A fast and deterministic solution to linear systems with displacement structure

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Context

And notations

Goal

Design algorithms with good complexity bounds for fundamental operations on polynomials and matrices

Exact Computations

Base Field $\mathbb{K}$

$\mathbb{Z}/p\mathbb{Z}$, $\mathbb{F}_{p^e}$, $\mathbb{Q}$, ...

Algebraic Complexity

Upper bound on the number of operations in $\mathbb{K}$

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Algebraic Complexity

Upper bound on the number of operations in $\mathbb{K}$

Notations:

- $\tilde{O}$: asymptotic bound hiding logarithmic factors.
- $\mathcal{M}(d)$: cost of multiplying two polynomials of degree less than d in $\mathbb{K}[x]$.
  $\rightsquigarrow$ $O(d\log(d)\log(\log(d))) = \tilde{O}(d)$ on all fields $\mathbb{K}$ [Cantor, Kaltofen 1991].
- ω: exponent of matrix multiplication, $2 < \omega \leq 3$, best known $\omega \approx 2.371$.

Structured matrices

Displacement rank approach

Example: Toeplitz matrix

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Example: Toeplitz matrix

$$\overset{\mathbb{K}^{4\times 4}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}}$$

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$\mathsf{A} \in \mathbb{K}^{m\times n}$

Displacement rank:

The **rank** of the image of A through the displacement operator ϕ.

Toeplitz-like Structure:

$$\phi : \begin{array}{ll} \mathbb{K}^{m\times n} & \longrightarrow \mathbb{K}^{m\times n} \\ \mathsf{A} & \longmapsto \mathsf{A} - \mathbb{S}_m \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}} \end{array}$$

$$\mathbb{S}_n = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{K}^{n\times n}$$

Structured matrices

Displacement rank approach

Example: Toeplitz matrix

$$\overset{\mathbb{K}^{4\times 4}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}} - \overset{\mathbb{S}_m \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & a_{-1} & a_0 & a_1 \\ 0 & a_{-2} & a_{-1} & a_0 \end{bmatrix}} = \overset{\text{Displacement result}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & 0 & 0 & 0 \\ a_{-2} & 0 & 0 & 0 \\ a_{-3} & 0 & 0 & 0 \end{bmatrix}}$$

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Generators:

$(\mathsf{G}, \mathsf{H}) \in \mathbb{K}^{m\times \alpha} \times \mathbb{K}^{n\times \alpha}$ are called ϕ-generators of length α for A if

$$\phi(\mathsf{A}) = \mathsf{G} \cdot \mathsf{H}^{\mathsf{T}}.$$

Structured matrices

Displacement rank approach

Example: Toeplitz matrix

$$\underset{\mathbb{K}^{4\times4}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}} - \underset{\mathbb{S}_m \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & a_{-1} & a_0 & a_1 \\ 0 & a_{-2} & a_{-1} & a_0 \end{bmatrix}} = \underset{\text{Displacement result}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & 0 & 0 & 0 \\ a_{-2} & 0 & 0 & 0 \\ a_{-3} & 0 & 0 & 0 \end{bmatrix}}$$

$$\underset{\mathsf{G}}{\begin{bmatrix} a_0 & 1 \\ a_{-1} & 0 \\ a_{-2} & 0 \\ a_{-3} & 0 \end{bmatrix}} \times \underset{\mathsf{H}^{\mathsf{T}}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \end{bmatrix}} = \underset{\text{Displacement result}}{\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & 0 & 0 & 0 \\ a_{-2} & 0 & 0 & 0 \\ a_{-3} & 0 & 0 & 0 \end{bmatrix}}$$

$\mathsf{G} \in \mathbb{K}^{4\times2}$ $\mathsf{H} \in \mathbb{K}^{4\times2}$ Displacement rank $\alpha = 2$

Structured matrices

Classical families of structured matrices

Toeplitz matrices ($\alpha = 2$)

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Toeplitz-like matrices:

$$\phi(\mathsf{A}) = \mathsf{A} - \mathbb{S}_m \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}}$$

Vandermonde matrices ($\alpha = 1$)

$$\begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{bmatrix}$$

Vandermonde-like matrices:
For a repetition-free $\mathsf{x} = [x_1, \ldots, x_m]$,

$$\phi(\mathsf{A}) = \mathsf{A} - \mathbb{D}(\mathsf{x}) \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}}$$

Cauchy matrices ($\alpha = 1$)

$$\begin{bmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \frac{1}{x_1 - y_3} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \frac{1}{x_2 - y_3} \\ \frac{1}{x_3 - y_1} & \frac{1}{x_3 - y_2} & \frac{1}{x_3 - y_3} \end{bmatrix}$$

Cauchy-like matrices:
For disjoint and repetition-free
$\mathsf{x} = [x_1, \ldots, x_m]$, $\mathsf{y} = [y_1, \ldots, y_n]$,

$$\phi(\mathsf{A}) = \mathbb{D}(\mathsf{x}) \cdot \mathsf{A} - \mathsf{A} \cdot \mathbb{D}(\mathsf{y})$$

Structured linear system

Background and motivations

$\mathsf{A} \in \mathbb{K}^{m \times n}$

Problem 1:

$\mathsf{LinearSystem}[\phi, \alpha](\mathsf{G}, \mathsf{H}, \mathsf{v})$

Using ϕ-generators of A: $\mathsf{G} \in \mathbb{K}^{m \times \alpha}, \mathsf{H} \in \mathbb{K}^{n \times \alpha}, \mathsf{v} \in \mathbb{K}^{m \times 1}$

find nonzero $\mathsf{u} \in \mathbb{K}^{n \times 1}$ such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

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$\rightsquigarrow$ ϕ has to be invertible.

Toeplitz-like matrices

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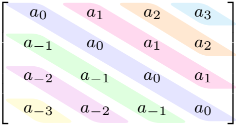
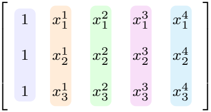


$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Vandermonde-like matrices

$$\phi(\mathsf{A}) = \mathsf{A} - \mathbb{D}(\mathsf{x}) \cdot \mathsf{A} \cdot \mathbb{S}_n^{\mathsf{T}}$$

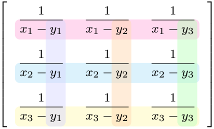


$$\begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{bmatrix}$$

$\rightsquigarrow$ x is repetition-free.

Cauchy-like matrices

$$\phi(\mathsf{A}) = \mathbb{D}(\mathsf{x}) \cdot \mathsf{A} - \mathsf{A} \cdot \mathbb{D}(\mathsf{y})$$



$$\begin{bmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \frac{1}{x_1 - y_3} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \frac{1}{x_2 - y_3} \\ \frac{1}{x_3 - y_1} & \frac{1}{x_3 - y_2} & \frac{1}{x_3 - y_3} \end{bmatrix}$$

$\rightsquigarrow$ x and y are disjoint and repetition-free.

Structured linear system

Background and motivations

$\mathsf{A} \in \mathbb{K}^{m\times n}$

Problem 1:

LinearSystem$[\phi, \alpha]$(G, H, v)

Using ϕ-generators of A: G $\in \mathbb{K}^{m\times\alpha}$, H $\in \mathbb{K}^{n\times\alpha}$, v $\in \mathbb{K}^{m\times 1}$

find nonzero u $\in \mathbb{K}^{n\times 1}$ such that A $\cdot$ u $=$ v

State of the art:

$\bar{m} = \max(m, n)$

Authors / Year	**Structure**	**Type**	**Complexity**
[Kailath, Kung, Morf 1979]	Toeplitz-like	deterministic	$O(\alpha\bar{m}^2)$
[Bitmead, Anderson 1980], [Morf 1980] Under generic rank profile assumption	Toeplitz-like	genericity	$O(\alpha^2\mathcal{M}(\bar{m})\log(\bar{m}))$
[Kaltofen 1994, 1995]	Toeplitz-like	randomized	$O(\alpha^2\mathcal{M}(\bar{m})\log(\bar{m}))$
[Pan 1990], [Gohberg, Olshevsky 1994]	Vandermonde-like, Cauchy-like	randomized	Reduction to Toeplitz-like case
[Cardinal 1999], [Pan, Zheng 2000]	Vandermonde-like, Cauchy-like	randomized	$O(\alpha^2\mathcal{M}(\bar{m})\log(\bar{m}))$
[Bostan, Jeannerod, Mouilleron, Schost 2008, 2017]	General	randomized	$O(\alpha^{\omega-1}\mathcal{M}(\bar{m})\log(\bar{m}))$

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[Bostan, Jeannerod, Mouilleron, Schost 2008, 2017]	General	randomized	$O(\alpha^{\omega-1} \mathcal{M}(\bar{m}) \log(\bar{m}))$

The algorithms are randomized!

Structured linear system

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Randomization:

Kaltofen's Leading Principal Inverse algorithm

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{\text{Multiplication by random matrices}} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

Generic rank profile

One of the issues with randomization

The field $\mathbb{K}$ has to be large enough for the algorithms to work with positive probability.
e.g.,15000×15000 system with $\alpha = 30$ solved in < 1 sec, but requires $|\mathbb{K}| > 2.25 \cdot 10^8 = 15000^2$.

Main result

Fast deterministic algorithm

$\mathsf{A} \in \mathbb{K}^{m\times n}$

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find nonzero $\mathsf{u} \in \mathbb{K}^{n\times 1}$ such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

Theorem

For $\phi \in \{$Toeplitz-like, Vandermonde-like, Cauchy-like$\}$,

Over any field $\mathbb{K}$, *we can* **deterministically** *solve* $\mathsf{LinearSystem}[\phi, \alpha]$ *in*

$$O(\alpha^{\omega-1}(\mathcal{M}(m)\log(m) + \mathcal{M}(n)\log(n)^2)) = \tilde{O}(\alpha^{\omega-1} \cdot \max(m, n)).$$

Main result

Fast deterministic algorithm

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find nonzero $\mathsf{u} \in \mathbb{K}^{n\times 1}$ such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

Problem 2:

Nullspace$[\phi, \alpha]$(G, H)

Using generators of A: $\mathsf{G} \in \mathbb{K}^{m\times\alpha}, \mathsf{H} \in \mathbb{K}^{n\times\alpha}$

find a basis of the nullspace $\{\mathsf{u} \in \mathbb{K}^{n\times 1} : \mathsf{A} \cdot \mathsf{u} = 0\}$

Theorem

For $\phi \in \{$*Toeplitz-like, Vandermonde-like, Cauchy-like*$\}$*,*

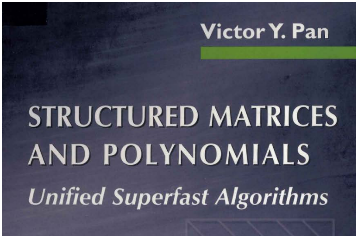
Over any field $\mathbb{K}$*, we can* **deterministically** *solve* LinearSystem$[\phi, \alpha]$ *and* Nullspace$[\phi, \alpha]$ *in*

$$O(\alpha^{\omega-1}(\mathcal{M}(m)\log(m) + \mathcal{M}(n)\log(n)^2)) = \tilde{O}(\alpha^{\omega-1} \cdot \max(m, n)).$$

Key tools: Approximation problems

Structured vs polynomial

Link between the two approaches



Victor Y. Pan

STRUCTURED MATRICES AND POLYNOMIALS

Unified Superfast Algorithms

$$u(x) \cdot v(x) \bmod x^n$$

$$\begin{bmatrix} u_1 & 0 & \cdots & 0 \\ u_2 & u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_{n-1} & \cdots & u_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Polynomial approach

Polynomial entries

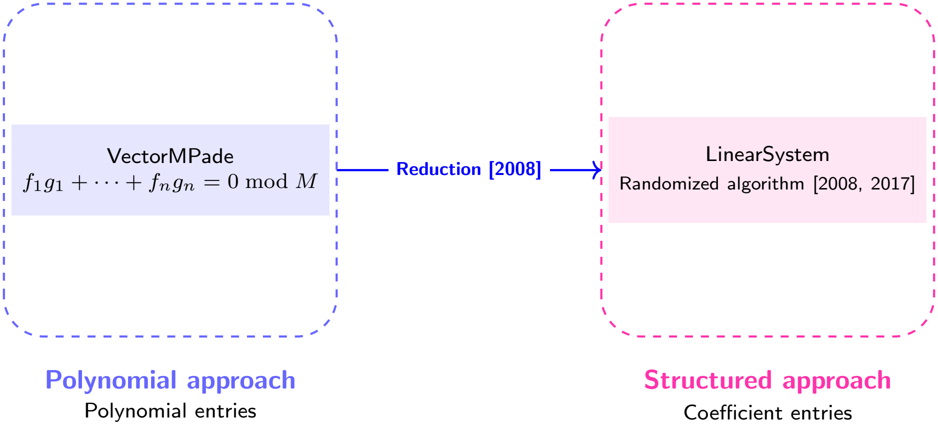
Structured approach

Coefficient entries

Structured vs polynomial

Link between the two approaches

Previous work:
[Bostan, Jeannerod, Mouilleron, Schost 2008, 2017]



VectorMPade
$f_1 g_1 + \cdots + f_n g_n = 0 \bmod M$

Reduction [2008]

LinearSystem
Randomized algorithm [2008, 2017]

Polynomial approach
Polynomial entries

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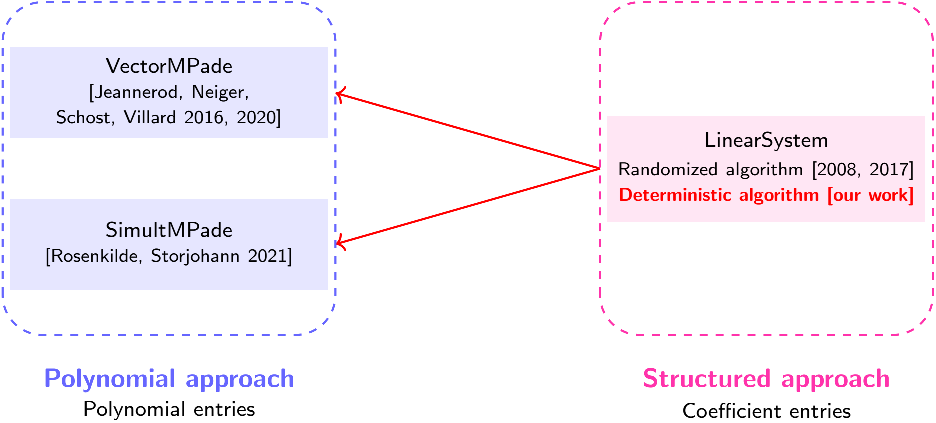
Contribution:



VectorMPade
[Jeannerod, Neiger, Schost, Villard 2016, 2020]

SimultMPade
[Rosenkilde, Storjohann 2021]

LinearSystem
Randomized algorithm [2008, 2017]
Deterministic algorithm [our work]

Polynomial approach
Polynomial entries

Structured approach
Coefficient entries

Vector M-Padé approximation

Problem definition and main results

VectorMPade$(M \in \mathbb{K}[x]_n, \mathsf{F} \in \mathbb{K}[x]^{1\times\alpha}, v \in \mathbb{K}[x])$

Find a small-degree basis of solutions for

$\{\mathsf{p} \in \mathbb{K}[x]^\alpha \mid \mathsf{F} \cdot \mathsf{p} = v \bmod M\}$

$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

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Fundamental problems:

- Hermite-Padé approximation when $M = x^n$,
- Berlekamp–Massey algorithm when $M = x^n$ and $\alpha = 2$,
- Rational interpolation when $M = (x - a_1)\cdots(x - a_n)$ with a_i's distinct.

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State of the art:

Authors / Year	Modulus	Type	Complexity
[Beckermann, Labahn 1994] [Giorgi, Jeannerod, Villard 2003]	x^n or $\prod_{i=1}^{n}(x-x_i)$	homogeneous	$\tilde{O}(\alpha^\omega n)$
[Storjohann 2006] [Zhou, Labahn 2012] [Jeannerod, Neiger, Schost, Villard 2020]	x^n	homogeneous	$O(\alpha^\omega \mathcal{M}(n/\alpha)\log(n))$

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Nonhomogeneous case ($v \neq 0$) with an arbitrary M.

Vector M-Padé approximation

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$$f_1 p_1 + f_2 p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

- Arbitrary modulus: Reduction to the case $M = x^n$ in [Neiger 2016].

Nonhomogeneous case ($v \neq 0$) with an arbitrary M.

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$$f_1p_1 + f_2p_2 + \cdots + f_\alpha p_\alpha = v \bmod M$$

- Arbitrary modulus: Reduction to the case $M = x^n$ in [Neiger 2016].
- Nonhomogeneous case:

Input:

$\mathsf{E}\cdot\mathsf{q} = 0 \bmod x^n$

$\mathsf{E} = \begin{bmatrix}\mathsf{F} & -v\end{bmatrix} \in \mathbb{K}[x]^{1\times(\alpha+1)}$

$$\rightsquigarrow$$

Output:

$\mathsf{Q} = \begin{bmatrix}\mathsf{P} & \mathsf{s} \\ 0 & 1\end{bmatrix}$ Popov form

Nonhomogeneous case ($v \neq 0$) with an arbitrary M.

Simultaneous M-Padé approximation

Problem definition and main results

SimultMPade$(M \in \mathbb{K}[x]_n, \mathsf{F} \in \mathbb{K}[x]^{\alpha \times 1}, \mathsf{v} \in \mathbb{K}[x]^{\alpha}, d = (d_1, \dots, d_\alpha))$

Find a basis of solutions for

$$\{p \in \mathbb{K}[x]^{\alpha}, \mathsf{q} \in \mathbb{K}[x]^{\alpha}, \deg(q_i) \leq d_i \mid \mathsf{F} \cdot p = (\mathsf{q} + \mathsf{v}) \bmod M\}$$

$$\begin{aligned} f_1 p =& (q_1 + v_1) \bmod M \\ f_2 p =& (q_2 + v_2) \bmod M \\ &\vdots \\ f_\alpha p =& (q_\alpha + v_\alpha) \bmod M \end{aligned}$$

Simultaneous M-Padé approximation

Problem definition and main results

$$\mathsf{SimultMPade}(M \in \mathbb{K}[x]_n, \mathsf{F} \in \mathbb{K}[x]^{\alpha\times 1}, \mathsf{v} \in \mathbb{K}[x]^{\alpha}, d = (d_1, \dots, d_\alpha))$$

Find a basis of solutions for

$$\{p \in \mathbb{K}[x]^{\alpha}, \mathsf{q} \in \mathbb{K}[x]^{\alpha}, \deg(q_i) \leq d_i \mid \mathsf{F} \cdot p = (\mathsf{q} + \mathsf{v}) \bmod M\}$$

$$\begin{aligned} f_1 p &= (q_1 + v_1) \bmod M \\ f_2 p &= (q_2 + v_2) \bmod M \\ &\vdots \\ f_\alpha p &= (q_\alpha + v_\alpha) \bmod M \end{aligned}$$

State of the art:

- [Van Barel, Bultheel 1992] in $O(\alpha^3 n^2)$,
- [Olesh, Storjohann 2007] in $\tilde{O}(\alpha^\omega n)$,
- [Rosenkilde, Storjohann 2016, 2021] in $\underbrace{O(\alpha^\omega \mathcal{M}(n/\alpha)\log(1+n/\alpha) + \alpha\mathcal{M}(n)\log^2(n))}_{\tilde{O}(\alpha^{\omega-1}n)}$.

$\rightsquigarrow$ computes p only, not the q_i's.

Structured linear system: Toeplitz-like case

Polynomial formulation

Step zero

LinearSystem$[$Toeplitz-like$, \alpha]($G, H, v$)$

Using generators of A, find u such that A $\cdot$ u $=$ v

A $\cdot$ u $=$ v

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \big(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\big) \bmod x^m = v(x)$$

Polynomial formulation

Step zero

LinearSystem$[$Toeplitz-like$, \alpha](\mathsf{G}, \mathsf{H}, \mathsf{v})$
Using generators of A, find u such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$$\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} \mathsf{G}_{1i} & 0 & \cdots & 0 \\ \mathsf{G}_{2i} & \mathsf{G}_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{G}_{mi} & \mathsf{G}_{(m-1)i} & \cdots & \mathsf{G}_{1i} \end{bmatrix} \cdot \begin{bmatrix} \mathsf{H}_{ni} & \mathsf{H}_{(n-1)i} & \cdots & \mathsf{H}_{1i} \\ 0 & \mathsf{H}_{ni} & \cdots & \mathsf{H}_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{H}_{ni} \end{bmatrix} \cdot \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_n \end{bmatrix} = \begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \vdots \\ \mathsf{v}_m \end{bmatrix}$$

Square Σ-LU decomposition of $\mathsf{A} \in \mathbb{K}[x]^{n \times n}$ [Pan 2001]

Given $\phi(\mathsf{A}) = \mathsf{G} \cdot \mathsf{H}^\mathsf{T}$, we have $\mathsf{A} = \sum_{i=1}^{\alpha} \mathbb{L}(\mathsf{G}_{*i}) \cdot \mathbb{U}(\mathsf{H}_{*i})$.

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that A $\cdot$ u $=$ v

$m \leq n$

$\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Σ-LU decomposition of A $\in \mathbb{K}[x]^{m \times n}$ [Pan, Wang 2003]

Given $\phi(\mathsf{A}) = \mathsf{G} \cdot \mathsf{H}^\mathsf{T}$, we have $\mathsf{A} = \sum_{i=1}^{\alpha} \mathbb{L}_{\min(m,n)}(\mathsf{G}_{*i}) \cdot \mathbb{U}_{\min(m,n)}(\mathsf{H}_{*i})$.

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$m \leq n$

$\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \text{ mod } x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{array}{l} g_i(x) \\ \begin{bmatrix} \mathsf{G}_{1i} & 0 & \cdots & 0 \\ \mathsf{G}_{2i} & \mathsf{G}_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{G}_{mi} & \mathsf{G}_{(m-1)i} & \cdots & \mathsf{G}_{1i} \end{bmatrix} \end{array} \cdot \begin{array}{c} h_i(x) \\ \begin{bmatrix} \mathsf{H}_{ni} & \mathsf{H}_{(n-1)i} & \cdots & \mathsf{H}_{(n-m+1)i} & \cdots & \mathsf{H}_{1i} \\ 0 & \mathsf{H}_{ni} & \cdots & \mathsf{H}_{(n-m+2)i} & \cdots & \mathsf{H}_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathsf{H}_{ni} & \cdots & \mathsf{H}_{mi} \end{bmatrix} \end{array} \cdot \begin{array}{c} u(x) \\ \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_n \end{bmatrix} \end{array} = \begin{array}{c} v(x) \\ \begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \vdots \\ \mathsf{v}_m \end{bmatrix} \end{array}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $A \cdot u = v$

$m \leq n$

$A \cdot u = v$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m = v(x)$$

$h_i(x)u(x)$ quo x^{n-1}

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$\begin{bmatrix} H_{ni}u_1 & + & H_{(n-1)i}u_2 & + & \cdots & + & H_{(n-m+1)i}u_m & + & \cdots & + & H_{1i}u_n \\ & & H_{ni}u_2 & + & \cdots & + & H_{(n-m+2)i}u_m & + & \cdots & + & H_{2i}u_n \\ & & & & & & & & & & \vdots \\ & & & & & & H_{ni}u_m & + & \cdots & + & H_{mi}u_n \end{bmatrix}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)
Using generators of A, find u such that $A \cdot u = v$

$m \leq n$

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$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m = v(x)$$

$h_i(x)u(x)$ quo x^{n-1}

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} G_{1i} & 0 & \cdots & 0 \\ G_{2i} & G_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{mi} & G_{(m-1)i} & \cdots & G_{1i} \end{bmatrix} \cdot \begin{bmatrix} H_{ni} & H_{(n-1)i} & \cdots & H_{(n-m+1)i} & \cdots & H_{1i} \\ 0 & H_{ni} & \cdots & H_{(n-m+2)i} & \cdots & H_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{ni} & \cdots & H_{mi} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$g_i(x) \cdot (h_i(x)u(x) \text{ quo } x^{n-1}) \bmod x^m$

$$\begin{bmatrix} G_{1i}p_1 \\ G_{2i}p_1 + G_{1i}p_2 \\ \vdots \\ G_{mi}p_1 + G_{(m-1)i}p_2 + \cdots + G_{1i}p_m \end{bmatrix}$$

$$\begin{bmatrix} H_{ni}u_1 + H_{(n-1)i}u_2 + \cdots + H_{(n-m+1)i}u_m + \cdots + H_{1i}u_n \\ H_{ni}u_2 + \cdots + H_{(n-m+2)i}u_m + \cdots + H_{2i}u_n \\ \vdots \\ H_{ni}u_m + \cdots + H_{mi}u_n \end{bmatrix}$$

Polynomial formulation

Step zero

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$m \leq n$

$\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$$\Leftrightarrow \sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m = v(x)$$

$$\Leftrightarrow \sum_{i=1}^{\alpha} \begin{bmatrix} \mathsf{G}_{1i} & 0 & \cdots & 0 \\ \mathsf{G}_{2i} & \mathsf{G}_{1i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{G}_{mi} & \mathsf{G}_{(m-1)i} & \cdots & \mathsf{G}_{1i} \end{bmatrix} \cdot \begin{bmatrix} \mathsf{H}_{ni} & \mathsf{H}_{(n-1)i} & \cdots & \mathsf{H}_{(n-m+1)i} & \cdots & \mathsf{H}_{1i} \\ 0 & \mathsf{H}_{ni} & \cdots & \mathsf{H}_{(n-m+2)i} & \cdots & \mathsf{H}_{2i} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathsf{H}_{ni} & \cdots & \mathsf{H}_{mi} \end{bmatrix} \cdot \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_n \end{bmatrix} = \begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \vdots \\ \mathsf{v}_m \end{bmatrix}$$

LinearSystem[Toeplitz-like, α](G, H, v)

Find u such that

$\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

Polynomial formulation ↔

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \big(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\big) = v(x) \bmod x^m$$

$$\Leftrightarrow \begin{bmatrix} g_1(x) & \cdots & g_\alpha(x) \end{bmatrix} \cdot \left(\begin{bmatrix} h_1(x) \\ \vdots \\ h_\alpha(x) \end{bmatrix} \cdot u(x) \text{ quo } x^{n-1} \right) = v(x) \bmod x^m$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \big(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\big) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{\mathsf{F} \cdot \mathsf{p} = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_\alpha(x) \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_\alpha(x) \end{bmatrix} = v(x) \bmod x^m$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{\mathsf{F} \cdot \mathsf{p} = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_\alpha(x) \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_\alpha(x) \end{bmatrix} = v(x) \bmod x^m$$

- $\mathsf{P} \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{\mathsf{q} \in \mathbb{K}[x]^{\alpha \times 1} \mid \mathsf{F} \cdot \mathsf{q} = 0 \bmod x^m\}$
- $\mathsf{s} \in \mathbb{K}[x]^{\alpha \times 1}$ such that $\mathsf{F} \cdot \mathsf{s} = v(x) \bmod x^m$

$$\begin{bmatrix} p_1(x) \\ \vdots \\ p_\alpha(x) \end{bmatrix} = \mathsf{P} \cdot \lambda + \mathsf{s}$$

Vector M-Padé approximation

First step

Find $u(x)$ such that
$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{\mathsf{F} \cdot \mathsf{p} = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_\alpha(x) \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{1\times\alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_\alpha(x) \end{bmatrix} = v(x) \bmod x^m$$

- $\mathsf{P} \in \mathbb{K}[x]^{\alpha\times\alpha}$ a Popov basis of $\{\mathsf{q} \in \mathbb{K}[x]^{\alpha\times 1} \mid \mathsf{F} \cdot \mathsf{q} = 0 \bmod x^m\}$
- $\mathsf{s} \in \mathbb{K}[x]^{\alpha\times 1}$ such that $\mathsf{F} \cdot \mathsf{s} = v(x) \bmod x^m$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_\alpha(x) \end{bmatrix} u(x) \text{ quo } x^{n-1} = \mathsf{P} \cdot \lambda + \mathsf{s}$$

Find $u(x)$ such that
$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

$$\xleftrightarrow{\text{VectorMPade}}$$

Find $u(x)$ such that
$$\mathsf{h} \cdot u(x) \text{ quo } x^{n-1} = \mathsf{P} \cdot \lambda + \mathsf{s}$$

Vector M-Padé approximation

First step

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

VectorMPade: Find basis of solutions for $\{\mathsf{F} \cdot \mathsf{p} = v(x) \bmod x^m\}$

$$\Leftrightarrow \underbrace{\begin{bmatrix} g_1(x) & \cdots & g_\alpha(x) \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{1 \times \alpha}} \cdot \begin{bmatrix} p_1(x) \\ \vdots \\ p_\alpha(x) \end{bmatrix} = v(x) \bmod x^m$$

- $\mathsf{P} \in \mathbb{K}[x]^{\alpha \times \alpha}$ a Popov basis of $\{\mathsf{q} \in \mathbb{K}[x]^{\alpha \times 1} \mid \mathsf{F} \cdot \mathsf{q} = 0 \bmod x^m\}$
- $\mathsf{s} \in \mathbb{K}[x]^{\alpha \times 1}$ such that $\mathsf{F} \cdot \mathsf{s} = v(x) \bmod x^m$

$$\begin{bmatrix} \operatorname{rev}_{n-1}(h_1(x)) \\ \vdots \\ \operatorname{rev}_{n-1}(h_\alpha(x)) \end{bmatrix} \operatorname{rev}_{n-1}(u(x)) \bmod x^n = \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) = v(x) \bmod x^m$$

$$\xleftrightarrow{\text{VectorMPade}}$$

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \operatorname{rev}_{n-1}(u(x)) \bmod x^n = \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \ = \ \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{s}} - \bar{\mathsf{P}} \cdot \bar{\lambda} = 0 \bmod x^n$$

$$\Leftrightarrow \begin{bmatrix} \bar{\mathsf{h}} & -\bar{\mathsf{s}} & -\bar{\mathsf{P}} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{rev}_{n-1}(u(x)) \\ 1 \\ \bar{\lambda} \end{bmatrix} = 0 \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \ = \ \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{s}} - \bar{\mathsf{P}} \cdot \bar{\lambda} = 0 \bmod x^n$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \bar{\mathsf{h}} & -\bar{\mathsf{s}} & -\bar{\mathsf{P}} \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{\alpha \times (\alpha+2)}} \cdot \begin{bmatrix} \mathrm{rev}_{n-1}(u(x)) \\ 1 \\ \bar{\lambda} \end{bmatrix} = 0 \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n = \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{s}} - \bar{\mathsf{P}} \cdot \bar{\lambda} = 0 \bmod x^n$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \bar{\mathsf{h}} & -\bar{\mathsf{s}} & -\bar{\mathsf{P}} \end{bmatrix}}_{\mathsf{F} \in \mathbb{K}[x]^{\alpha \times (\alpha+2)}} \cdot \begin{bmatrix} \mathrm{rev}_{n-1}(u(x)) \\ 1 \\ \bar{\lambda} \end{bmatrix} = 0 \bmod x^n$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

Definition

P is in Popov form ⇔ clm(P) is invertible and rlm(P) $= \mathbb{I}_\alpha$.

Example.

$$\mathsf{P} = \begin{bmatrix} \underbrace{\begin{matrix} 3 + 2x + 0x^2 \\ 1 + 1x^2 \end{matrix}}_{\text{degree } 2} & \underbrace{\begin{matrix} 1 + 1x^3 \\ 1 + 4x + 0x^3 \end{matrix}}_{\text{degree } 3} \end{bmatrix} \qquad \mathrm{clm}(\mathsf{P}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ invertible}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

Definition

P is in Popov form $\Leftrightarrow$ clm(P) is invertible and rlm(P) $= \mathbb{I}_\alpha$.

Example.

$$\mathsf{P} = \begin{bmatrix} 3 + 2x + 0x^2 & 1 + 1x^3 \\ 1 + 1x^2 & 1 + 4x + 0x^3 \end{bmatrix}$$

degree 2 degree 3

$$\mathrm{clm}(\mathsf{P}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ invertible

$$\bar{\mathsf{P}} = \begin{bmatrix} 3x^3 + 2x + 0 & x^3 + 1 \\ x^2 + 1 & x^3 + 4x^2 + 0 \end{bmatrix}$$

$\bar{\mathsf{P}}(0)$ is invertible $\Rightarrow \det(\bar{\mathsf{P}})(0) \neq 0 \Rightarrow$

$\det(\bar{\mathsf{P}})$ is coprime with x.

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$\bar{\mathsf{P}}$ invertible $\bmod x^n$:

$$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}} \bmod x^n$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$\bar{\mathsf{P}}$ invertible $\bmod x^n$:

$$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}} \bmod x^n$$

K basis of the right kernel

$$\{\mathsf{k} \in \mathbb{K}[x]^{\alpha+1} \mid \begin{bmatrix}\bar{\mathsf{P}} & -\bar{\mathsf{h}}\end{bmatrix} \cdot \mathsf{k} = 0\}$$

Compressing the problem

Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$\bar{\mathsf{P}}$ invertible $\mathrm{mod}\ x^n$:

$$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}} \bmod x^n$$

K basis of the right kernel

$$\{\mathsf{k} \in \mathbb{K}[x]^{\alpha+1} \mid \underbrace{\begin{bmatrix}\bar{\mathsf{P}} & -\bar{\mathsf{h}}\end{bmatrix}}_{\text{full rank}} \cdot \mathsf{k} = 0\}$$

$$\mathsf{K} \;=\; \begin{bmatrix}\mathsf{u} \\ \mu\end{bmatrix}$$

Compressing the problem

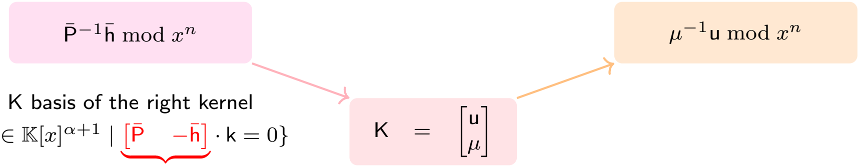
Second step

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \; = \; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$\bar{\mathsf{P}}$ invertible $\bmod x^n$:



$$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}} \bmod x^n$$

$$\mu^{-1}\mathsf{u} \bmod x^n$$

K basis of the right kernel

$$\{\mathsf{k} \in \mathbb{K}[x]^{\alpha+1} \mid \underbrace{\begin{bmatrix} \bar{\mathsf{P}} & -\bar{\mathsf{h}} \end{bmatrix}}_{\text{full rank}} \cdot \mathsf{k} = 0\}$$

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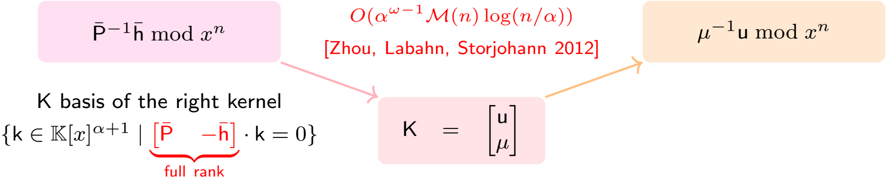

$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}} \bmod x^n$

$O(\alpha^{\omega-1}\mathcal{M}(n)\log(n/\alpha))$

[Zhou, Labahn, Storjohann 2012]

$\mu^{-1}\mathsf{u} \bmod x^n$

K basis of the right kernel

$$\{\mathsf{k} \in \mathbb{K}[x]^{\alpha+1} \mid \underbrace{\begin{bmatrix} \bar{\mathsf{P}} & -\bar{\mathsf{h}} \end{bmatrix}}_{\text{full rank}} \cdot \mathsf{k} = 0\}$$

$$\mathsf{K} \;=\; \begin{bmatrix} \mathsf{u} \\ \mu \end{bmatrix}$$

$$\begin{array}{c} \text{Find } u(x) \text{ such that} \\ \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n = \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}} \end{array} \xleftrightarrow{\text{Compression}} \begin{array}{c} \text{Find } u(x) \text{ such that} \\ \bar{\mathsf{P}}^{-1}\bar{\mathsf{h}}\; \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1}\bar{\mathsf{s}} = \bar{\lambda} \bmod x^n \end{array}$$

Simultaneous M-Padé approximation

SimultMPade

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) = (\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} + \bar{\lambda}) \bmod x^n$$

Simultaneous M-Padé approximation

SimultMPade

Find $u(x)$ such that
$$\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \underbrace{\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}}}_{\mathsf{F} \in \mathbb{K}[x]^{\alpha \times 1}} \cdot \mathrm{rev}_{n-1}(u(x)) = (\underbrace{\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}}}_{\mathsf{v} \in \mathbb{K}[x]^{\alpha}} + \bar{\lambda}) \bmod x^n$$

$$f_1(x) \cdot \mathrm{rev}_{n-1}(u(x)) = (v_1(x) + \bar{\lambda}_1(x)) \bmod x^n$$

$$f_2(x) \cdot \mathrm{rev}_{n-1}(u(x)) = (v_2(x) + \bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_\alpha(x) \cdot \mathrm{rev}_{n-1}(u(x)) = (v_\alpha(x) + \bar{\lambda}_\alpha(x)) \bmod x^n$$

Simultaneous M-Padé approximation

SimultMPade

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \underbrace{\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}}}_{\mathsf{F} \in \mathbb{K}[x]^{\alpha \times 1}} \cdot \mathrm{rev}_{n-1}(u(x)) = (\underbrace{\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}}}_{\mathsf{v} \in \mathbb{K}[x]^{\alpha}} + \bar{\lambda}) \bmod x^n$$

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$$f_2(x) \cdot \mathrm{rev}_{n-1}(u(x)) = (v_2(x) + \bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_\alpha(x) \cdot \mathrm{rev}_{n-1}(u(x)) = (v_\alpha(x) + \bar{\lambda}_\alpha(x)) \bmod x^n$$

$$\mathsf{P} = \left[\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \begin{array}{l} \text{degree } \delta_1 \\ \text{degree } \delta_2 \\ \\ \text{degree } \delta_\alpha \end{array}$$

with respect to the degrees $d = (n - \delta_1, \ldots, n - \delta_2, n - \delta_\alpha)$.

SimultMPade: Find a partial basis of solutions for $\{\mathsf{F} \cdot \mathrm{rev}_{n-1}(u(x)) = (\mathsf{v} + \bar{\lambda}) \bmod x^n\}$

Simultaneous M-Padé approximation

SimultMPade

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1}\cdot\bar{\mathsf{h}}\cdot\mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1}\cdot\bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$$\Leftrightarrow \underbrace{\bar{\mathsf{P}}^{-1}\cdot\bar{\mathsf{h}}}_{\mathsf{F}\in\mathbb{K}[x]^{\alpha\times 1}}\cdot\mathrm{rev}_{n-1}(u(x)) = (\underbrace{\bar{\mathsf{P}}^{-1}\cdot\bar{\mathsf{s}}}_{\mathsf{v}\in\mathbb{K}[x]^{\alpha}}+\bar{\lambda}) \bmod x^n$$

$$f_1(x)\cdot\mathrm{rev}_{n-1}(u(x)) = (v_1(x)+\bar{\lambda}_1(x)) \bmod x^n$$

$$f_2(x)\cdot\mathrm{rev}_{n-1}(u(x)) = (v_2(x)+\bar{\lambda}_2(x)) \bmod x^n$$

$$\vdots$$

$$f_\alpha(x)\cdot\mathrm{rev}_{n-1}(u(x)) = (v_\alpha(x)+\bar{\lambda}_\alpha(x)) \bmod x^n$$

$$\mathsf{P} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{array}{l} \text{degree } \delta_1 \\ \text{degree } \delta_2 \\ \\ \text{degree } \delta_\alpha \end{array}$$

with respect to the degrees $d = (n-\delta_1, \dots, n-\delta_2, n-\delta_\alpha)$.

SimultMPade: Find a partial basis of solutions for $\{\mathsf{F}\cdot\mathrm{rev}_{n-1}(u(x)) = (\mathsf{v}+\bar{\lambda}) \bmod x^n\}$

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1}\bar{\mathsf{h}}\ \mathrm{rev}_{n-1}(u(x)) - \bar{\mathsf{P}}^{-1}\bar{\mathsf{s}} = \bar{\lambda} \bmod x^n$$

$$\xrightarrow{\text{SimultMPade}}$$

Output $\mathrm{rev}_{n-1}(u(x))$

Toeplitz-like linear system

Overview of the deterministic algorithm

LinearSystem[Toeplitz-like, α](G, H, v)

Using generators of A, find u such that $\mathsf{A} \cdot \mathsf{u} \;=\; \mathsf{v}$

$\sum_{i=1}^{\alpha} \mathbb{L}_m(\mathsf{G}_{*i}) \cdot \mathbb{U}_m(\mathsf{H}_{*i}) \cdot \mathsf{u} = \mathsf{v}$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} g_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod x^m \;=\; v(x)$$

VectorMPade $\tilde{O}(\alpha^{\omega-1} \cdot m)$
[Jeannerod, Neiger, Villard 2020]

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n \;=\; \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

Compression $\tilde{O}(\alpha^{\omega-1} \cdot n)$
[Zhou, Labahn, Storjohann 2012]

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \;=\; (\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} + \bar{\lambda}) \bmod x^n$$

SimultMPade $\tilde{O}(\alpha^{\omega-1} \cdot n)$
[Rosenkilde, Storjohann 2021]

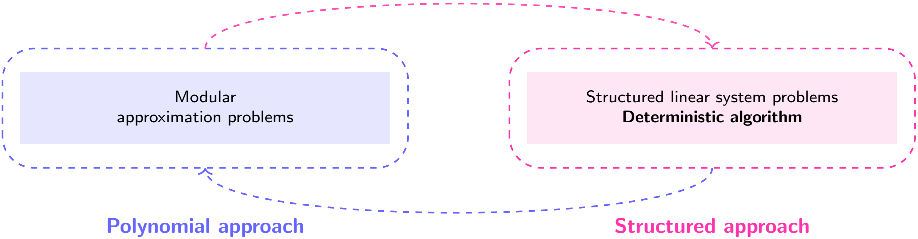
Output $\mathrm{rev}_{n-1}(u(x))$

$\tilde{O}(\alpha^{\omega-1} \cdot n)$
Deterministic

Conclusion

Conclusion

Summary and future work



Modular
approximation problems

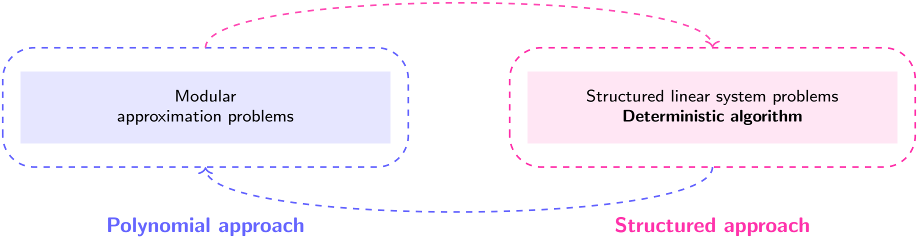
Structured linear system problems
Deterministic algorithm

Polynomial approach

Structured approach

Conclusion

Summary and future work



Modular
approximation problems

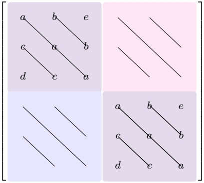
Structured linear system problems
Deterministic algorithm

Polynomial approach

Structured approach

Perspectives:

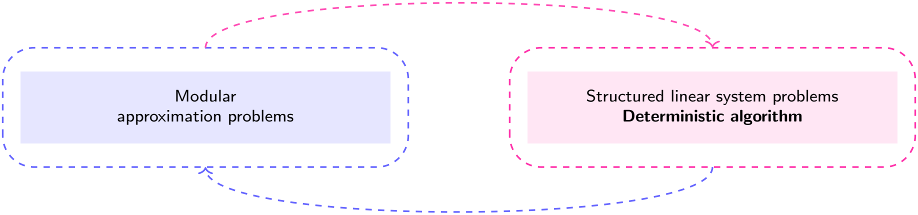
- Extend our result to more general displacement operators,
- Compute rank, inverse (Jeanneord 2026) and determinant,
- Study cases with multiple nested structures
  e.g., $\mathsf{F} = [1, p, p^2, \ldots, p^{m-1}]$ for some $p \in \mathbb{K}[x]$.



BTTB

Conclusion

Summary and future work



Modular
approximation problems

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Perspectives:

- Extend our result to more general displacement operators,
- Compute rank, inverse (Jeannerod 2026) and determinant,
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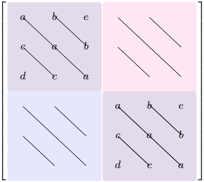
  e.g., F $= [1, p, p^2, \ldots, p^{m-1}]$ for some $p \in \mathbb{K}[x]$.



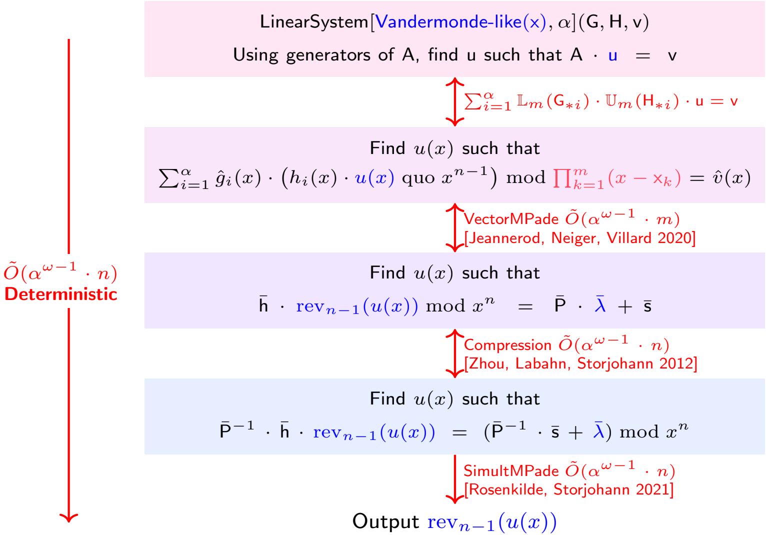
BTTB

Thank you!

Appendix

Vandermonde-like linear system

Overview of the deterministic algorithm



LinearSystem[Vandermonde-like(x), α](G, H, v)

Using generators of A, find u such that A $\cdot$ u $=$ v

$\sum_{i=1}^{\alpha} \mathbb{L}_m(\mathsf{G}_{*i}) \cdot \mathbb{U}_m(\mathsf{H}_{*i}) \cdot \mathsf{u} = \mathsf{v}$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} \hat{g}_i(x) \cdot \left(h_i(x) \cdot u(x) \text{ quo } x^{n-1}\right) \bmod \prod_{k=1}^{m}(x - \mathsf{x}_k) = \hat{v}(x)$$

VectorMPade $\tilde{O}(\alpha^{\omega-1} \cdot m)$
[Jeannerod, Neiger, Villard 2020]

Find $u(x)$ such that

$$\bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) \bmod x^n = \bar{\mathsf{P}} \cdot \bar{\lambda} + \bar{\mathsf{s}}$$

Compression $\tilde{O}(\alpha^{\omega-1} \cdot n)$
[Zhou, Labahn, Storjohann 2012]

Find $u(x)$ such that

$$\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{h}} \cdot \mathrm{rev}_{n-1}(u(x)) = (\bar{\mathsf{P}}^{-1} \cdot \bar{\mathsf{s}} + \bar{\lambda}) \bmod x^n$$

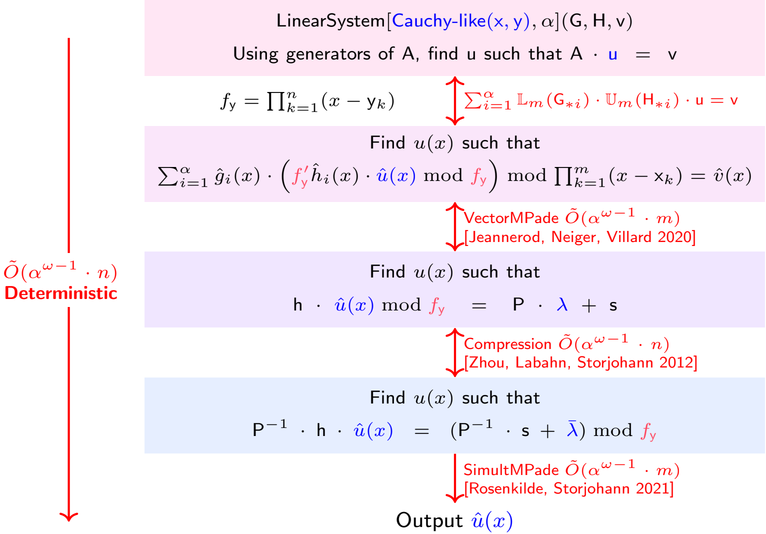
SimultMPade $\tilde{O}(\alpha^{\omega-1} \cdot n)$
[Rosenkilde, Storjohann 2021]

Output $\mathrm{rev}_{n-1}(u(x))$

$\tilde{O}(\alpha^{\omega-1} \cdot n)$
Deterministic

Cauchy-like linear system

Overview of the deterministic algorithm



$\mathsf{LinearSystem}[\mathsf{Cauchy\text{-}like}(\mathsf{x}, \mathsf{y}), \alpha](\mathsf{G}, \mathsf{H}, \mathsf{v})$

Using generators of A, find u such that $\mathsf{A} \cdot \mathsf{u} = \mathsf{v}$

$f_{\mathsf{y}} = \prod_{k=1}^{n}(x - \mathsf{y}_k)$

$\sum_{i=1}^{\alpha} \mathbb{L}_m(\mathsf{G}_{*i}) \cdot \mathbb{U}_m(\mathsf{H}_{*i}) \cdot \mathsf{u} = \mathsf{v}$

Find $u(x)$ such that

$$\sum_{i=1}^{\alpha} \hat{g}_i(x) \cdot \left(f_{\mathsf{y}}' \hat{h}_i(x) \cdot \hat{u}(x) \bmod f_{\mathsf{y}}\right) \bmod \prod_{k=1}^{m}(x - \mathsf{x}_k) = \hat{v}(x)$$

VectorMPade $\tilde{O}(\alpha^{\omega-1} \cdot m)$
[Jeannerod, Neiger, Villard 2020]

Find $u(x)$ such that

$$\mathsf{h} \cdot \hat{u}(x) \bmod f_{\mathsf{y}} = \mathsf{P} \cdot \lambda + \mathsf{s}$$

Compression $\tilde{O}(\alpha^{\omega-1} \cdot n)$
[Zhou, Labahn, Storjohann 2012]

Find $u(x)$ such that

$$\mathsf{P}^{-1} \cdot \mathsf{h} \cdot \hat{u}(x) = (\mathsf{P}^{-1} \cdot \mathsf{s} + \bar{\lambda}) \bmod f_{\mathsf{y}}$$

SimultMPade $\tilde{O}(\alpha^{\omega-1} \cdot m)$
[Rosenkilde, Storjohann 2021]

Output $\hat{u}(x)$

$\tilde{O}(\alpha^{\omega-1} \cdot n)$
Deterministic

More structured-like matrices

We distinguish two types of displacement operators: the Stein operator $\Delta[\mathsf{P},\mathsf{Q}]$ and the Sylvester operator $\nabla[\mathsf{P},\mathsf{Q}]$, where $\mathsf{P} \in \mathbb{K}^{m\times m}$ and $\mathsf{Q} \in \mathbb{K}^{n\times n}$ are fixed matrices. They are defined as follows

$$\Delta[\mathsf{P},\mathsf{Q}] : \begin{array}{lll} \mathbb{K}^{m\times n} & \longrightarrow & \mathbb{K}^{m\times n} \\ \mathsf{A} & \longmapsto & \mathsf{A}-\mathsf{PAQ} \end{array} \qquad \nabla[\mathsf{P},\mathsf{Q}] : \begin{array}{lll} \mathbb{K}^{m\times n} & \longrightarrow & \mathbb{K}^{m\times n} \\ \mathsf{A} & \longmapsto & \mathsf{PA}-\mathsf{AQ} \end{array} .$$

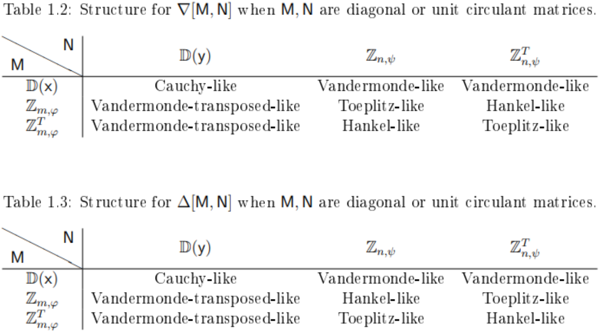


Table 1.2: Structure for $\nabla[\mathsf{M},\mathsf{N}]$ when M,N are diagonal or unit circulant matrices.

N M	$\mathbb{D}(\mathsf{y})$	$\mathbb{Z}_{n,\psi}$	$\mathbb{Z}_{n,\psi}^T$
$\mathbb{D}(\mathsf{x})$	Cauchy-like	Vandermonde-like	Vandermonde-like
$\mathbb{Z}_{m,\varphi}$	Vandermonde-transposed-like	Toeplitz-like	Hankel-like
$\mathbb{Z}_{m,\varphi}^T$	Vandermonde-transposed-like	Hankel-like	Toeplitz-like

Table 1.3: Structure for $\Delta[\mathsf{M},\mathsf{N}]$ when M,N are diagonal or unit circulant matrices.

N M	$\mathbb{D}(\mathsf{y})$	$\mathbb{Z}_{n,\psi}$	$\mathbb{Z}_{n,\psi}^T$
$\mathbb{D}(\mathsf{x})$	Cauchy-like	Vandermonde-like	Vandermonde-like
$\mathbb{Z}_{m,\varphi}$	Vandermonde-transposed-like	Hankel-like	Toeplitz-like
$\mathbb{Z}_{m,\varphi}^T$	Vandermonde-transposed-like	Toeplitz-like	Hankel-like

Figure: [Mouilleron 2017]