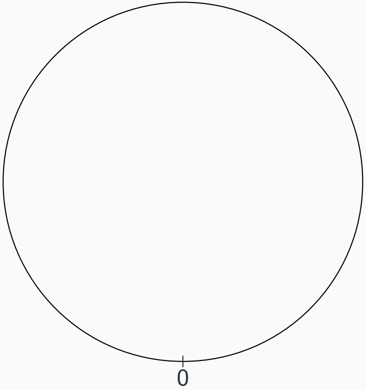
Stochastic diffusion model on the unit circle

Zoé Varin

March 17th, 2025

Joint work with Jean-François Marckert

Definition of the model



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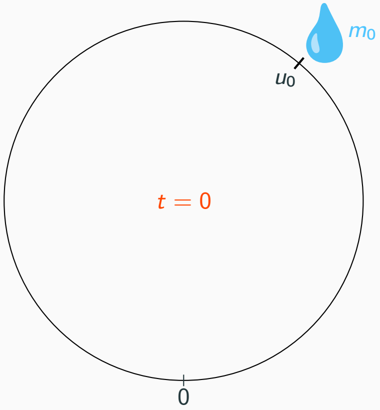

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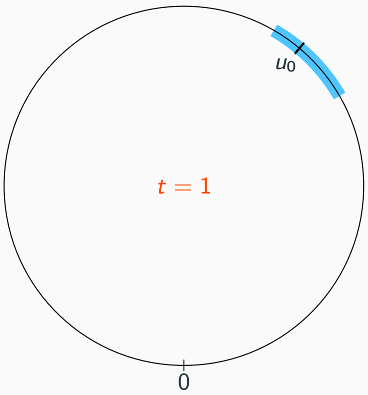

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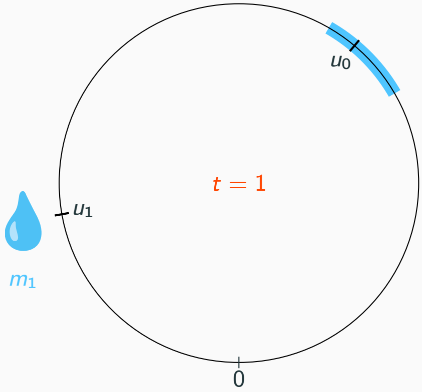

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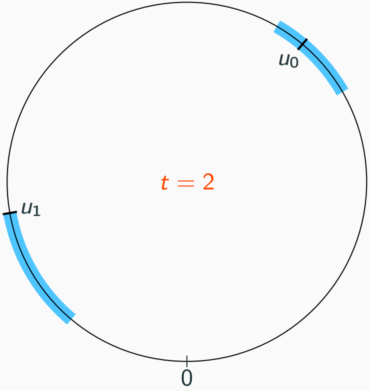

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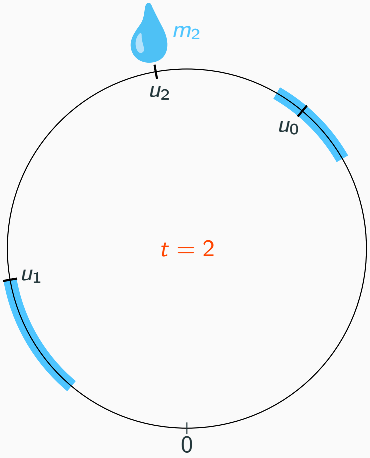

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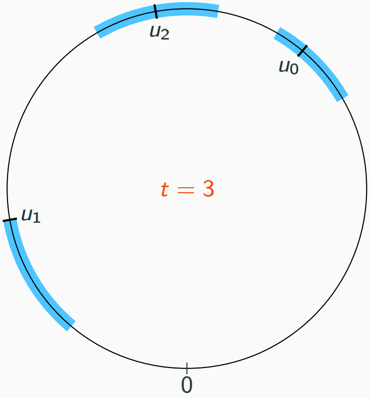

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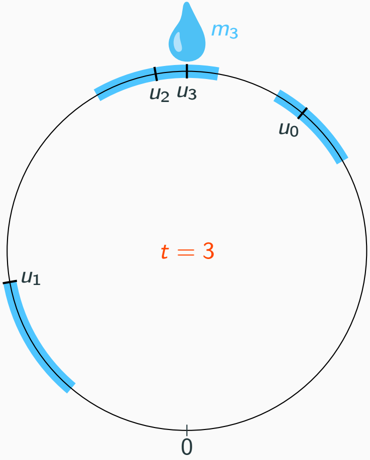

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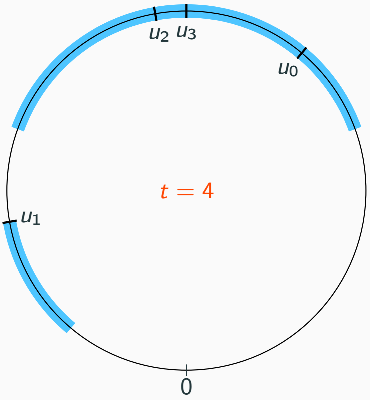

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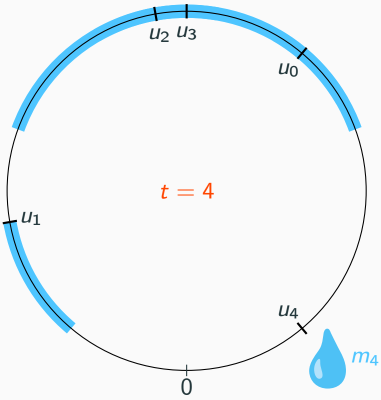

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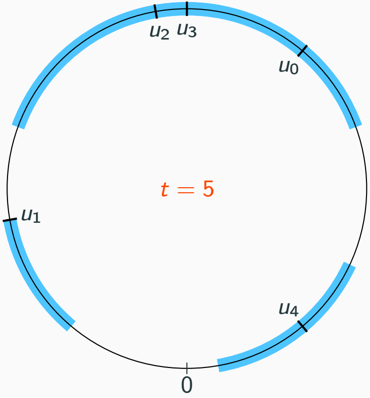

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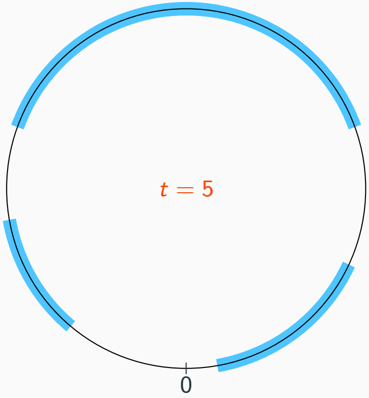

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- **occupied space** $O^{(k)}$ of size $\mathrm{Leb}\left(O^{(k)}\right) = \sum_{i=0}^{k-1} m_i$
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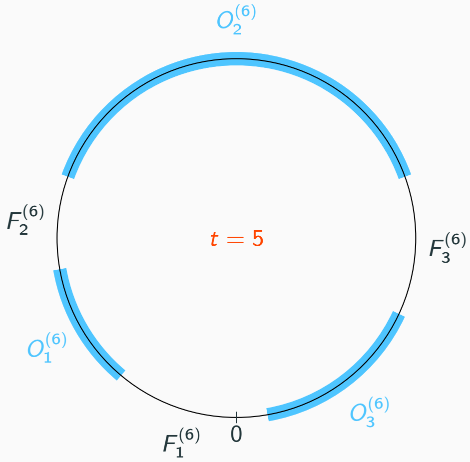

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- more precisely, $N^{(k)}$ **blocks** of each type

$\Longrightarrow$ $(O_i^{(k)}, F_i^{(k)})_{1 \leq i \leq N^{(k)}}$ ordered around the circle with $0 \in O_1^{(k)} \cup F_1^{(k)}$

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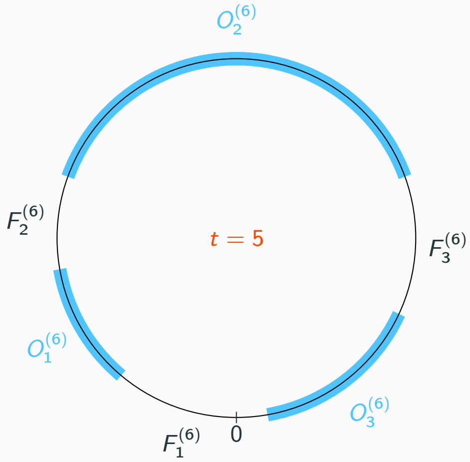

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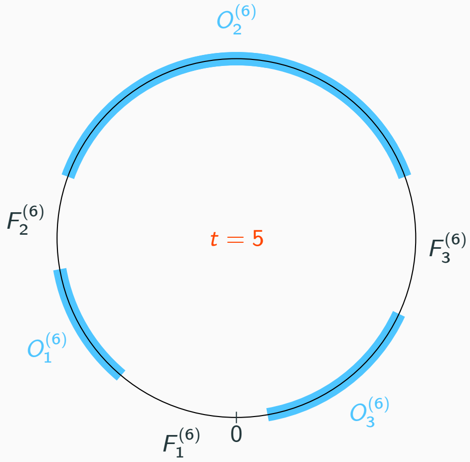
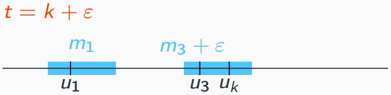


Hypotheses: local and continuous diffusion



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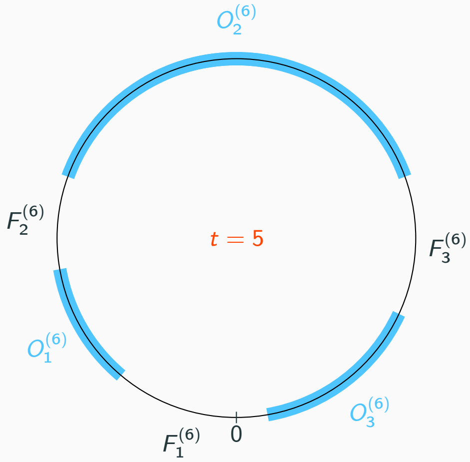

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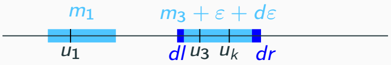
$t = k + \varepsilon + d\varepsilon$ $\quad$ $dl + dr = d\varepsilon$



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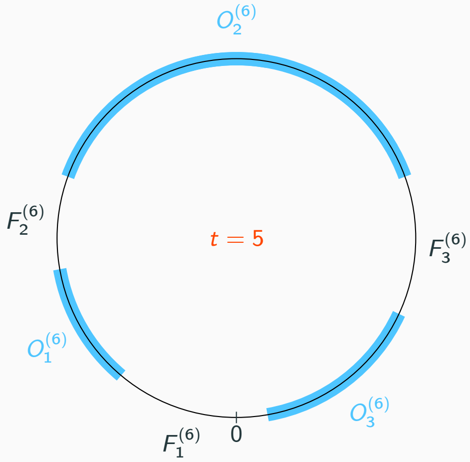

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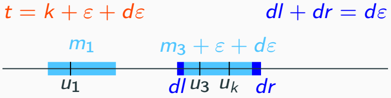

- invariance by translation of the process
- dl and dr only depend on what is inside the current component of u_k (one of the $O_i^{(k+\varepsilon)}$)

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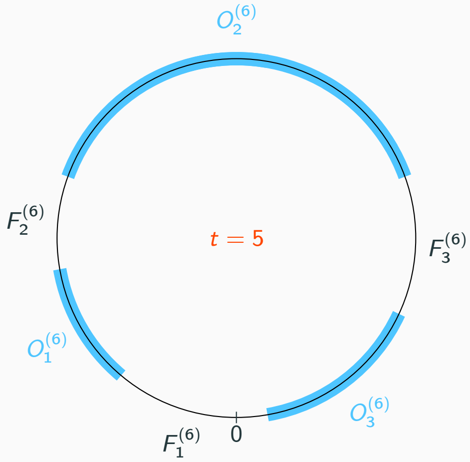

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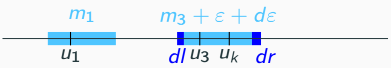

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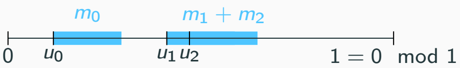
Examples:

- **Right diffusion at constant speed:** $\overrightarrow{O^{(k)}}, \overrightarrow{F^{(k)}}$



(Caravans, Bertoin, Miermont [BM06])

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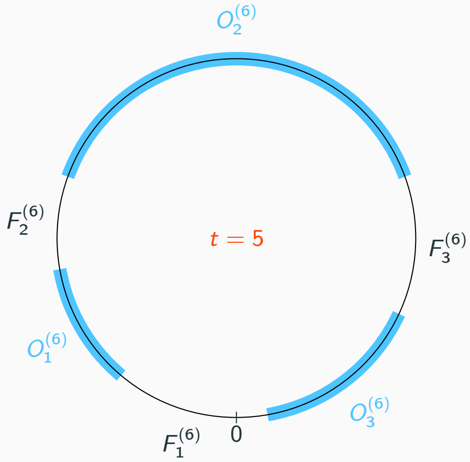

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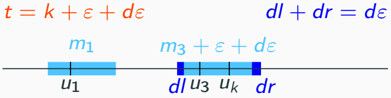

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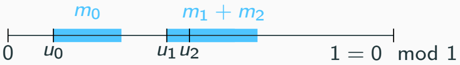

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- Diffusion to the closest side
- "short-sighted jam spreader"

A universality result

We consider $\frac{\sigma.|F^{(k)}|}{R} = \left(\frac{|F^{(k)}_{\sigma_i}|}{R}\right)_{1\leq i\leq N^{(k)}}$, for $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$. Let $R = 1 - \sum_{i=0}^{k-1} m_i$.

Theorem (Marckert, ZV)

Independently of the diffusion policy,

- ***Number of blocks:*** $N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$
- ***Lengths of the free blocks:*** $\frac{\boldsymbol{\sigma}.|F^{(k)}|}{R} \sim \text{Dirichlet}(N^{(k)}; 1, \ldots, 1)$
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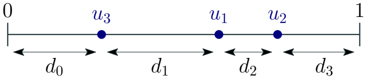
$u_1, u_2, u_3 \sim \mathcal{U}([0,1])$



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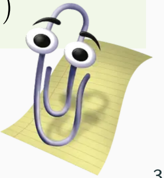

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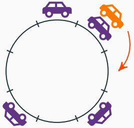

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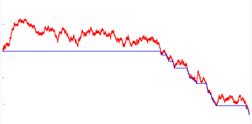

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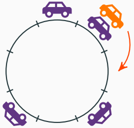

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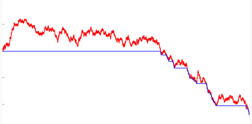

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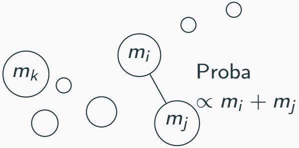

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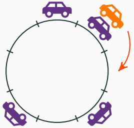

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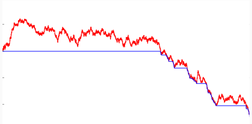

Generalized parking

- Parking on $\mathbb{Z}$ (Przykucki, Roberts, Scott [PRS23])
- Parking on (random) trees (Contat et. al.)
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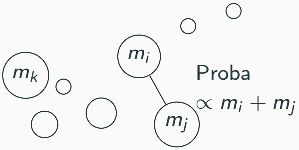

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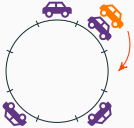

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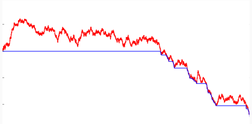

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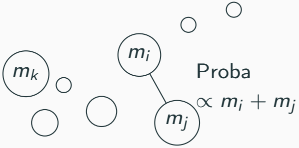

Continuous version of the classical parking

Caravans (Bertoin, Miermont [BM06])

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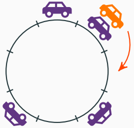

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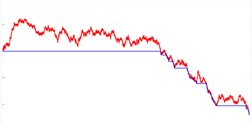

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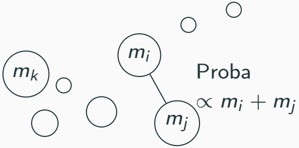

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Main principle

Consider 4 uniform points on $[0, 1]$.

Conditional on :

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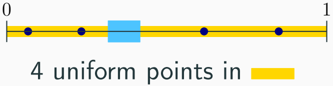

Main principle

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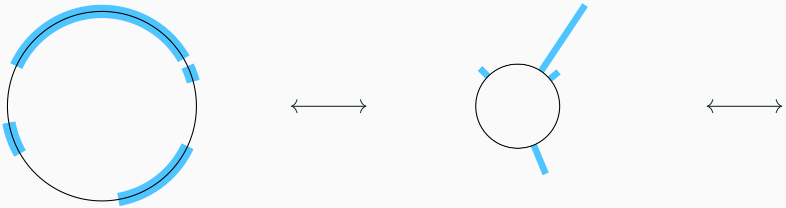
Intuition on the proof

Peaks representation:



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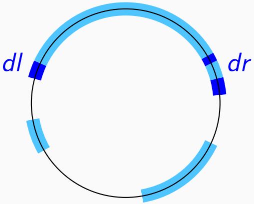

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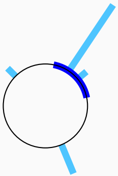

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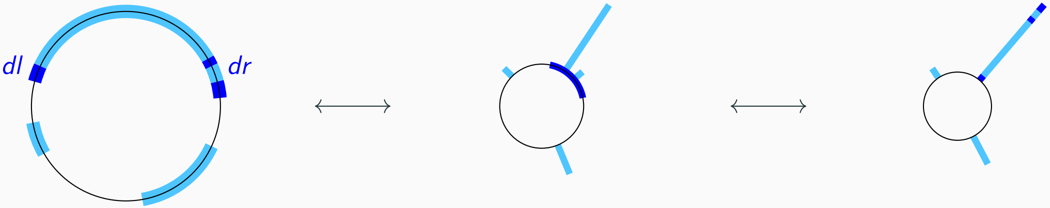

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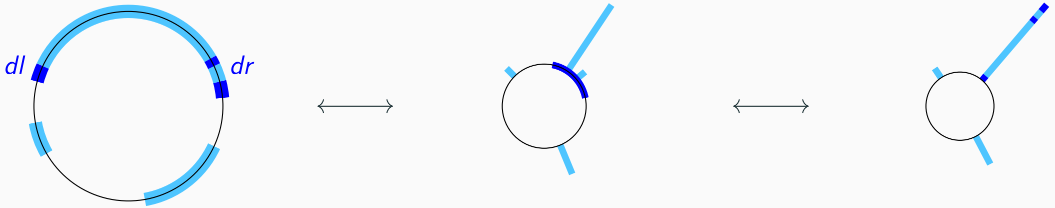

allow to see that, at any time:

- the positions of the peaks are uniform on the smaller cycle $\mathcal{C}_R$ of size $R = 1 - \sum m_i$
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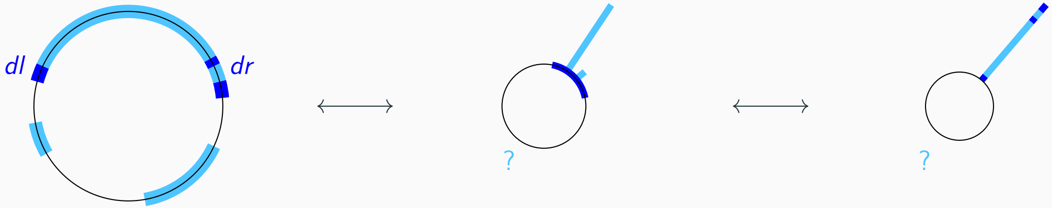

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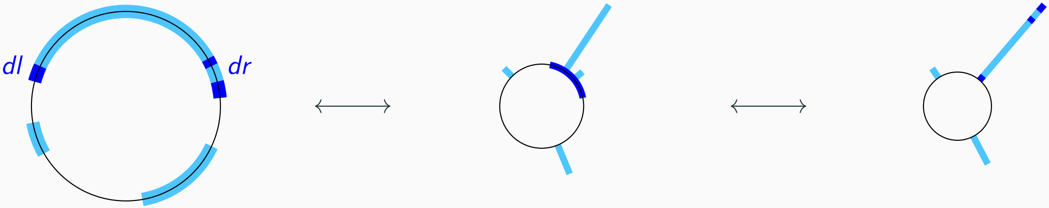

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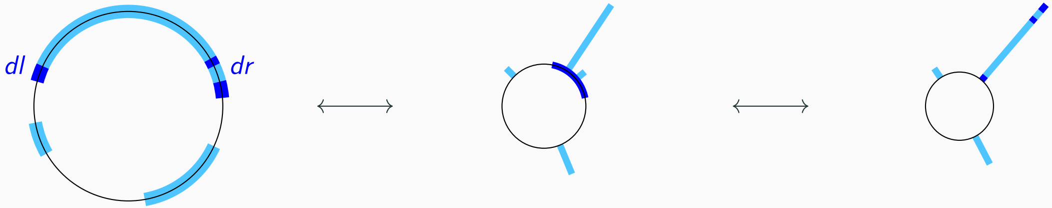

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- the positions of the peaks are uniform on the smaller cycle $\mathcal{C}_R$ of size $R = 1 - \sum m_i$
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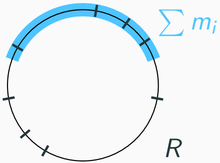
- even more surprisingly, for the peaks **number** and **positions**: do not depend on which peak is extended by the diffusion

Distribution of the number of blocks $N^{(k)}$

Theorem

$$\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_0, \cdots, m_{k-1})\right) = \mathcal{L}\left(|F^{(k)}| \;\middle|\; (\sum m_i, 0, \cdots, 0)\right)$$



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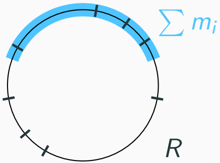
Theorem (Distribution of $N^{(k)}$)

Let $B(k-1, R) \sim \text{Binomial}(k-1, R)$*, then*

$$N^{(k)} \overset{(d)}{=} 1 + B(k-1, R)$$

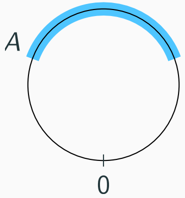


$$N^{(k)} = 1 + \sum_{j=1}^{k-1} \mathbb{1}_{u_k \notin \blacksquare}$$

Distribution of the occupied blocks

One block case:

$$\mathbb{P}\left(N^{(k)} = 1\right) = \left(\sum m_i\right)^{k-1} =: Q\left(\sum m_i, k\right)$$



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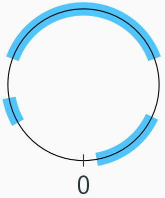

General case:

Theorem

$$\mathbb{P}\left(|O^{(k)}|=(M_0,\dots,M_{b-1})\right)=\mathrm{T}(M_0,\dots,M_{b-1})\sum_{P\in\mathcal{P}(k,b)}\left[\prod_{\ell=0}^{b-1}Q(M_j,|P_j|)\,\mathbb{1}_{\sum_{i\in P_\ell}m_i=M_\ell}\right]$$

where

- $\mathcal{P}(k,b)$ *is the set of partitions* $P=(P_0,\dots,P_{b-1})$ *of* $\{1,\dots,k-1\}$ *into* b *non empty parts,*

- $\mathrm{T}(M_0,\dots,M_{b-1})=M_0\frac{(1-\Sigma M_\ell)^{b-1}}{(b-1)!}+\frac{(1-\Sigma M_\ell)^b}{b!}$.

Summary of universality results

Theorem

For any continuous model with valid spreading policy, the following distributions are ***explicit*** *and* ***independent of the spreading policy****:*

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Corollary: results on $O^{(k)}, F^{(k)}$ for one spreading policy are valid for any spreading policy !

Asymptotic results

With n (random) masses, $n \to \infty$, for example

- $\forall i, m_i = 1/n$,
- $\forall i, m_i = \ell_i / n$ (where ℓ_i are i.i.d. with $\mathbb{E}[\ell_i] < \infty$ and $\mathbb{V}[\ell_i] < \infty$), before $t = \sup\{k : \sum_{i=0}^{k} m_i < 1\}$,

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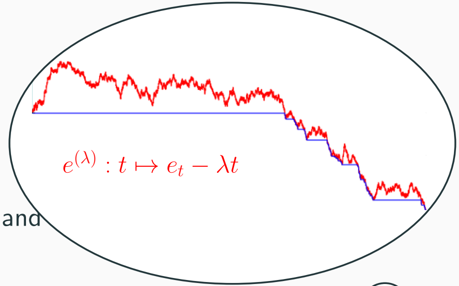
Corollary (Bertoin, Miermont [BM06]; Marckert, V.)

$$\left(\text{LargestBlock}^{(\text{i})}, i \geq 1\right) \xrightarrow[n\to\infty]{(d)} \left(\text{SortedExc}(e^{(\lambda)})_i, i \geq 1\right)$$



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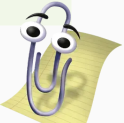
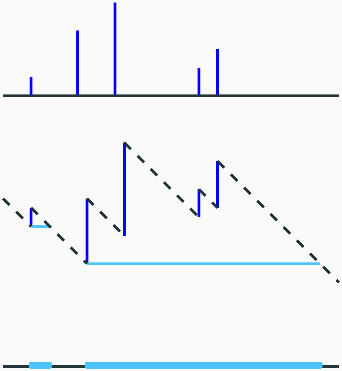
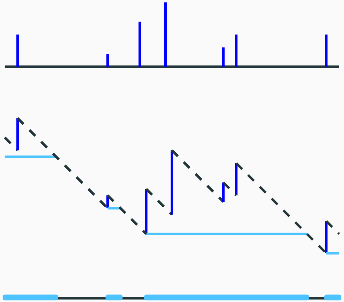


Illustration:



Key tool : the "collecting path"

Illustration:



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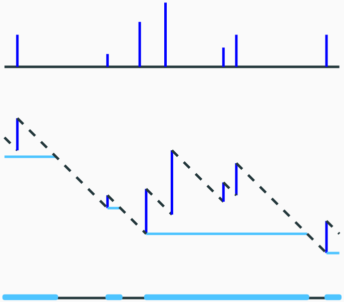

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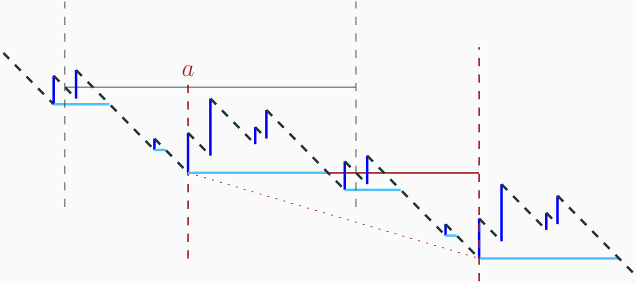

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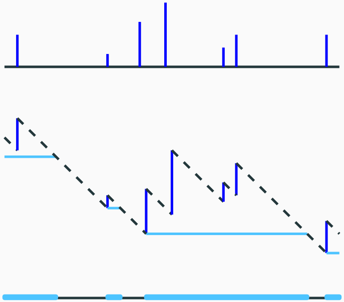

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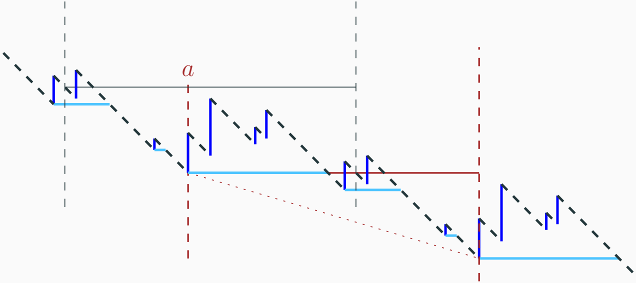
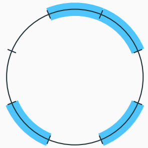


Convergence: when $\bar{S}$ converges to $e^{(\lambda)}$,

$$\left(\text{SortedExc}(\bar{S})_i\right)_{1\le i\le j} \xrightarrow[n\to\infty]{(d)} \left(\text{SortedExc}(e^{(\lambda)})_i\right)_{1\le i\le j}.$$

Other results

Discrete version of the process on $\mathcal{C}_n := \{0/n, \cdots, (n-1)/n\} \subset \mathcal{C}$



- Similar universality results

When $k = n - \lambda\sqrt{n}$ and $\forall i, m_i = 1/n$:

- Same asymptotics for the large blocks
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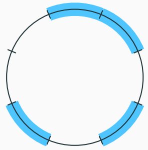

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Cost of parking procedures:

- Standard parking : if car k falls in a block of size B_k, it costs $C_k = \lfloor U.B_k \rfloor$, with $U \sim \mathcal{U}([0,1])$. Chassaing-Louchard [CL02]:

$$\frac{1}{n^{3/2}} \sum_{k=1}^{\lfloor n-\lambda\sqrt{n} \rfloor} C_k \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t^{(\lambda)} dt$$

- Parking with different parking policies : see Marckert-V. for some of them.

The end

💧 **Thank you !** 💧

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