Fast basecases for arbitrary-size multiplication

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Fast multiple-precision arithmetic – why?

We care about fast multiple-precision arithmetic. *Why?*

Examples include:

- Correctly rounded floating point arithmetic without the use of lookup tables (e.g. MPFR [1])
- Verifying the Riemann hypothesis up to very big numbers [2]
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Basics of Multiple-Precision Arithmetic

Fundamentals are these schoolbook $\mathcal{O}(n)$ operations:

- Left and right shift: $r \leftarrow \lfloor a \cdot 2^e \rfloor$
- Addition and subtraction: $r \leftarrow a \pm b$
- $m \times 1$-multiplication: $r \leftarrow a \cdot b_0$ (`mul_1`)
- Addition of $m \times 1$-multiplication: $r \leftarrow r + a \cdot b_0$ (`addmul_1`)

Schoolbook $m \times n$-multiplication is then

$r \leftarrow a \cdot b_0$ // `mul_1`
for $i \leftarrow 1$ *to* $n-1$ **do**
  $r \leftarrow r + (a \cdot b_i) \cdot \beta^i$ // `addmul_1`
end

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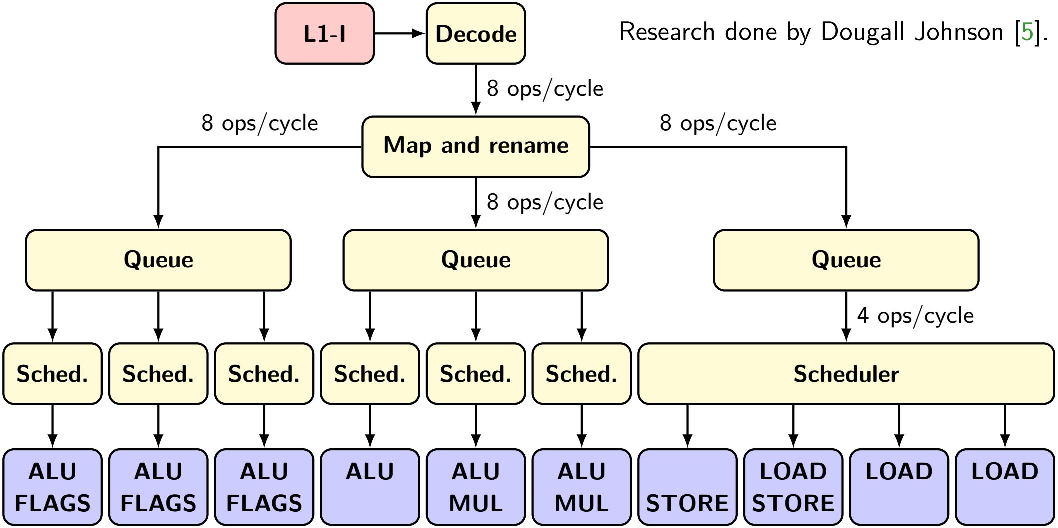
Apple M1 Pipeline (Simplified)

Research done by Dougall Johnson [5].



CPU Pipeline

Simple version:

1. Read some instructions from memory
2. Schedule the instructions to the correct unit
3. Units executes instructions

This scheme allows for:

- Concurrent execution of multiple instructions
- Out-of-order execution

But one has to be aware of dependency chains.

Example: (Dependency chain)
Consider the algorithm

$$\begin{aligned} x_0 &\leftarrow a_0\,, \\ x_1 &\leftarrow x_0 + a_1\,, \\ x_2 &\leftarrow x_1 + a_2\,. \end{aligned}$$

x_0 needs to be evaluated before x_1 can be computed. And x_1 needs to be evaluated before x_2 can be computed. This is called a *dependency chain*.

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Lower bound of GMP's addmul_1

```
L(top): ldp     u0, u1, [up], #16        adds    x0, r0, x0
        ldp     u2, u3, [up], #16        adcs    u0, r1, u0
        ldp     r0, r1, [rp]             adcs    u1, r2, u1
        ldp     r2, r3, [rp,#16]         adcs    u2, r3, u2
        mul     x0, u0, v0               adc     u3, u3, zero
        umulh   u0, u0, v0               adds    x0, x0, CY
        mul     x1, u1, v0               adcs    u0, u0, x1
        umulh   u1, u1, v0               adcs    u1, u1, x2
        mul     x2, u2, v0               adcs    u2, u2, x3
        umulh   u2, u2, v0               adc     CY, u3, zero
        mul     x3, u3, v0               stp     x0, u0, [rp], #16
        umulh   u3, u3, v0               stp     u1, u2, [rp], #16
                                         sub     n, n, #1
                                         cbnz    n, L(top)
```

Unit type (amount)	Cycles / 4 words
LOAD/STORE (3/2)	
MUL (2)	
ALU+FLAGS (3)	

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ALU+FLAGS (3)	~~4~~ 5

5 cycles per 4 words?

Benchmarks says yes!

What can be improved?

GMP's `addmul_1` will do $\frac{k+1}{k}$ cycles per word *asymptotically* on Apple M1, where k is the number of unrolls.

To improve this, we fully unroll one size parameter in the full multiplication:

- Reduces overhead,
- Avoids breaking carry chains, hopefully lets

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On x86 CPUs (Intel and AMD), we completely unroll both size parameters.

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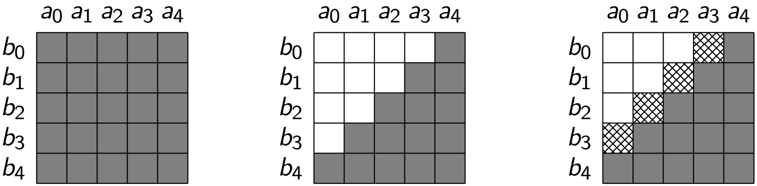
High multiplication

High multiplication is multiplication where we scrap the lower part of the result.
Important use cases are floating point arithmetic and modular arithmetic.



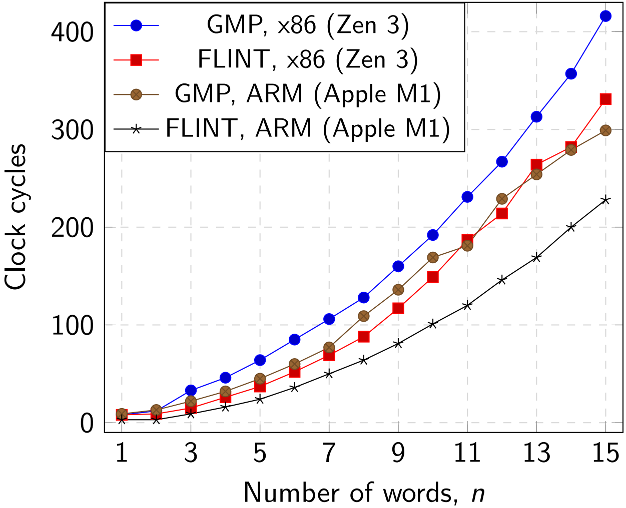
Full multiplication | Sloppy approximate | Precise approximate

– scrapped

– high multiplication between two words u and v: $\lfloor uv/\beta \rfloor$

– full multiplication

Results, full multiplication (throughput)



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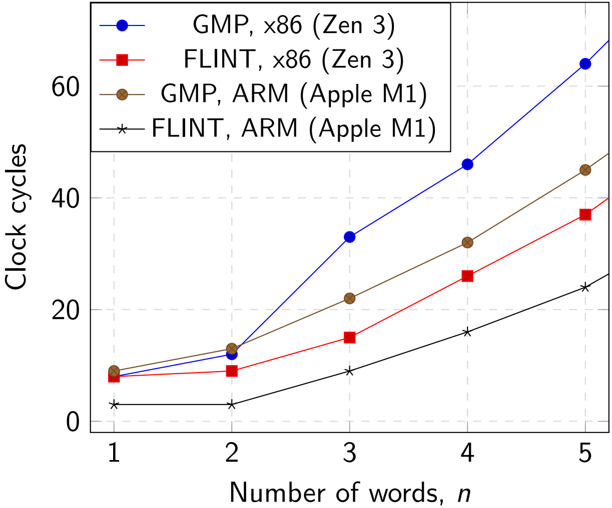

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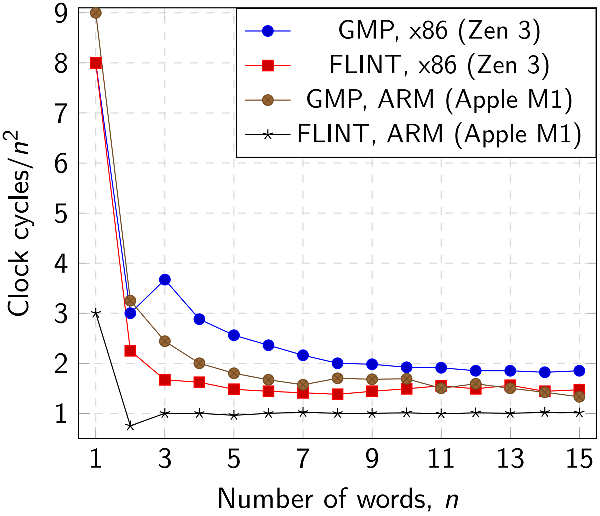
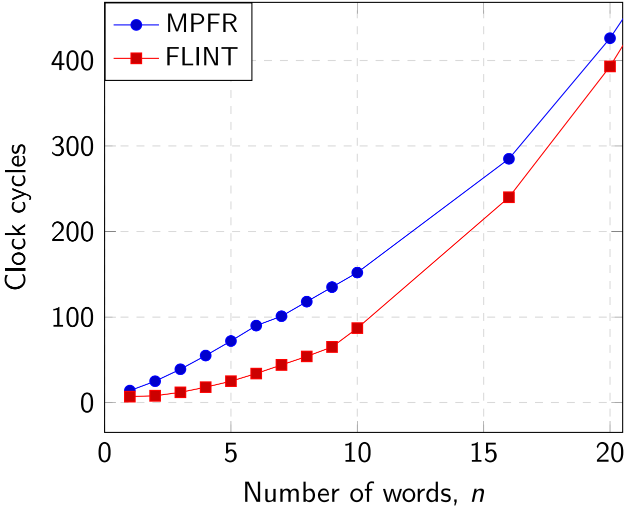


10^7 multiplications with lengths $m, n \in \{1, 2, \ldots, N\}$, uniformly random

	GMP (`mpn_mul`)				Ours (`flint_mpn_mul`)			
N	Time	C	J	I	Time	C	J	I
Random, x86-64 (Zen 3)								
8	0.32 s	18.3%	22.3%	0%	0.18 s	20.9%	48.4%	0.00%
16	0.55 s	10.0%	18.2%	0%	0.43 s	14.3%	33.1%	3.05%
32	1.39 s	10.5%	12.7%	0%	1.32 s	10.7%	16.7%	0.41%
64	4.48 s	11.5%	11.6%	0%	4.29 s	12.9%	14.3%	0.12%
Random, ARM64 (M1)								
8	0.30 s	11.4%	0.00%	0.00%	0.23 s	11.2%	41.7%	0.01%
16	0.50 s	10.9%	0.00%	0.00%	0.43 s	10.5%	41.6%	0.00%
32	1.31 s	9.6%	0.00%	0.00%	1.13 s	10.0%	13.9%	0.00%
64	4.16 s	8.3%	0.20%	0.02%	3.82 s	9.8%	4.2%	0.06%

Table: Conditional branch misprediction rates “C”, indirect jump address misprediction rates “J” and instruction cache miss rates “I”.

Results, high multiplication on Zen 3 (throughput)



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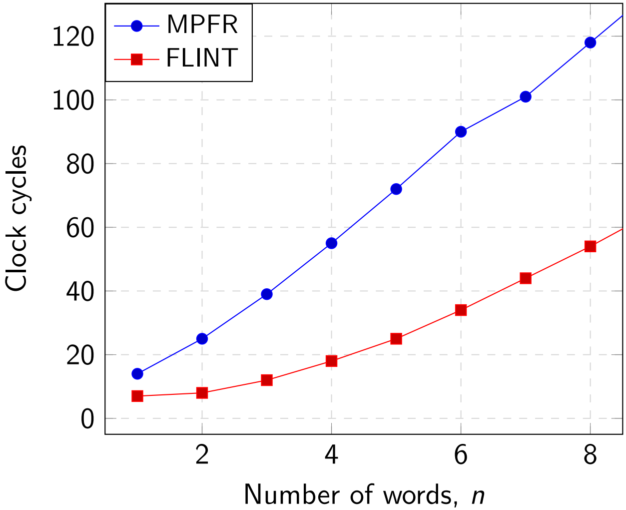

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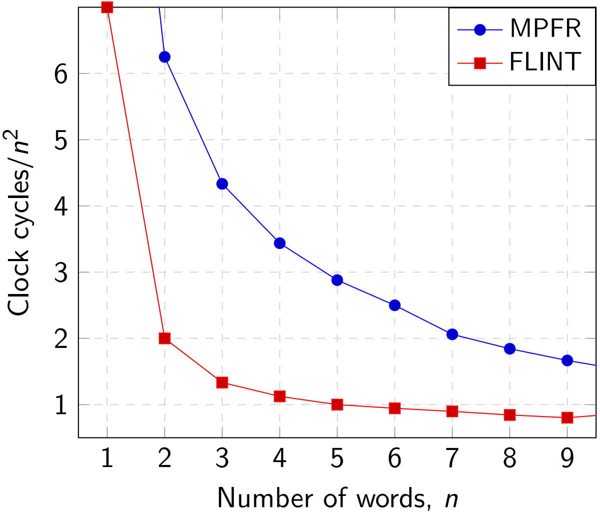
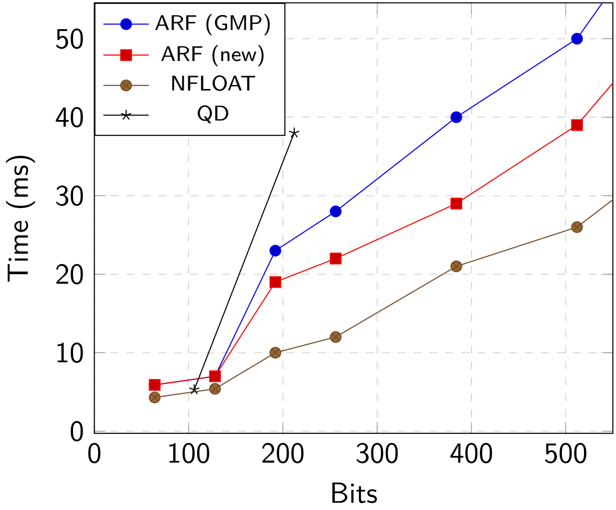


Multiply two 100×100 FP matrices using dot products (Zen 3)



Conclusions and thoughts

- Critical functions require hardware awareness!
- Straight line programs (SLPs) can be important to reduce overhead when going from native data types to multiple precision arithmetic
- Handwritten assembly still remain critical for multiple precision arithmetic

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Bibliography

[1] Laurent Fousse et al. "MPFR: A multiple-precision binary floating-point library with correct rounding". In: *ACM Trans. Math. Softw.* 33.2 (June 2007), 13–es. ISSN: 0098-3500. DOI: 10.1145/1236463.1236468.

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