Effective bounds for polynomial systems defined over the rationals

Dedicated to my advisor
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The Bézout inequality

The Bézout inequality (Heintz'1983)

- $f_1, \dots, f_s \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \dots, x_n]$ with

$$V := V_{\mathbb{C}}(f_1, \dots, f_s) \neq \emptyset \quad \textbf{finite}$$

- $d_j := \deg(f_j)$ s.t. $d := d_1 \geq d_2 \geq \cdots \geq d_{s-1}$

Then

$$\deg(V) \leq d_1 \cdots d_{n-1} d_s \leq d^{n-1} d_s$$

Height of a finite variety (Philippon'1995)

- $V = \{\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \,:\, \boldsymbol{\zeta} \in V\} \neq \emptyset$ **finite**
- $\boldsymbol{U} = (U_0, U_1, \ldots, U_n)$ new variables

$$U_0 + \zeta_1 U_1 + \cdots + \zeta_n U_n$$

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$$\prod_{\boldsymbol{\zeta} \in V} \left(U_0 + \zeta_1 U_1 + \cdots + \zeta_n U_n\right) \in \mathbb{Q}[\boldsymbol{U}]$$

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$$\mathrm{Ch}_V(\boldsymbol{U}) := c \prod_{\boldsymbol{\zeta} \in V} \big(U_0 + \zeta_1 U_1 + \cdots + \zeta_n U_n\big) \; \in \; \mathbb{Q}[\boldsymbol{U}]$$

with $c \in \mathbb{Z} \setminus \{0\}$ s.t. Ch_V is a *primitive* polynomial in $\mathbb{Z}[\boldsymbol{U}]$

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Then, $\mathrm{h}(V)$, defined through a Mahler measure of Ch_V, satisfies

- $\displaystyle\sum_{\zeta \in V} \log(\|(1, \boldsymbol{\zeta})\|_2) \leq \mathrm{h}(V)$
- $\left|\mathrm{h}(V) - \mathrm{h}(\mathrm{Ch}_V)\right| \leq 3 \log(n+1) \deg(V)$

An arithmetic Bézout inequality (K-Pardo-Sombra'01)

- $f_1, \dots, f_s \in \mathbb{Z}[\boldsymbol{x}]$ with $V \neq \emptyset$ **finite**
- $d_j := \deg(f_j)$ s.t. $d := d_1 \geq d_2 \geq \cdots \geq d_{s-1}$
- $h_j := \mathrm{h}(f_j)$ s.t. $h := \max\{h_1, \dots, h_{s-1}\}$

Then

$$\mathrm{h}(V) \leq \Big(\frac{h_s}{d_s} + (n-1)\frac{h}{d_{n-1}} + 2n\log(n+1)\Big) d_1 \cdots d_{n-1} d_s$$

$$\leq d^{n-1} h_s + (n-1) d^{n-2} d_s h + 2n\log(n+1) d^{n-1} d_s$$

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In particular, for any $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in V$,

$$\log(|\zeta_i|) \leq d^{n-1} h_s + (n-1) d^{n-2} d_s h + 2n \log(n+1) d^{n-1} d_s$$

The Nullstellensatz

- $I \subset \mathbb{C}[\mathbf{x}]$ ideal

Then

- *Weak NSS:*

$$V_{\mathbb{C}}(I) = \emptyset \quad \iff \quad 1 \in I$$

- *Strong NSS:*

$$I(V_{\mathbb{C}}(I)) = \sqrt{I}$$

where $\sqrt{I} := \{f \in \mathbb{C}[\mathbf{x}] \, : \, \exists\, N \in \mathbb{N} \text{ s.t. } f^N \in I\}$
and given $X \subset \mathbb{C}^n$, $I(X) := \{f \in \mathbb{C}[\mathbf{x}] \, : \, f(\zeta) = 0, \ \forall\, \zeta \in X\}$

An arithmetic Nullstellensatz (D'Andrea-K-Sombra'13)

- $f_1, \dots, f_s \in \mathbb{Z}[\boldsymbol{x}]$ with $V = \emptyset$
- $d_j := \deg(f_j)$ s.t. $d := d_1 \geq d_2 \geq \cdots \geq d_{s-1}$
- $h_j := \mathrm{h}(f_j)$ s.t. $h := \max\{h_1, \dots, h_{s-1}\}$

Then there exist $a \in \mathbb{N}$ and $g_1, \dots, g_s \in \mathbb{Z}[\boldsymbol{x}]$ s.t.

$$a = g_1 f_1 + \cdots + g_s f_s$$

with

- $\deg(g_i f_i) \leq d_1 \cdots d_r d_s$
- $\mathrm{h}(a), \mathrm{h}(g_i) + \mathrm{h}(f_i) \leq d^r h_s + r d^{r-1} d_s h + 15\, n\big(\log(n+3) + \log\max\{1, s-n\}\big) d^r d_s$

where $r = \min\{s-1, n\}$