Effective bounds for polynomial systems defined over the rationals

(Part II)

Dedicated to my advisor
Joos Heintz
27 Oct. 1945 - 3 Oct. 2024

Teresa Krick
Universidad de Buenos Aires & CONICET

JNCF 2025

An arithmetic Shape Lemma (Baldi-K-Mourrain'24)

- $I = (f_1, \ldots, f_s) \subset \mathbb{Q}[\boldsymbol{x}]$ **radical zero-dimensional** ideal
- with $f_1, \ldots, f_s \in \mathbb{Z}[\boldsymbol{x}]$ and $D := \deg(V_{\mathbb{C}}(I))$
- $d_j := \deg(f_j)$ s.t. $d := d_1 \geq d_2 \geq \cdots \geq d_{s-1}$
- $h_j := \mathrm{h}(f_j)$ s.t. $h := \max\{h_1, \ldots, h_{s-1}\}$

Then, there exists an isomorphism of algebras

$$\begin{aligned} \varphi : \ \mathbb{Q}[\boldsymbol{x}]/I \quad &\xrightarrow{\simeq} \quad \mathbb{Q}[t]/(\omega_0) \\ x_i \quad &\mapsto \quad \omega_i(t)/\omega_0'(t) \quad \bmod \ \omega_0 \quad \text{for } 1 \leq i \leq n \end{aligned}$$

where $\omega_0, \omega_1, \ldots, \omega_n \in \mathbb{Z}[t]$ satisfy

- ω_0 squarefree
- $\deg(\omega_0) = D, \deg(\omega_i) < D$
- $\mathrm{h}(\omega_i) \leq d^{n-1} h_s + (n-1) d^{n-2} d_s h + 2n \log(n+1) d^{n-1} d_s + 4D \log\big((n+1)D\big)$

Height of the remainder (Baldi-K-Mourrain'24)

- $I = (f_1, \dots, f_s) \subset \mathbb{Q}[\boldsymbol{x}]$ **radical zero-dimensional** ideal
- with $f_1, \dots, f_s \in \mathbb{Z}[\boldsymbol{x}]$ and $\delta := \deg(B)$, B basis of $\mathbb{Q}[\boldsymbol{x}]/I$
- $d := \max_j\{\deg(f_j)\}$ and $h := \max_j\{\mathrm{h}(f_j)\}$
- $p \in \mathbb{Z}[\boldsymbol{x}]$ with $d_p := \deg(p)$, $h_p := \mathrm{h}(p)$

Then, there exist $a \in \mathbb{Z} \setminus \{0\}$ and $N(p) \in \mathbb{Z}[\boldsymbol{x}]$ such that

- $\overline{p} = N(p)/a$
- $\mathrm{h}(a) \leq nd^{2n-1}\delta h + 5n\log((n+2)d)d^{2n}\delta$
- $$\mathrm{h}(N(p)) \leq h_p + nd^{n-1}(d_p + d^n\delta)h + 5n\log((n+2)d)d^n(d_p + d^n\delta)$$

Effective NSS (Jelonek'05)

- $f_1, \ldots, f_{n+1} \in \mathbb{C}[\boldsymbol{x}]$ s.t. $V_{\mathbb{C}}(f_1, \ldots, f_{n+1}) = \emptyset$
- (unknown) g_1, g_2 s.t. $1 = g_1 f_1 + \cdots + g_{n+1} f_{n+1}$
- Morphism of algebras

$$\begin{array}{rcll} \Phi : \; R := \mathbb{C}[\boldsymbol{x}, z_1, \ldots, z_{n+1}] & \longrightarrow & \mathbb{C}[\boldsymbol{x}, z] & \\ x_i & \mapsto & x_i & \text{for } 1 \le i \le n \\ z_j & \mapsto & z\, f_j(\boldsymbol{x}) & \text{for } 1 \le j \le n+1 \end{array}$$

Effective NSS (Jelonek'05)

- $f_1, \dots, f_{n+1} \in \mathbb{C}[\boldsymbol{x}]$ s.t. $V_{\mathbb{C}}(f_1, \dots, f_{n+1}) = \emptyset$
- (unknown) g_1, g_2 s.t. $1 = g_1 f_1 + \cdots + g_{n+1} f_{n+1}$
- Morphism of algebras

$$\begin{array}{rclll} \Phi : \ R := \mathbb{C}[\boldsymbol{x}, z_1, \dots, z_{n+1}] & \longrightarrow & \mathbb{C}[\boldsymbol{x}, z] & \\ x_i & \mapsto & x_i & \text{for } 1 \leq i \leq n \\ z_j & \mapsto & z\, f_j(\boldsymbol{x}) & \text{for } 1 \leq j \leq n+1 \end{array}$$

$$\Phi\big(g_1(\boldsymbol{x}) z_1 + \cdots + g_{n+1}(\boldsymbol{x}) z_{n+1}\big) = z$$

Effective NSS (Jelonek'05)

- $f_1, \ldots, f_{n+1} \in \mathbb{C}[\boldsymbol{x}]$ s.t. $V_{\mathbb{C}}(f_1, \ldots, f_{n+1}) = \emptyset$
- (unknown) g_1, g_2 s.t. $1 = g_1 f_1 + \cdots + g_{n+1} f_{n+1}$
- Morphism of algebras

$$\begin{array}{rclll} \Phi : \ R := \mathbb{C}[\boldsymbol{x}, z_1, \ldots, z_{n+1}] & \longrightarrow & \mathbb{C}[\boldsymbol{x}, z] & \\ x_i & \mapsto & x_i & \text{for } 1 \leq i \leq n \\ z_j & \mapsto & z\, f_j(\boldsymbol{x}) & \text{for } 1 \leq j \leq n+1 \end{array}$$

$\Phi\big(g_1(\boldsymbol{x}) z_1 + \cdots + g_{n+1}(\boldsymbol{x}) z_{n+1}\big) = z \ \Rightarrow \ \Phi$ epi

Effective NSS (Jelonek'05)

- $f_1, \ldots, f_{n+1} \in \mathbb{C}[\boldsymbol{x}]$ s.t. $V_{\mathbb{C}}(f_1, \ldots, f_{n+1}) = \emptyset$
- (unknown) g_1, g_2 s.t. $1 = g_1 f_1 + \cdots + g_{n+1} f_{n+1}$
- Morphism of algebras

$$\begin{array}{rcll} \Phi : \; R := \mathbb{C}[\boldsymbol{x}, z_1, \ldots, z_{n+1}] & \longrightarrow & \mathbb{C}[\boldsymbol{x}, z] & \\ x_i & \mapsto & x_i & \text{for } 1 \leq i \leq n \\ z_j & \mapsto & z\, f_j(\boldsymbol{x}) & \text{for } 1 \leq j \leq n+1 \end{array}$$

$$\Phi\big(g_1(\boldsymbol{x})z_1 + \cdots + g_{n+1}(\boldsymbol{x})z_{n+1}\big) = z \;\Rightarrow\; \Phi \text{ epi} \;\Rightarrow\; R/\ker(\Phi) \simeq \mathbb{C}[\boldsymbol{x}, z]$$

An arithmetic Perron theorem

(D'Andrea-K-Sombra'13)

- $f_1, \dots, f_{n+1} \in \mathbb{Z}[\boldsymbol{x}]$
- $d := \max_j \{\deg(f_j)\}$ and $h := \max_j \{\mathrm{h}(f_j)\}$

Then, there exists a polynomial

$$P \in \mathbb{Z}[y_1, \dots, y_{n+1}] \setminus \{0\}$$

such that

- $P(f_1, \dots, f_{n+1}) = 0$
- $\deg(P) \leq d^{n+1}$
- $\mathrm{h}(P) \leq (n+1) d^n h + (n+2) \log(2n+8) d^{n+1}$

MERCI !!!!!!