupoidal assemblies?

Do these categories yield models of type theory?

Related WIP (with Alyssa Renata): a 2D / proof-relevant realizability tripos.