Real Root Classification via Hermite's Quadratic Forms

Recent Advances and Applications

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March 2, 2026

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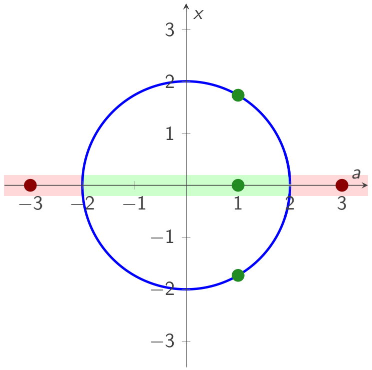
Real root classification

Parameter $\boldsymbol{y} = a$. Variable $\boldsymbol{x}$.
Polynomial $f = x^2 + a^2 - 4$.
Real root classification of $f_a = 0$:



0 real roots:
$\mathcal{T}_0 = \{a \in \mathbb{R} : a > -2 \wedge a < 2\}$

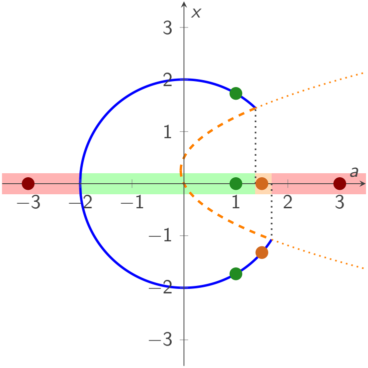
2 real roots:
$\mathcal{T}_2 = \{a \in \mathbb{R} : a < -2 \vee a > 2\}$

Real root classification

Parameter $\boldsymbol{y} = a$. Variable $\boldsymbol{x}$.

Polynomials $f = x^2 + a^2 - 4$, $g = 2x^2 - x - 2a$.

Real root classification of $f_a = 0 \land g_a > 0$:



0 real roots:

$\mathcal{T}_0 = \{a \in \mathbb{R} : (a < -2 \lor 2a > 3) \land h_a > 0\}$

1 real root:

$\mathcal{T}_1 = \{a \in \mathbb{R} : h_a < 0\}$

2 real roots:

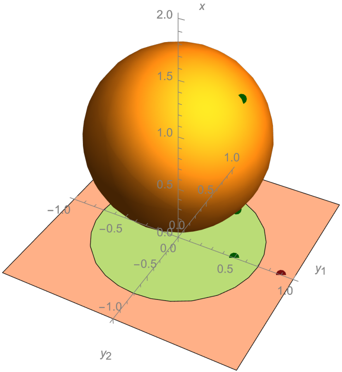
$\mathcal{T}_2 = \{a \in \mathbb{R} : (a > -2 \land 2a < 3) \land h_a > 0\}$

where

$h_a = 4a^4 + 8a^3 - 27a^2 - 32a + 60$

Real root classification

Parameters $\boldsymbol{y} = y_1, y_2$. Variable $\boldsymbol{x}$.
Polynomials $f = x^2 + y_1^2 + y_2^2 - 1$.
Real root classification of $f_{y_1,y_2} = 0$:



0 real roots:
$\mathcal{T}_0 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 > 1\}$

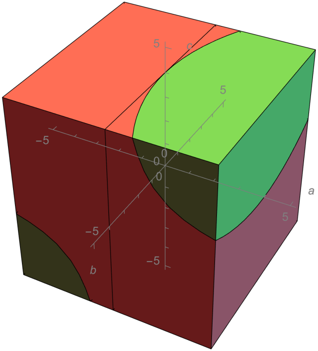
2 real roots:
$\mathcal{T}_2 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$

Real root classification

Parameters $\boldsymbol{y} = a, b, c$. Variable $\boldsymbol{x}$.

Polynomials $f = ax^2 + bx + c$.

Partition of $\mathbb{R}^3$ w.r.t. # of real roots of $f_{a,b,c} = 0$:



0 real roots:

$\mathcal{T}_0 = \{(a, b, c) \in \mathbb{R}^3 : b^2 - 4ac < 0\}$

2 real roots:

$\mathcal{T}_2 = \{(a, b, c) \in \mathbb{R}^3 : b^2 - 4ac > 0 \wedge a \neq 0\}$

Problem

Parameters $\boldsymbol{y} = y_1, \ldots, y_t$. Variables $\boldsymbol{x} = x_1, \ldots, x_n$.
Polynomials $\boldsymbol{f} = f_1, \ldots, f_p \in \mathbb{Q}[\boldsymbol{y}, \boldsymbol{x}]^p$, $\boldsymbol{g} = g_1, \ldots, g_s \in \mathbb{Q}[\boldsymbol{y}, \boldsymbol{x}]^s$.

Assumption

For generic $y \in \mathbb{C}^t$, the system $\boldsymbol{f}_y = 0$ has δ complex solutions, counted with multiplicities.

Problem

Determine disjoint sets $(\mathcal{T}_r)_{0 \leq r \leq \delta}$ whose union is dense in $\mathbb{R}^t$, s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

State of the art

Problem

Determine $(\mathcal{T}_r)_{0 \leq r \leq \delta}$ s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

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Problem

Determine $(\mathcal{T}_r)_{0 \leq r \leq \delta}$ s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

- Tarski's QE algorithm: $\notin \bigcup_{k \in \mathbb{N}} k\text{-EXP}$ [Tarski 1951]

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- Tarski's QE algorithm: $\notin \bigcup_{k \in \mathbb{N}} k$-EXP [Tarski 1951]
- Cylindrical algebraic decomposition (CAD): $(sd)^{2^{O(n+t)}}$
  [Collins 1975] [McCallum 1988] [Hong 1990] [Brown 2001]

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- Border/Discriminant polynomials:
  - Regular chains [Yang-Xia 2005] [Liang-Jeffrey-Maza 2008]
  - Algebraic elimination with GB: [Moroz 2006] [Lazard-Rouillier 2007]
    $(n+s)^{O(n+t)} d^{O(n^2+nt)}$ for borders + $(sd)^{2^{O(t)}}$ for classification.
    Complexity under genericity assumptions.

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    $(n+s)^{O(n+t)}d^{O(n^2+nt)}$ for borders + $(sd)^{2^{O(t)}}$ for classification.
    Complexity under genericity assumptions.
  - Parametric Hermite matrices: simply exponential in n and t^2
    [Le-Safey El Din 2022] [Gaillard-Safey El Din 2024]

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Determine $(\mathcal{T}_r)_{0 \leq r \leq \delta}$ s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

- 0 sign constraints: $n^{O(t)} d^{O(nt)}$ [Le-Safey El Din 2022]
  Implementation: ParamHermite.mla (in Maple with FGb).
- s sign constraints: $s^{O(t^2)} n^{O(t)} d^{O(nt+t^2)}$ [Gaillard-Safey El Din 2024]
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Question

Can we improve the exponential dependency in t^2?

Contributions

Problem

Determine $(\mathcal{T}_r)_{0 \le r \le \delta}$ s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

Contributions [Safey El Din-Wang 2026*]

- New algorithm for s sign constraints: $n^{O(t)} d^{O(nt \log s)}$.
- Efficient implementation in Julia, to appear in AlgebraicSolving.jl.
- Applications to convergence analysis of numerical optimization schemes
- Tackles problems previously unreachable

Real root counting via Tarski-queries

Tarski-queries

Let $\boldsymbol{f} \in \mathbb{Q}[\boldsymbol{x}]^p$, $g \in \mathbb{Q}[\boldsymbol{x}]$. Compute $\#(g > 0, \boldsymbol{f})$.

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Tarski-query of g for $\boldsymbol{f}$

$\mathrm{TaQ}(g, \boldsymbol{f}) := \#(g > 0, \boldsymbol{f}) - \#(g < 0, \boldsymbol{f})$.

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Let $\boldsymbol{f} \in \mathbb{Q}[\boldsymbol{x}]^p$, $g \in \mathbb{Q}[\boldsymbol{x}]$. Compute $\#(g > 0, \boldsymbol{f})$.

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$\mathrm{TaQ}(g^2, \boldsymbol{f})$

$\#(g > 0, \boldsymbol{f}) + \#(g < 0, \boldsymbol{f})$

Takeaway

2 Tarski-queries $\rightsquigarrow \#(g > 0, f)$.

Tarski-queries

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Tarski-query of g for $\boldsymbol{f}$

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$\mathrm{TaQ}(g^2, \boldsymbol{f})$

$\#(g > 0, \boldsymbol{f}) + \#(g < 0, \boldsymbol{f})$

2 Tarski-queries $\mathrm{TaQ}(g^\alpha, \boldsymbol{f})$ of indices $\alpha \in \{1, 2\}$ $\rightsquigarrow$ 2 counts $\#(\mathrm{sign}(g) = \sigma, \boldsymbol{f})$ of signs $\sigma \in \{+, -\}$

Takeaway

2 Tarski-queries $\rightsquigarrow$ $\#(g > 0, f)$.

Let $\boldsymbol{f} \in \mathbb{Q}[\boldsymbol{x}]^p$, $\boldsymbol{g} \in \mathbb{Q}[\boldsymbol{x}]^s$. Compute $\#(\boldsymbol{g} > 0, \boldsymbol{f})$.

Tarski-query of g for $\boldsymbol{f}$

$\mathrm{TaQ}(g, \boldsymbol{f}) := \#(g > 0, \boldsymbol{f}) - \#(g < 0, \boldsymbol{f})$.

$\mathrm{TaQ}(g^2, \boldsymbol{f})$

$\#(g > 0, \boldsymbol{f}) + \#(g < 0, \boldsymbol{f})$

2^s Tarski-queries $\mathrm{TaQ}(\boldsymbol{g}^{\boldsymbol{\alpha}}, \boldsymbol{f})$ of indices $\boldsymbol{\alpha} \in \{1, 2\}^s$ $\rightsquigarrow$ 2^s counts $\#(\mathrm{sign}(\boldsymbol{g}) = \boldsymbol{\sigma}, \boldsymbol{f})$ of signs $\boldsymbol{\sigma} \in \{+, -\}^s$

Takeaway

2^s Tarski-queries $\rightsquigarrow \#(\boldsymbol{g} > 0, \boldsymbol{f})$.

A problem has been detected and this talk has been suspended to prevent damage to your brain.

The problem seems to be caused by the following:
COMPLEXITY_IS_EXPONENTIAL_IN_S

Compression of Tarski-queries

Let $\boldsymbol{f} \in \mathbb{Q}[\boldsymbol{x}]^p$, $\boldsymbol{g} \in \mathbb{Q}[\boldsymbol{x}]^s$. Compute $\#(\boldsymbol{g} > 0, \boldsymbol{f})$.

Compression of Tarski-queries

Let $\boldsymbol{f} \in \mathbb{Q}[\boldsymbol{x}]^p$, $\boldsymbol{g} \in \mathbb{Q}[\boldsymbol{x}]^s$. Compute $\#(\boldsymbol{g} > 0, \boldsymbol{f})$.

$\boldsymbol{f}$ has at most δ real solutions $\rightsquigarrow$ $\boldsymbol{g}$ has at most δ realizable signs!

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Realizable signs

$\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f}) := \{\boldsymbol{\sigma} \in \{+,-\}^s \mid \#(\mathrm{sign}(\boldsymbol{g}) = \boldsymbol{\sigma}, \boldsymbol{f}) \neq 0\}$.

At most δ counts
of signs $\boldsymbol{\sigma} \in \mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f})$

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Realizable signs

$\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f}) := \{\boldsymbol{\sigma} \in \{+, -\}^s \mid \#(\mathrm{sign}(\boldsymbol{g}) = \boldsymbol{\sigma}, \boldsymbol{f}) \neq 0\}$.

At most δ Tarski-queries of indices $\boldsymbol{\alpha} \in \mathrm{Ada}(\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f}))$ $\rightsquigarrow$ At most δ counts of signs $\boldsymbol{\sigma} \in \mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f})$

Takeaway

Once $\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f})$ is known, at most δ Tarski-queries $\rightsquigarrow$ $\#(\boldsymbol{g} > 0, \boldsymbol{f})$.

Tarski-queries
via Hermite's quadratic forms

Hermite's quadratic forms

Let $\mathbb{K}$ be a field, $\boldsymbol{f} \in \mathbb{K}[\boldsymbol{x}]^p$ with $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle) = \delta$.

Hermite's quadratic forms

Let $\mathbb{K}$ be a field, $\boldsymbol{f} \in \mathbb{K}[\boldsymbol{x}]^p$ with $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle) = \delta$.

Multiplication map

$$\begin{aligned} M_g^{\boldsymbol{f}} : \mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle &\to \mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle \\ \overline{h} &\mapsto \overline{gh} \end{aligned}$$

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Hermite's quadratic form

$$\begin{aligned} H_g^{\boldsymbol{f}} : \mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle &\to \mathbb{K} \\ \overline{h} &\mapsto \mathrm{Tr}(M_{gh^2}^{\boldsymbol{f}}) \end{aligned}$$

Hermite's quadratic forms

Let $\mathbb{K} = \mathbb{Q}(a, b, c)$, $\boldsymbol{f} = ax^2 + bx + c$ with $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle) = 2$.

Multiplication map

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$H_1^{ax^2+bx+c}$ in $\mathcal{B} = \{1, x\}$

$$H_1^{\boldsymbol{f}} = \begin{bmatrix} \mathrm{Tr}(M_1^{\boldsymbol{f}}) & \mathrm{Tr}(M_x^{\boldsymbol{f}}) \\ \mathrm{Tr}(M_x^{\boldsymbol{f}}) & \mathrm{Tr}(M_{x^2}^{\boldsymbol{f}}) \end{bmatrix}$$

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$M_g^{ax^2+bx+c}$ with $g = 1$

$$M_1^{\boldsymbol{f}} : \overline{1} \mapsto 1 + 1\overline{x}$$
$$\overline{x} \mapsto 0 + 1\overline{x}$$

$H_1^{ax^2+bx+c}$ in $\mathcal{B} = \{1, x\}$

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$M_g^{ax^2+bx+c}$ with $g = x$

$$M_x^{\boldsymbol{f}} : \overline{1} \mapsto 0 + 1\overline{x}$$
$$\overline{x} \mapsto -\frac{c}{a} - \frac{b}{a}\overline{x}$$

$H_1^{ax^2+bx+c}$ in $\mathcal{B} = \{1, x\}$

$$H_1^{\boldsymbol{f}} = \begin{bmatrix} 2 & \mathrm{Tr}(M_x^{\boldsymbol{f}}) \\ \mathrm{Tr}(M_x^{\boldsymbol{f}}) & \mathrm{Tr}(M_{x^2}^{\boldsymbol{f}}) \end{bmatrix}$$

Hermite's quadratic forms

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$M_g^{ax^2+bx+c}$ with $g = x^2$

$$M_{x^2}^{\boldsymbol{f}} : \overline{1} \mapsto -\frac{c}{a} - \frac{b}{a}\overline{x}$$
$$\overline{x} \mapsto \frac{bc}{a^2} + \frac{b^2 - ac}{a^2}\overline{x}$$

$H_1^{ax^2+bx+c}$ in $\mathcal{B} = \{1, x\}$

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$H_1^{ax^2+bx+c}$ in $\mathcal{B} = \{1, x\}$

$$H_1^{ax^2+bx+c} = \begin{bmatrix} 2 & -\frac{b}{a} \\ -\frac{b}{a} & \frac{b^2-2ac}{a^2} \end{bmatrix}$$

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Let $\mathbb{K} = \mathbb{Q}$, $\boldsymbol{f} \in \mathbb{K}[\boldsymbol{x}]^p$ with $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/\langle \boldsymbol{f} \rangle) = \delta$.

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Theorem [Hermite 1853]

When $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, $\mathrm{TaQ}(g, \boldsymbol{f}) = \mathrm{Sig}(H_g^{\boldsymbol{f}})$.

Takeaways

For real root counting with s sign constraints:

- Naive approach: 2^s Tarski-queries.
- Once the realisable signs $\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f})$ are known:
  at most δ Tarski-queries, with indices in $\mathrm{Ada}(\mathrm{SIGN}(\boldsymbol{g}, \boldsymbol{f}))$.

Tarski-queries $=$ signatures of Hermite's quadratic forms.

Real root classification via Hermite's quadratic form

Discriminant variety

Let $\boldsymbol{y} = a, b, c$, $\boldsymbol{f} = ax^2 + bx + c$. Classify $y \in \mathbb{R}^3$ w.r.t. $\mathrm{Sig}(H_1^{\boldsymbol{f}_y})$.

Discriminant variety

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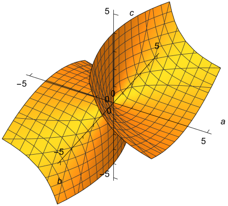
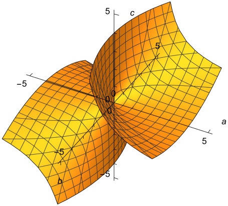
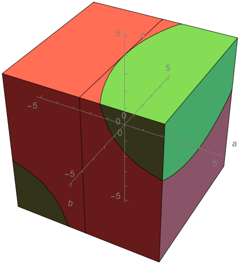


$$\mathcal{V} = \{y \in \mathbb{R}^3 \mid \det(H_1^{\boldsymbol{f}_y}) = 0\}$$

Discriminant variety

Let $\boldsymbol{y} = a, b, c$, $\boldsymbol{f} = ax^2 + bx + c$. Classify $y \in \mathbb{R}^3$ w.r.t. $\mathrm{Sig}(H_1^{\boldsymbol{f}_y})$.





$\mathcal{V} = \{y \in \mathbb{R}^3 \mid \det(H_1^{\boldsymbol{f}_y}) = 0\}$ $\qquad$ $\mathcal{S} = \{y \in \mathbb{R}^3 \mid \det(H_1^{\boldsymbol{f}_y}) \neq 0\}$

$\mathrm{Sig}(H_1^{\boldsymbol{f}_y})$ is constant on each connected component $\mathcal{E}_i$ of $\mathcal{S}$.

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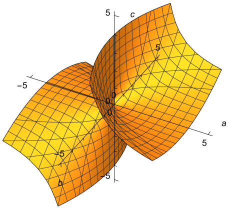

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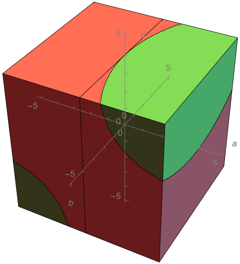

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$\mathrm{Sig}(H_1^{\boldsymbol{f}_y})$ is constant on each connected component $\mathcal{E}_i$ of $\mathcal{S}$.

Sampling one point η_i per $\mathcal{E}_i$ $\rightsquigarrow$ $\mathrm{Sig}(H_1^{\boldsymbol{f}_{\eta_i}}) = \mathrm{Sig}(H_1^{\boldsymbol{f}_y})$ for all $y \in \mathcal{E}_i$.

Real root classification

Problem

Determine $(\mathcal{T}_r)_{0\leq r\leq\delta}$ s.t. for all $y \in \mathcal{T}_r$, $\#(\boldsymbol{g}_y > 0, \boldsymbol{f}_y) = r$.

Let $(\eta_i)_{1\leq i\leq N}$ be points in $\mathbb{R}^t$, s.t. each η_i represents a component $\mathcal{E}_i$ where $(\mathrm{Sig}(H^{\boldsymbol{f}_y}_{\boldsymbol{g}^{\boldsymbol{\alpha}}}))_{\boldsymbol{\alpha}\in\{1,2\}^s}$ is constant.

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Denote by $\Sigma_i := \mathrm{SIGN}(\boldsymbol{g}_{\eta_i}, \boldsymbol{f}_{\eta_i})$.

At most δ Tarski-queries of indices $\boldsymbol{\alpha} \in \mathrm{Ada}(\Sigma_i)$ $\rightsquigarrow$ $r_i := \#(\boldsymbol{g}_{\eta_i} > 0, \boldsymbol{f}_{\eta_i})$

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To describe all $\mathcal{E}_i$, we can use parametric Hermite matrices of indices in $A = \bigcup_{1 \leq i \leq N} \mathrm{Ada}(\Sigma_i)$, with $\#A \in \delta^{O(\log s)}$.

A problem has been detected and this talk has been suspended to prevent damage to your brain.

The problem seems to be caused by the following:
ALGORITHM_IS_NOT_CORRECT

Real root classification

Different realisable signs $\not\leftrightarrow$ different Tarski-queries.

$\boldsymbol{f}_{\eta_1} = 0$ has 2 real roots,
$\mathrm{sign}(\boldsymbol{g}_{\eta_1}(x_{1,1})) = (+,+)$,
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The indices $\mathrm{Ada}(\Sigma_1)$ and $\mathrm{Ada}(\Sigma_2)$ are the same.

The Tarski-queries of these indices are also the same.

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State of the art [Gaillard-Safey El Din 2024]

$A = \mathrm{Ada}(\bigcup_{1\le i\le N} \Sigma_i)$, with $\#A \in s^{O(t)} d^{O(n+t)}$.

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COMPLEXITY_IS_EXPONENTIAL_IN_T_SQUARED

Real root classification

Semi-algebraic description of each $\mathcal{E}_i$ takes $\gg (\#A)^t$ time to compute.
To have a good complexity, $\#A$ must not be exponential in t.

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Idea [Safey El Din-Wang 2026*]

$A \supset \mathrm{Ada}(\Sigma_i \cup \Sigma_j)$ guarantees that $\mathcal{E}_i$ and $\mathcal{E}_j$ are disjoint.

To guarantee that $(\mathcal{E}_i)_{1 \leq i \leq N}$ are disjoint, it suffices to guarantee the property for all pairs $(\mathcal{E}_i, \mathcal{E}_j)$.

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Theorem [Safey El Din-Wang 2026*]

$A = \bigcup_{1 \leq i,j \leq N} \mathrm{Ada}(\Sigma_i \cup \Sigma_j)$, with $\#A \in \delta^{O(\log s)}$.

Applications

Implementation details

Implementation in Julia with Nemo.jl.

- Interfaced with AlgebraicSolving.jl for Gröbner bases over $\mathbb{Q}$ + RAGlib for computing sampling points efficiently.
- Extensive use of multivariate rational interpolation [Cuyt-Lee 2011] for Gröbner bases over $\mathbb{Q}(\boldsymbol{y})$, Hermite matrices and resultants.
- Full multi-threading support.

Performance

- On random small examples: computing all matrices takes $\ll$ 5s, benchmark matches [Gaillard-Safey El Din 2024].
- On structured examples: much faster in many cases, often the only method that succeeds.

Parametric linear matrix inequalities

$A \in \mathbb{Q}[\boldsymbol{y}, \boldsymbol{x}]^{m \times m}$, symmetric, with entries of degree at most 1 in x.

Theorem [Naldi-Safey El Din-Taylor-Wang 2025]

Determining a dense subset of $\{y \in \mathbb{R}^t \mid \exists x \in \mathbb{R}^n : A_y(x) \succeq 0\}$ reduces to real root classification with at most m sign constraints.

Applications in optimization: [Drori-Teboulle 2014], [Kim-Fessler 2016], [Taylor-Hendrickx-Glineur 2018], [Lieder 2021], [Drori-Taylor 2023], ...

δ is polynomial in n when m is fixed. $\rightsquigarrow$ structured examples!

Benchmarks

PRB-t-n	HQF	QE1	QE2
MKN11	0.2 s	5.7 s	0.06 s
RBN11	0.2 s	7.1 s	0.04 s
GRD12	1.0 s	∞	0.5 s
GRD13	11 s	∞	∞
GRD14	1.1 min	∞	∞
GRD21	0.6 s	1.3 s	0.1 s
GRD22	3.6 s	∞	42 min
GRD23	2.3 min	∞	∞
GRD24	8.5 min	∞	∞
PPM21	0.2 s	0.3 s	0.005 s
PPM31	1.1 s	0.4 s	0.007 s
DRS32	46 min	∞	∞
DRS42	8.3 h	∞	∞
DRS33	2.1 h	∞	∞
DRS43	1.8 d	∞	∞

∞ = timeout after 7 days

HQF: HQF in Julia

QE1: QE in Maple
QE2: QE in Wolfram

MKN = Perturbation of Motzkin poly.
RBN = Perturbation of Robinson poly.

GRD = Gradient Descent
PPM = Proximal Point Method
DRS = Douglas-Rachford Splitting

Conclusion

Summary

- For real root classification via Hermite's quadratic forms, # of Hermite matrices required is now reduced to $\delta^{O(\log s)}$.
- With a new implementation in Julia, we can solve problems previously unreachable.

Open questions

- Can we further reduce # of Hermite matrices?
- What assumptions are truly necessary for a good complexity bound?
- What about bit complexity?
- Can we solve quantifier elimination problems using HQF?