A fast and deterministic solution to linear systems with displacement structure

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Abstract

The most efficient algorithms for solving linear systems with Toeplitz, Vandermonde or Cauchy-like structures [2, 1] achieve a complexity of $\tilde{O}(\alpha^{\omega-1}n)$, where n is the size of the system, α is the displacement rank of the matrix and $\omega > 2$ is a feasible exponent for matrix multiplication over the base field. Such algorithms rely on a structured version of the fast Gaussian elimination that requires the input matrix to satisfy the generic rank profile condition. When the latter is not satisfied, a randomized preprocessing step is used [4]. Moreover, they require a sufficiently large base field to work with positive probability.

In this work, we present a deterministic algorithm that solves such systems within a complexity that matches the best known bounds previously achievable only through randomization. Our algorithm works on any field, including small finite fields without using field extensions, thereby eliminating both the randomization requirement and the field size restriction. The key tools of our approach are polynomial matrix algorithms and in particular recent fast deterministic algorithms for vector and simultaneous M-Padé approximations.

The algorithm starts by finding the polynomial formulations of such systems and extracting a nonhomogeneous vector M-Padé approximation problem, for which we compute the basis of all solutions. There exists a fast deterministic algorithm for this task [3, 5]. Assuming $\alpha \in O(n)$, this step costs $O(\alpha^{\omega}\mathcal{M}(\frac{n}{\alpha})\log(n))$, where $\mathcal{M}(d)$ is the cost of multiplying two polynomials of degree less than d. Seeing each solution as a linear combination of the basis of solutions, we exploit their structure by reducing the problem to a nonhomogeneous simultaneous M-Padé approximation. In order to reduce the size of the input and thus not exceed the desired complexity bound, we compress the problem using kernel computations [7] and an efficient algorithm presented in [6]. The latter outputs a generating set of all solutions and runs in $O(\alpha^{\omega}\mathcal{M}(\frac{n}{\alpha})\log(1+\frac{n}{\alpha})^2 + \alpha^{\omega}\mathcal{M}(\frac{n}{\alpha})\log(n)^2)$ operations when $\alpha \in O(n)$.

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