

# Random high genus triangulations

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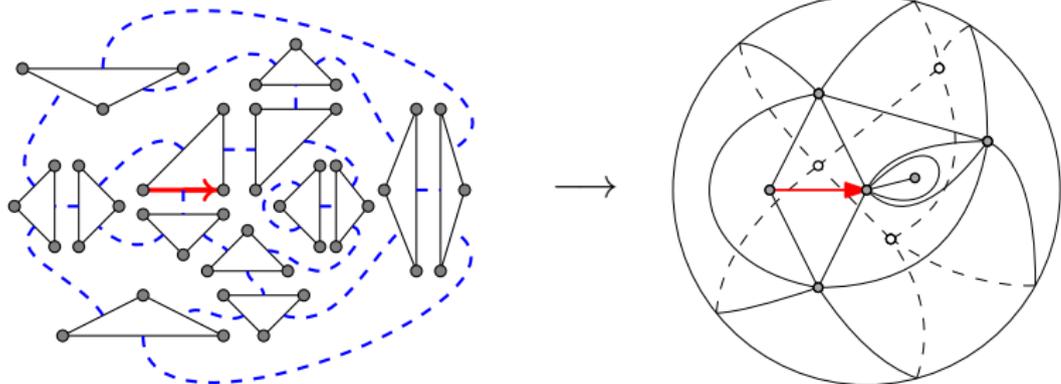
Journées ALEA 2026

CIRM, Marseille

Based on joint works with Baptiste Louf, Guillaume Chapuy and  
Tanguy Lions

- Random triangulations: discrete models of random two-dimensional geometry.
- Planar case: very active in the last 20 years, motivated by 2-dimensional quantum gravity, exact enumeration helps a lot.
- Higher genus: more recent. Enumeration is much more difficult, so the techniques are different.
- Goals of the talk:
  - "Random planar maps are fractals, random high genus maps are expanders".
  - "Dictionary" between combinatorial and probabilistic properties.
  - Systematic use of invariance under uniform rerooting.

# Finite triangulations



- A *triangulation* with  $2n$  faces is a set of  $2n$  triangles whose sides have been glued two by two to obtain a connected, orientable surface.
- The *genus*  $g$  of the triangulation is the number of holes of this surface ( $g = 0$  on the figure).
- Our triangulations are of *type I* (we may glue two sides of the same triangle), and *rooted* (oriented root edge).

- Let  $\mathcal{T}_{n,g}$  be the set of triangulations of genus  $g$  with  $2n$  faces, and  $\tau(n, g)$  its cardinal.
- Let also  $\tau_p(n, g)$  be the number of triangulations of size  $n$  and genus  $g$ , with a boundary of length  $p$ .
- In the planar case, exact formulas [Tutte, 60s]:

$$\tau(n, 0) = 2 \frac{4^n (3n)!!}{(n+1)! (n+2)!!} \underset{n \rightarrow +\infty}{\sim} \sqrt{\frac{6}{\pi}} (12\sqrt{3})^n n^{-5/2},$$

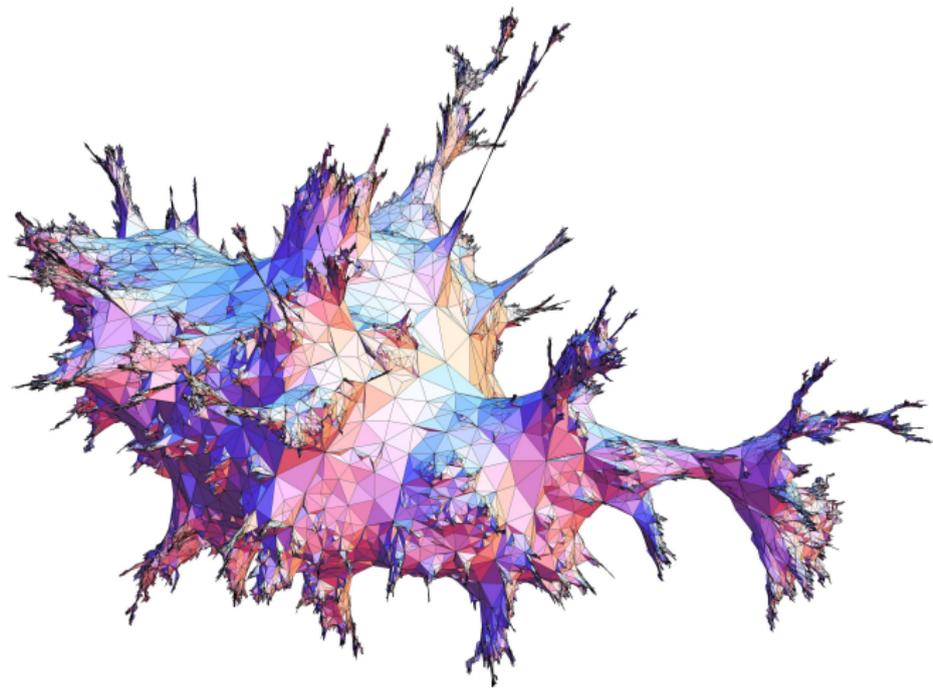
where  $n!! = n(n-2)(n-4)\dots$ . Also  $\tau_p(n, 0)$  is explicit.

- In general, double recurrence relations [Goulden–Jackson, 2008] (cf. Valentin's talk on Monday), but no close formula.
- $\tau(n, g) = n^{2g} e^{O(n)}$  by comparison with unicellular maps.
- Precise asymptotics when  $n \rightarrow +\infty$  with  $g$  fixed, but not when both  $n, g \rightarrow +\infty$ .

# The planar case: global properties

- Let  $T_{n,g}$  be a uniform triangulation in  $\mathcal{T}_{n,g}$ .
- Easy but important:  $T_{n,g}$  is *invariant under uniform rerooting*: if we pick a new root uniformly in  $T_{n,g}$  and forget the old one, what we get has the law of  $T_{n,g}$ .
- Graph distances in  $T_{n,0}$  are typically of order  $n^{1/4}$  [[Chassaing–Schaeffer 2004](#)].
- The same is true for  $T_{n,g}$  when  $g$  is fixed and  $n \rightarrow +\infty$  [[Bettinelli 2010](#)].
- Much more precise : scaling limits after renormalization [[Le Gall 2011](#), [Miermont 2011](#), [Bettinelli–Miermont 2022](#)].
- Main tool: bijections with labelled trees or labelled tree-like structures.

# A sample of $T_{32400,0}$



## The planar case: local properties

- *Local convergence*: two triangulations  $t$  and  $t'$  are close if there is a large  $r$  such that  $B_r(t) = B_r(t')$ , where  $B_r$  denotes the ball for the graph distance.

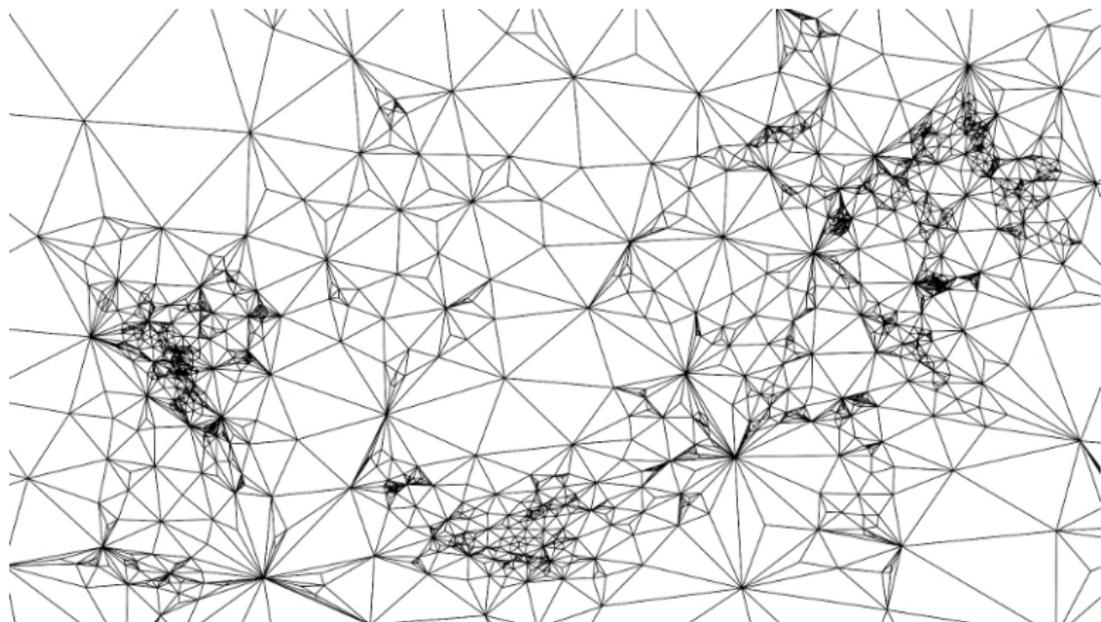
### Theorem (Angel–Schramm 2003)

We have the convergence in distribution

$$T_{n,0} \xrightarrow[n \rightarrow +\infty]{(d)} \mathbb{T}$$

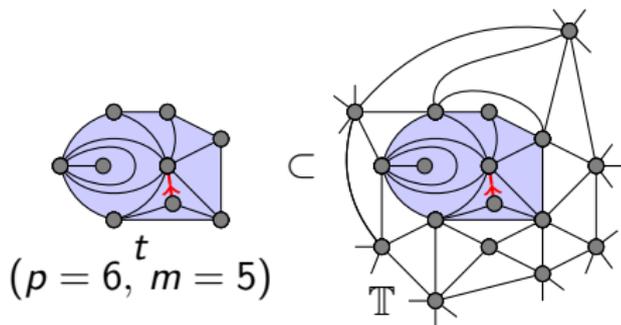
for the local topology, where  $\mathbb{T}$  is an infinite triangulation of the plane called the *UIPT* (Uniform Infinite Planar Triangulation).

- Moreover, the UIPT has volume growth of order  $r^4$  [Angel 2004].

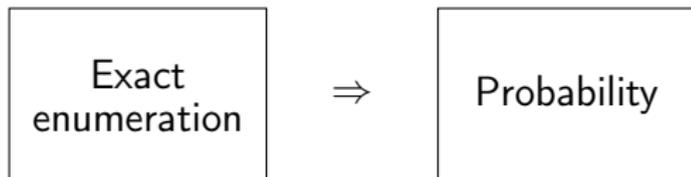


# Convergence to the UIPT $\mathbb{T}$ : sketch of the proof

- Let  $t$  be a small pattern with perimeter  $p$  and  $2m$  internal faces.



- Then  $P(t \subset T_{n,0}) = \frac{\tau_p(n-m,0)}{\tau(n,0)}$ , and the limit is given by the results of Tutte.



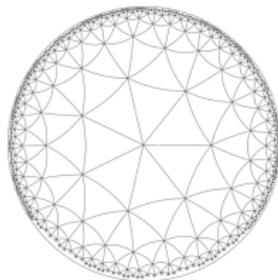
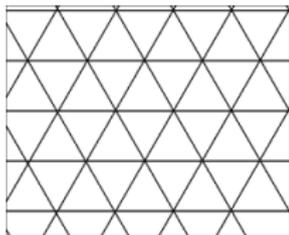
- In particular  $\mathbb{P}(t \subset \mathbb{T})$  is explicit, which allows to explore  $\mathbb{T}$  "face by face" in a Markovian way ("peeling").

## The non-planar case: what is going on?

- Euler formula:  $T_{n,g}$  has  $\#E = 3n$  edges and  $\#V = n + 2 - 2g$  vertices. In particular  $g \leq \frac{n}{2}$ .
- Hence, the *average degree* in  $T_{n,g}$  is

$$\frac{2\#E}{\#V} = \frac{6n}{n + 2 - 2g} \approx \frac{6}{1 - 2g/n}.$$

- Interesting regime:  $\frac{g}{n} \rightarrow \theta \in (0, \frac{1}{2})$ . The average degree in the limit is strictly between 6 and  $+\infty$ .
- The  $d$ -regular infinite triangulation for  $d > 6$  is hyperbolic, so we expect a *hyperbolic* behaviour.



# The high genus case: local limits

## Theorem (B.-Louf 2019)

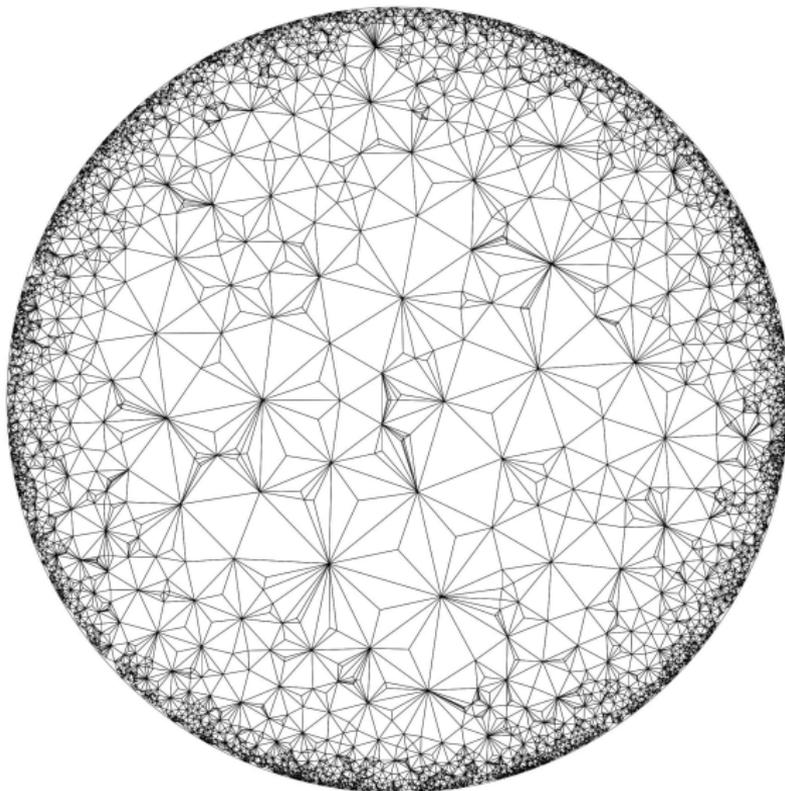
Let  $\frac{g_n}{n} \rightarrow \theta \in [0, \frac{1}{2})$ . Then we have the convergence

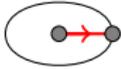
$$T_{n,g_n} \xrightarrow[n \rightarrow +\infty]{(d)} \mathbb{T}_\theta$$

in distribution for the local topology, where  $\mathbb{T}_\theta$  is a random infinite triangulation of the plane called *PSHT*.

- In particular  $\mathbb{T}_0$  is the UIPT.
- For  $\theta > 0$ , the triangulation  $\mathbb{T}_\theta$  is "hyperbolic": exponential volume growth, transience of the simple random walk...  
[Curien 2016]
- The case  $\theta = \frac{1}{2}$  is degenerate (vertices with "infinite degrees").
- The limit is planar, although  $T_{n,g_n}$  has a high genus! Also true e.g. for random regular graphs.

# A sample of a PSHT



- Natural idea for the proof: as in the planar case, use asymptotic results on the counting  $\tau_p(n, g_n)$ .
- But accurate asymptotics seem very hard to obtain!
- Reverse the planar argument: let  $t_0 =$   . Then

$$\mathbb{P}(t_0 \subset T_{n,g}) = \frac{\tau_1(n-1, g)}{\tau(n, g)} = \frac{\tau(n-1, g)}{\tau(n, g)}$$

by a simple root transformation (erase the boundary edge and glue two edges together).

- But we also know that if  $\frac{g}{n} \rightarrow \theta$ , then

$$\mathbb{P}(t_0 \subset T_{n,g}) \xrightarrow{n \rightarrow +\infty} \mathbb{P}(t_0 \subset \mathbb{T}_\theta) = \lambda(\theta),$$

where  $\lambda(\theta)$  satisfies an explicit equation.

## Local limits: back to combinatorics

- We know that  $\frac{\tau(n-1, g)}{\tau(n, g)} \approx \lambda\left(\frac{g}{n}\right)$  and  $\tau(2g-1, g)$  is explicit, so we can estimate  $\tau(n, g)$  with a telescopic product.

### Theorem (B.-Louf, 2019)

When  $\frac{g_n}{n} \rightarrow \theta \in \left[0, \frac{1}{2}\right]$ , we have

$$\tau(n, g_n) = n^{2g_n} \exp(f(\theta)n + o(n)),$$

where  $f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_{2\theta}^1 \log \frac{1}{\lambda(\theta/t)} dt$ , and  $\lambda(\theta)$  is as above.

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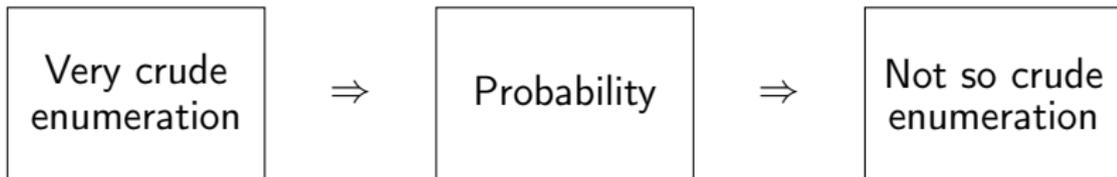
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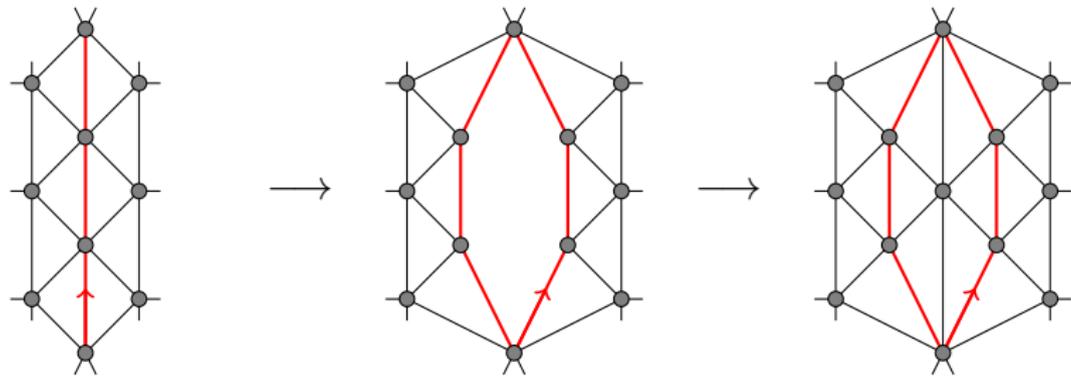
where  $f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_{2\theta}^1 \log \frac{1}{\lambda(\theta/t)} dt$ , and  $\lambda(\theta)$  is as above.

- Summary of the proof:



# Ideas of proofs 1: Tightness

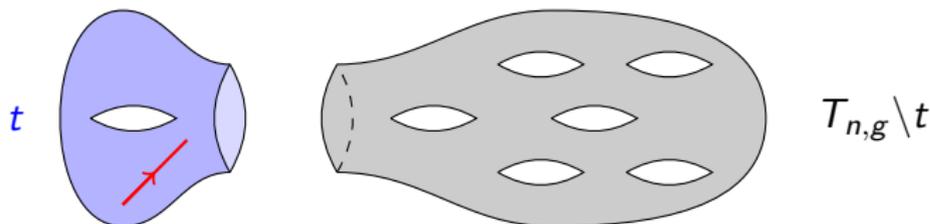
- Existence of limits along subsequences: not immediate because the ball of radius 10 could become too large as  $n \rightarrow \infty$ .
- The number of vertices at distance 10 is bounded by the number of simple paths of length 10 from the root.
- Use scissors: the expected number of such paths is  $\frac{\tau_{20}(n,g)}{\tau(n,g)}$ .



- Fill the hole:  $\frac{\tau_{20}(n,g)}{\tau(n,g)} \leq \frac{\tau(n+10,g)}{\tau(n,g)}$ , so we need  $\frac{\tau(n+1,g)}{\tau(n,g)}$  bounded.
- To prove it: local surgery to decrease the size.

## Ideas of proofs 2: Local planarity

- Let  $t$  be a nonplanar small triangulation (say, with genus 1).



- We can write

$$\mathbb{P}(t \subset T_{n,g}) = \frac{\tau_{|\partial t|}(n - |t|, g - 1)}{\tau(n, g)} \leq \frac{\tau(n - cst, g - 1)}{\tau(n, g)}$$

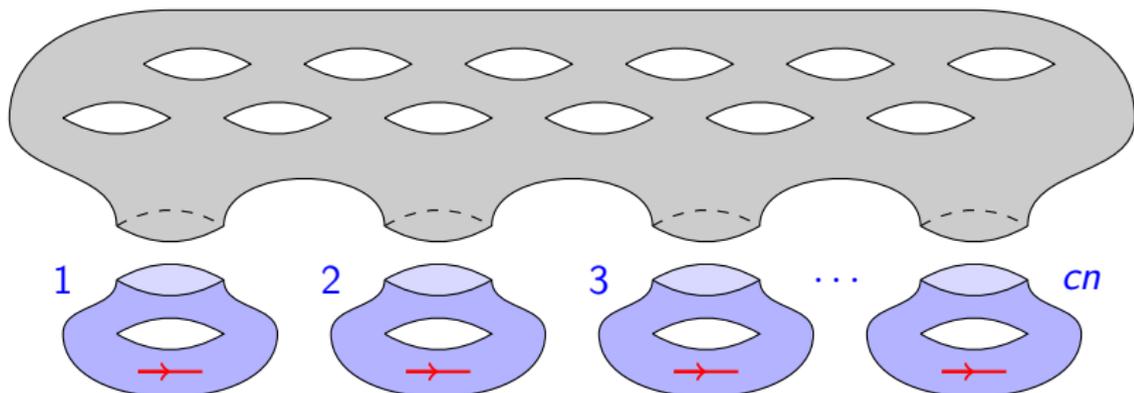
so we need  $\frac{\tau(n, g-1)}{\tau(n, g)} \rightarrow 0$ .

- Free from the Goulden–Jackson recursion:

$$\tau(n, g) = \frac{4}{n+1} \left( n(3n-2)(3n-4)\tau(n-2, g-1) + \sum_{\substack{n_1+n_2=n-2 \\ g_1+g_2=g}} (3n_1+2)(3n_2+2)\tau(n_1, g_1)\tau(n_2, g_2) \right).$$

## Ideas of proofs 2: Local planarity

- More robust argument [Lions 26]: use invariance under uniform rerooting!
- If  $\mathbb{P}(t \subset T_{n,g}) \geq c' > 0$ , this occurs  $cn$  times in  $T_{n,g}$ :



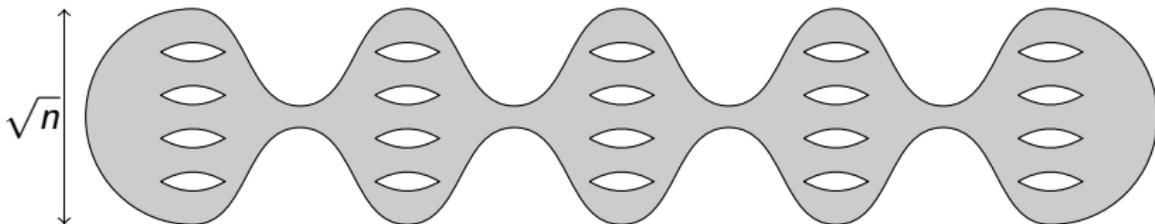
- Probability:  $\binom{n}{cn} \times \frac{\tau(n-cn|t|, g-cn)}{\tau(n,g)} = e^{O(n)} \frac{\tau(n, g-cn)}{\tau(n, g)}$ .
- But  $\tau(n, g) = n^{2g} e^{O(n)}$ , so this is  $e^{O(n)} \times n^{-2cn}$ .

## Ideas of proof 3: Planar considerations

- Let  $T$  be a limit of  $T_{n,g}$  along some subsequence. We know  $T_{n,g}$  is an infinite planar triangulation.
- $T$  must also satisfy a "spatial Markov property" inherited from finite models.
- Classify random infinite triangulations satisfying this property: this requires exact combinatorics results, but *only planar combinatorics!*
- The average vertex degree in  $T$  must be  $\frac{6}{1-2\theta}$  (Euler formula), which gives uniqueness of  $T$ .

# High genus triangulations: global properties

- For  $\frac{g}{n} \rightarrow \theta > 0$ , at large scale, we expect  $T_{n,g}$  to have "good expander properties".
- Heuristics:
  - hyperbolic local limit: easy to reach many points quickly,
  - comparison with unicellular maps [Ray 2014] or with models of random hyperbolic surfaces [Mirzakhani 2013],
  - random graphs (ex: uniform 3-regular random graph) are typically expanders and have high genus.
- Many nice properties to study: Graph distances? Isoperimetric inequalities? Spectral gap? Cut-off for the random walk? Percolation?
- Not immediate from the local behaviour:



- Distances are logarithmic and "almost all the same":

## Theorem (B.–Chapuy–Louf 2023)

For  $\theta \in (0, \frac{1}{2})$ , there are constants  $c_\theta, C_\theta > 0$  such that if  $\frac{g_n}{n} \rightarrow \theta$ , then

$$\mathbb{P}(c_\theta \log n \leq \text{diam}(T_{n,g_n}) \leq C_\theta \log n) \xrightarrow{n \rightarrow +\infty} 1.$$

## Theorem (B.–Chapuy–Louf 2023)

Let  $\frac{g_n}{n} \rightarrow \theta \in (0, \frac{1}{2})$  and let  $x_n, y_n, u_n, v_n$  be four independent uniform vertices of  $T_{n,g_n}$ . Then

$$d_{T_{n,g_n}}(x_n, y_n) - d_{T_{n,g_n}}(u_n, v_n)$$

is tight (i.e. of order  $O(1)$ ).

# Logarithmic distances: ideas of proof

- Lower bound: we already did the proof:

$$\mathbb{E} [|\text{ball of radius } r|] \leq \frac{\tau(n + 2r, g)}{\tau(n, g)} \leq C_\theta^r,$$

so the ball does not cover the triangulation for  $r < c_\theta \log n$ .

- Upper bound: isoperimetric inequality "with defects".
- Ideally: if  $A \subset T_{n,g}$ , then  $|\partial A| \geq c \min(|A|, |T_{n,g} \setminus A|)$ .
- But the object is locally random, so "local defects".

## Theorem (B.–Chapuy–Louf 2023)

For  $\theta \in (0, \frac{1}{2})$ , there are  $\delta_\theta, K_\theta > 0$  such that if  $\frac{g_n}{n} \rightarrow \theta$ , the following holds with high probability:

For any multicurve  $\eta$  which splits  $T_{n,g_n}$  into two connected components with  $n_1$  and  $n_2$  faces, if  $n_2 \geq n_1 \geq K_\theta \log n$ , then the length of  $\eta$  is at least  $\delta_\theta n_1$ .

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- Let  $x \in T_{n,g}$ . For  $r \geq K_\theta \log n$ , we have

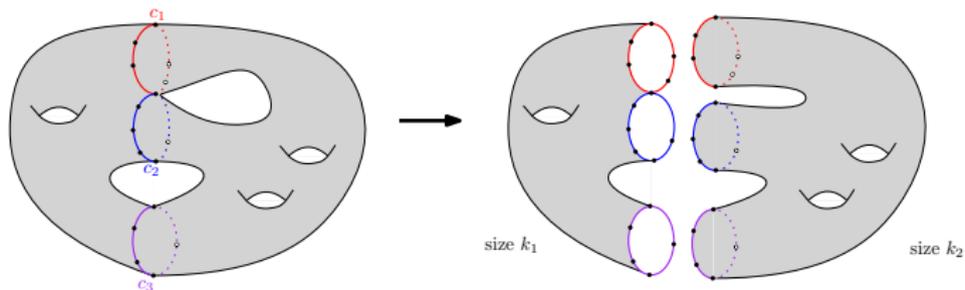
$$|B_{r+1}(x) \setminus B_r(x)| \approx |\partial B_r(x)| \geq \delta_\theta |B_r(x)|,$$

so  $|B_{r+1}(x)| \geq (1 + \delta)|B_r(x)|$  and  $|B_r(x)| \geq (1 + \delta)^r$ , so  $B_r(x) = T_{n,g}$  for  $r = C \log n$ , up to the small defects.

- Defects are actually not a problem: their volume is  $O(\log n)$  each, so their diameter as well.

# Sketch of proof of the isoperimetric inequality

- First moment computation on short separating multicurves:



- The expected number of such multicurves is

$$\frac{\tau_{p_1, \dots, p_k}(n_1, g_1) \tau_{p_1, \dots, p_k}(n_2, g_2)}{\tau(n, g)}$$

$$\text{for } \begin{cases} n_1 + n_2 = n, \\ n_2 \geq n_1 \geq K \log n, \\ g_1 + g_2 = g - k + 1, \\ p = p_1 + \dots + p_k \leq \delta n_1. \end{cases}$$

# Sketch of proof of the isoperimetric inequality

- Close boundaries as before: we then want to bound

$$\frac{\tau(n_1, g_1)\tau(n_2, g_2)}{\tau(n, g)}. \quad (1)$$

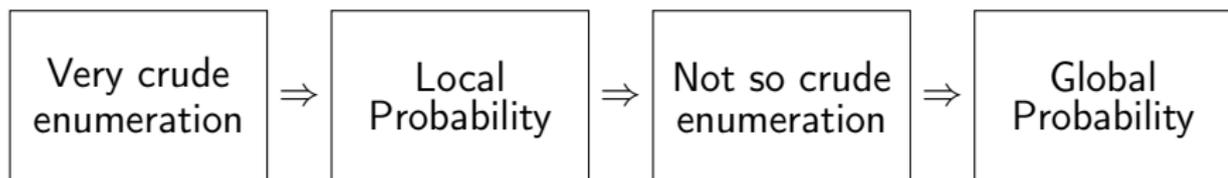
- Rough idea: this is of the form  $\frac{u(n_1)u(n_2)}{u(n_1+n_2)}$ , where  $u(n)$  counts triangulations of size  $n$ , but  $\tau(n, g) \approx n^{2g} e^{nf(g/n)}$ , so  $u(n)$  grows superexponentially.
- Take the genus  $g$  into account: concavity of  $f$ .

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## A few questions

- Work in progress [Lions 26+]: if  $\frac{g_n}{n} \rightarrow \theta$  and  $x_n, y_n$  are two uniform vertices of  $T_{n, g_n}$ , then

$$\frac{1}{\log n} d_{T_{n, g_n}}(x_n, y_n) \xrightarrow[n \rightarrow \infty]{P} D_\theta,$$

where  $D_\theta$  is linked to the rate of exponential growth of  $\mathbb{T}_\theta$ .

- Conjecture:  $\frac{1}{\log n} \text{diam}(T_{n, g_n}) \rightarrow D'_\theta$ , with  $D'_\theta = 3D_\theta$ .
- Defects imply that if  $D'_\theta$  exist, then  $D'_\theta > D_\theta$ .
- Intermediate regime  $1 \ll g \ll n$ ?
- Other expander properties ?

# Dictionary between combinatorics and probability

Combinatorics	Probability
Explicit computations	Easy local convergence
Convergence of $\frac{\tau(n+1,g)}{\tau(n,g)}$	Local convergence
$\frac{\tau(n+1,g)}{\tau(n,g)}$ is bounded	Existence of limits along subsequences
$\frac{\tau(n,g-1)}{\tau(n,g)} \rightarrow 0$ (or $\frac{\tau(n,g-cn)}{\tau(n,g)} \rightarrow 0$ quicker than exponential).	Local planarity
Bounds on sums of the form $\sum_{\substack{n_1+n_2=n \\ g_1+g_2=g}} \frac{\tau(n_1, g_1)\tau(n_2, g_2)}{\tau(n, g)}$	Absence of "bottlenecks"

THANK YOU!