Orders and shuffles on nestohedra

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based on "Tridendriform algebras on hypergraph polytopes" published in Algebraic combinatorics (2025)

Research School "Beyond Permutahedra and Associahedra"
CIRM, 1-5 December 2025

Outline

1. Shuffles of packed words and planar trees
2. Hypergraph polytopes (a.k.a. nestoedra)
3. Algebraic structures on faces of nestohedra

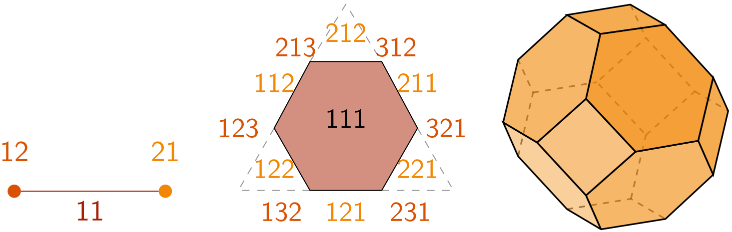
Shuffles of packed words and planar trees

Outline

Permutohedra [Schoute 1911]

vertices=permutation of $[\![1; n]\!]$

k-dimensional face $=$ a surjection from $[\![1; n]\!]$ to $[\![1; k]\!]$ (also know as packed word)



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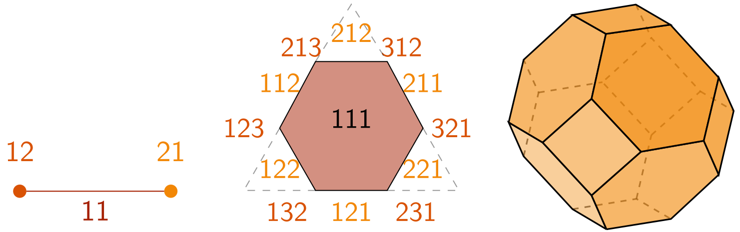


In this talk

A permutation $\pi \in \mathfrak{S}_n$ is identified with the word $\pi(1)\ldots\pi(n)$.

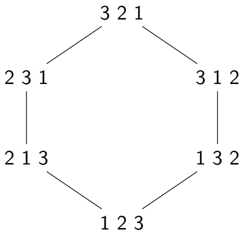
Weak Bruhat order [Verma 1968]

1234

Covering relations :

$\ldots ab \ldots \lhd \ldots ba \ldots$

with $a < b$



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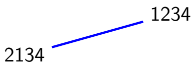

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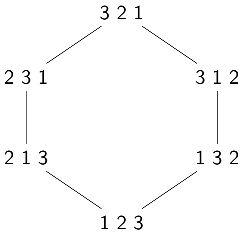

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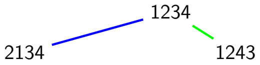

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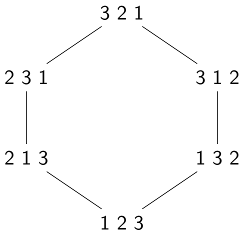

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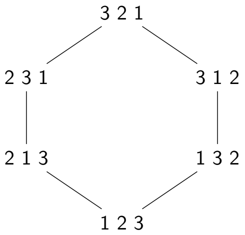

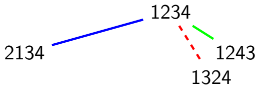

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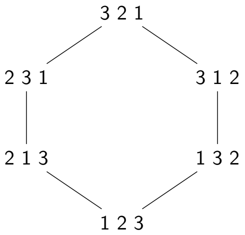

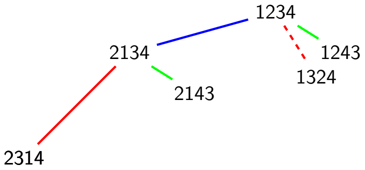

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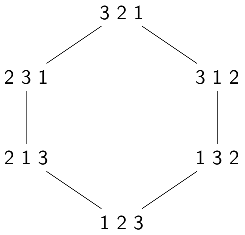

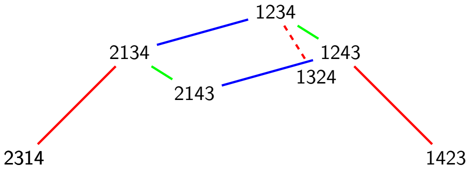

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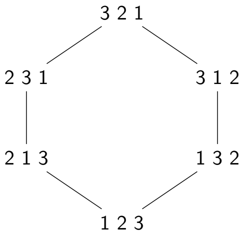
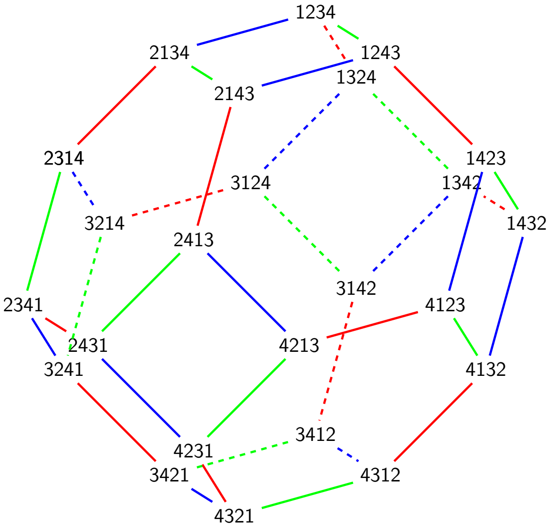




Shuffles of permutations

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ its set of (possibly empty) words.
For any $a, b \in \mathcal{A}$ and $m, n \in \mathcal{A}^*$ the shuffle product is:

$$a.m \sqcup\!\sqcup\, b.n = a.(m \sqcup\!\sqcup\, b.n) + b.(a.m \sqcup\!\sqcup\, n),$$

with $\varepsilon \sqcup\!\sqcup\, m = m \sqcup\!\sqcup\, \varepsilon = m$, where ε is the empty word.

For $\sigma \in \mathfrak{S}_q$, we denote by $\sigma + p$ the permutation of $\mathfrak{S}_{p+1,\dots,p+q}$ satisfying
$(\sigma + p)(i) = \sigma(i - p) + p$
Given two permutations $\pi \in \mathfrak{S}_p$ and $\sigma \in \mathfrak{S}_q$, Malvenuto-Reutenauer define the (horizontal) shuffle of permutations as:

$$\pi \sqcup\!\sqcup_h \sigma = \pi \sqcup\!\sqcup\, (\sigma + p)$$

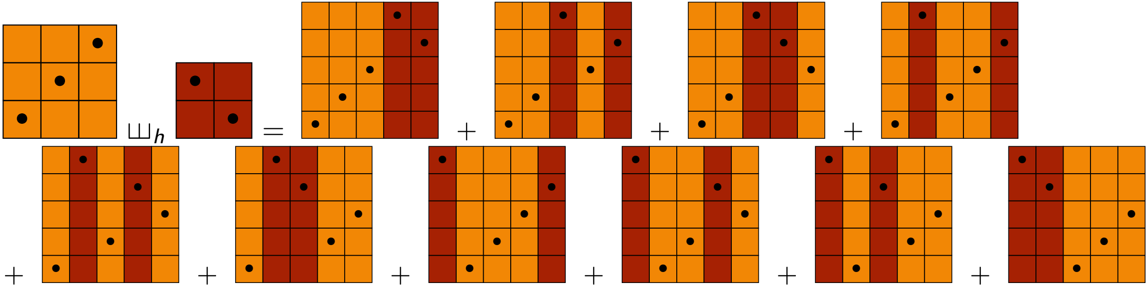
Example:

$123 \sqcup\!\sqcup_h 21 = 12354 + 12534 + 12543 + 15234 + 15243 + 15423 + 51234 + 51243 + 51423 + 54123$

Horizontal shuffle

Example:

$123 \sqcup\!\sqcup_h 21 = 12354 + 12534 + 12543 + 15234 + 15243 + 15423 + 51234 + 51243 + 51423 + 54123$



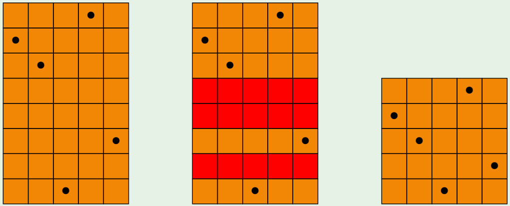
Malvenuto-Reutenauer Hopf algebra [$\mathfrak{S}Sym$: Malvenuto-Reutenauer 1995, "FQSym" : Duchamp-Hivert-Thibon, 2002]

$$\Delta(\sigma) = \sum_{p=0}^{n} \operatorname{std}\left(\sigma(1)\ldots\sigma(p)\right) \otimes \operatorname{std}\left(\sigma(p+1)\ldots\sigma(n)\right),$$

where std is the standardisation, i.e. $\operatorname{std}(t) = \phi \circ t$ with ϕ the unique increasing bijection from $\Im(t)$ to $[\![|\Im(t)|]\!]$.

Example:

$$\operatorname{std}(76183) = 43152$$



Example:

$\Delta(132) = \varepsilon \otimes 132 + 1 \otimes 21 + 12 \otimes 1 + 132 \otimes \varepsilon.$

Proposition

$\bigotimes_{n\geqslant 0} \mathbb{C}\mathfrak{S}_n$ endowed with the product $⧢_h$ and the coproduct Δ is a Hopf algebra, i.e. Δ is an algebra morphism.

Link between the shuffle of permutations and weak Bruhat order

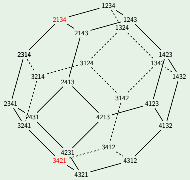
Proposition (Loday-Ronco 02)

There exists two (duplicial) products on permutations $\lhd$ and $\rhd$ s.t. for any permutations $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$:

$$\sigma \sqcup\!\sqcup_h \tau = \sum_{\sigma \lhd \tau \leqslant p \leqslant \sigma \rhd \tau} p,$$

where $\leqslant$ is the weak Bruhat order and $\sigma \lhd \tau = \sigma.(\tau + p)$ and $\sigma \rhd \tau = (\tau + p).\sigma$.

Example:



$$21 \sqcup\!\sqcup 12 = \sum_{2134 \leqslant p \leqslant 3421} p$$

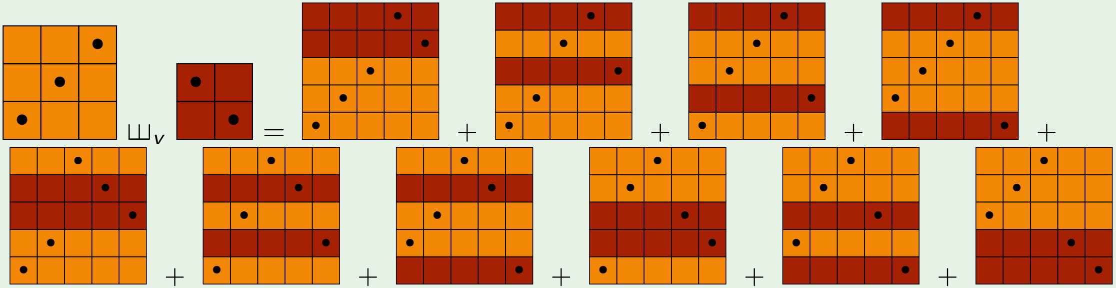
$$= 2134 + 2314 + 2341 + 3214 + 3241 + 3421$$

Vertical shuffle

$$\sigma \shuffle_v \tau = \sum_{\substack{\mathrm{std}(s)=\sigma \\ \mathrm{std}(t)=\tau}} s.t,$$

Example:

$123 \shuffle_v 21 = 12354 + 12453 + 13452 + 23451 + 12543 + 13542 + 23541 + 14532 + 24531 + 34521$



Shuffles of packed words/surjections [Perm : Chapoton 2000, WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005]

Definition

A packed word is a word w on $\mathbb{N}$ such that the set of letters in w is an interval $[\![1;k]\!]$. Equivalently, it is a surjection.

$$u \sqcup\!\sqcup\, v = \sum_{\substack{\mathrm{pack}(\alpha)=u \\ \mathrm{pack}(\beta)=v}} \alpha\beta,$$

Example :

$$11 \sqcup\!\sqcup\, 221 = 22221 + 33221 + 22331 + 11221 + 11332$$

Shuffles of packed words/surjections [Perm : Chapoton 2000, WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005]

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In fact, more than a shuffle product : tridendriform products (WQSym free tridendriform algebra on infinitely many generators [Vong, Burgunder-Curien-Ronco, 2015])

Hopf algebra of packed words/surjections [WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005, Perm : Chapoton 2000]

Given a packed word u, define:

$$\Delta(u) = \sum_{k=0}^{n} u_{|[\![1;k]\!]} \otimes u_{|[\![k+1;l]\!]}$$

Example:

$\Delta(11332) = \varepsilon \otimes 11332 + 11 \otimes 221 + 112 \otimes 11 + 11332 \otimes \varepsilon$

Link between the shuffle of packed words and an order on them [Palacios-Ronco 2006]

Palacios and Ronco introduced the following order for $A_i < A_{i+1}$ (i.e. for all $x \in A_i$ and $y \in A_{i+1}$, $x < y$):

$$(A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_n) \leqslant (A_1, \ldots, A_{i-1}, A_i \cup A_{i+1}, \ldots, A_n)$$
$$...i...i...(i+1)...(i+1)... \leqslant ...i...i...i...i...$$
$$(A_1, \ldots, A_{i-1}, A_i \cup A_{i+1}, \ldots, A_n) \leqslant (A_1, \ldots, A_{i-1}, A_{i+1}, A_i, \ldots, A_n)$$
$$...i...i...i...i... \leqslant ...(i+1)...(i+1)...i...i...$$

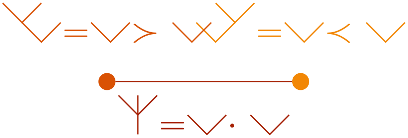
Proposition (Palacios-Ronco 02)

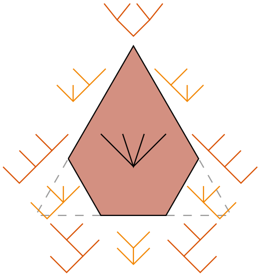
The shuffle product of packed words is the sum of element in an interval of this order.

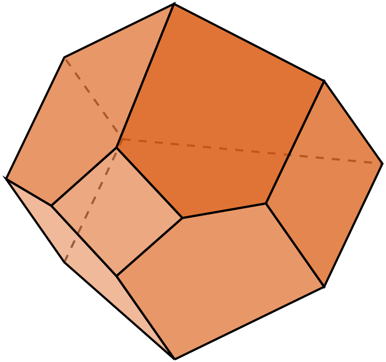
Another polytope in the title of the conference: Associahedra

vertices = planar binary trees

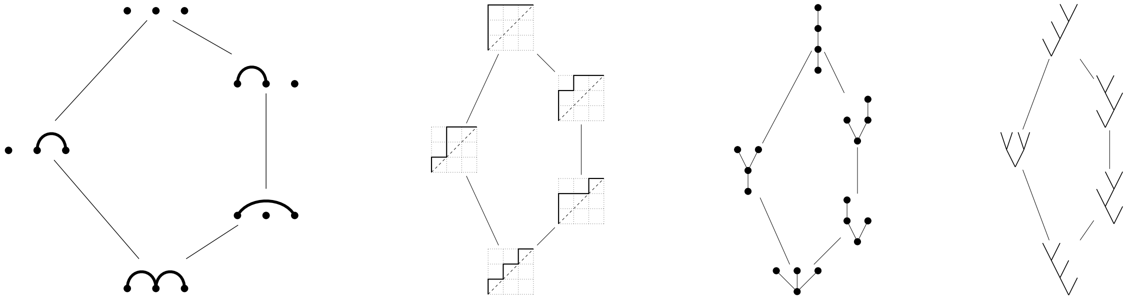
k-dimensional face = planar trees on $n - k$ nodes







Tamari order

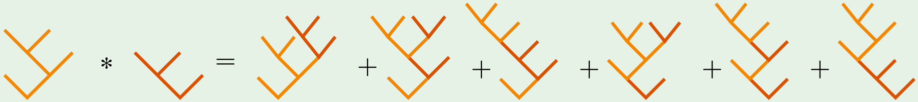


Shuffles of planar binary trees [Loday-Ronco 1998]

Denoted by $T = T_l \vee T_r$ a planar binary tree, Loday-Ronco introduced the following product in 1998:

$$T * S = T_l \vee (T_r * S) + (T * S_l) \vee S_r .$$

Example

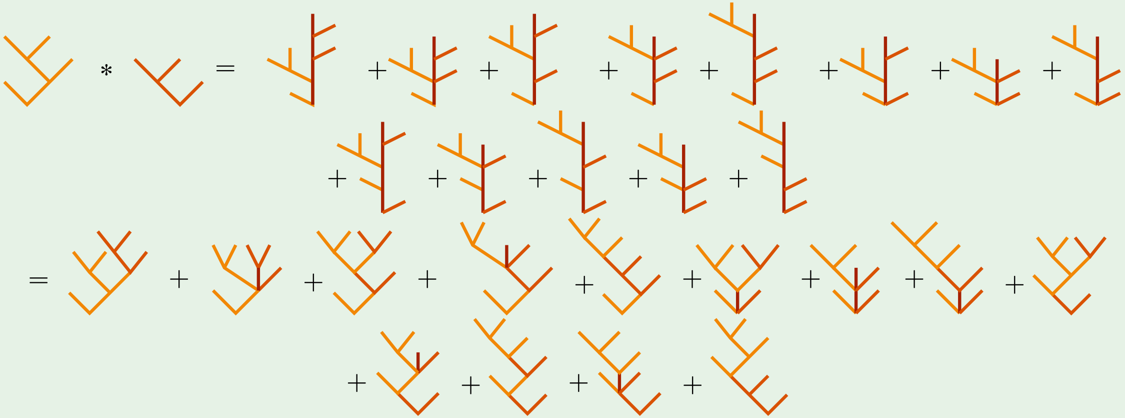


Proposition (Loday-Ronco 2002)

The shuffle product of two planar binary trees S and T can be expressed as a sum of elements in an interval whose bounds are given by some (duplicial) operations on S and T.

Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example



Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

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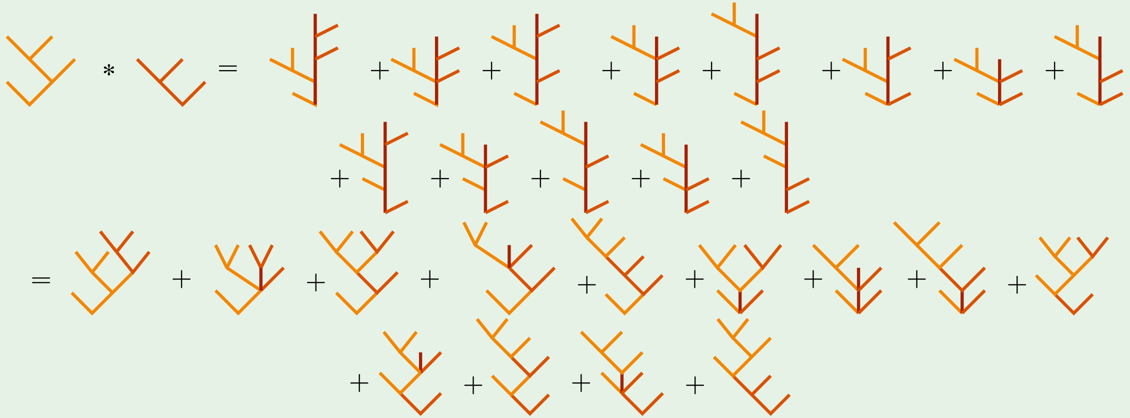
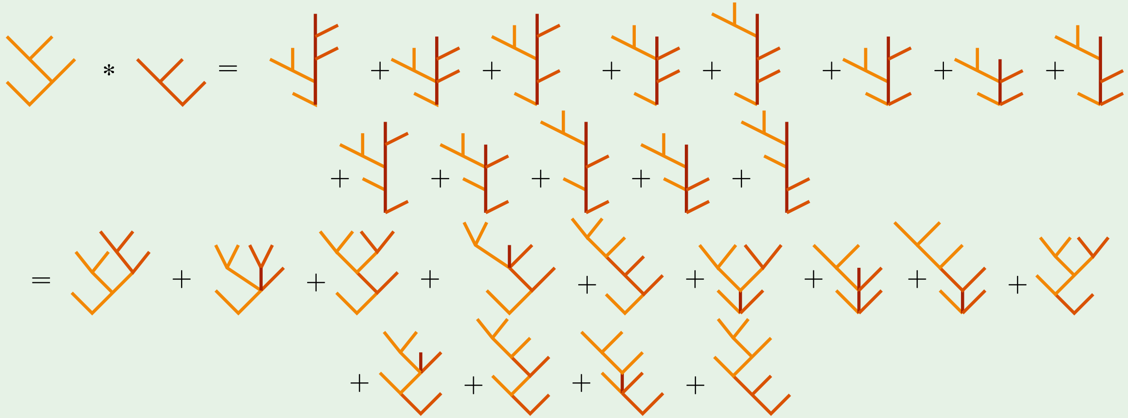


Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example



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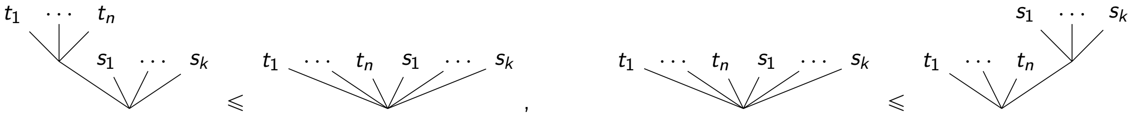
Link between the shuffle of planar trees and generalised Tamari order

Palacios and Ronco introduced the following order:



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The shuffle product of planar trees is the sum of element in an interval of this order.

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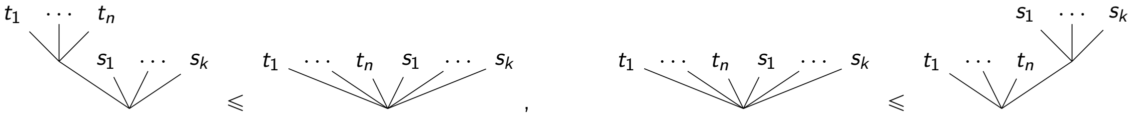

Proposition (Palacios-Ronco 02)

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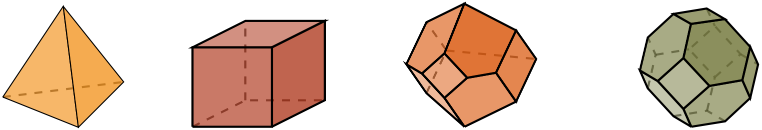
Goal of this talk

Explain to what extent these constructions (polytope $\leftrightarrow$ algebra on faces whose product is given by an order on faces) can be generalised to other polytopes.

Hypergraph polytopes (a.k.a. nestoedra)

Outline



Simplices | Hypercubes | Associahedra | Permutohedra

These four polytopes are instances of nestohedra / hypergraph polytopes

Could we define a shuffle product and an order on their faces ?

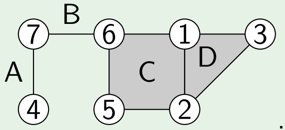
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

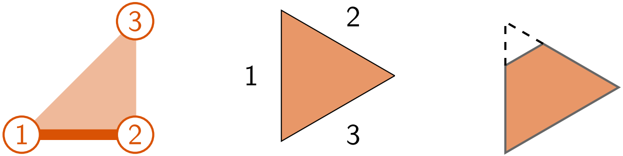
Example of an hypergraph on $[1;7]$



Warning:

All the hypergraphs considered in this talk are connected !

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])



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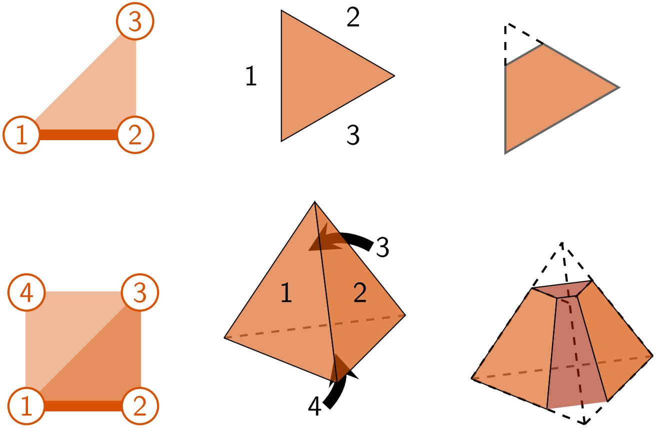
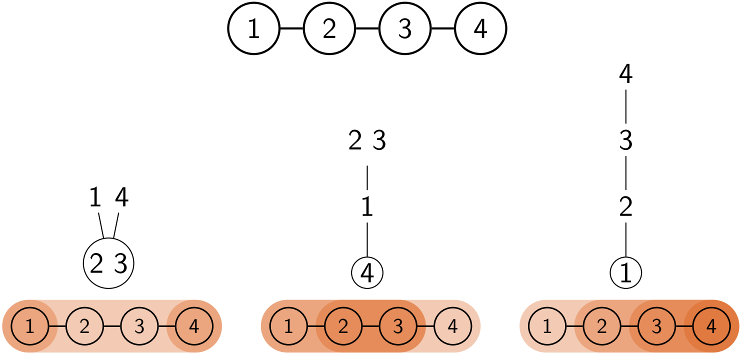


Correspondence Tubings = Constructs = Spines



Upper ideals in constructs = tubes

Constructs [Postnikov; Curien-Ivanovic-Obradović]

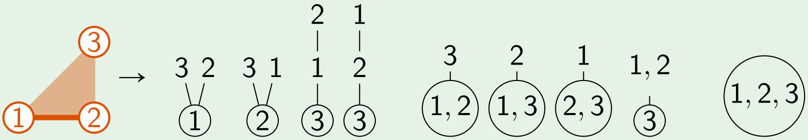
Constructs

A construct c of a hypergraph H is a rooted tree whose vertices form a partition of $V(H)$ and defined inductively by:

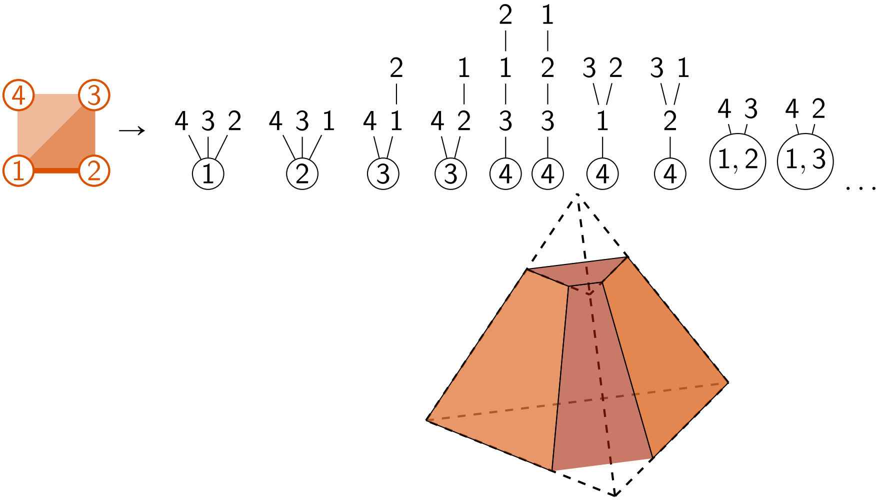
- c has only one node labelled by $V(H)$,
- or the root of c is $E \subseteq V(H)$
  and each of its children is a construct of a connected component of $H - E$.

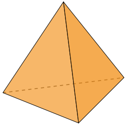
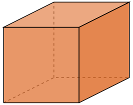
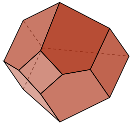
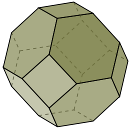
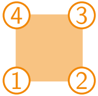
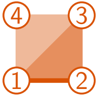
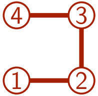
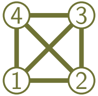
The set of constructs of a given hypergraph labels faces of the associated polytope.

First example:



Second example geometrically



Polytope	Simplex	Hypercube	Associahedron	Permutohedron
Photo				
Associated hypergraph				
Combinatorial objects	multipointed sets	left-comb tree	planar trees	packed words
Cardinality	$2^{n+1} - 1$ (A074909)	3^n (A013609)	Super-Catalan (A001003)	Fubini nbrs (A000670)

Algebraic structures on faces of nestohedra

Outline

Heuristics for a shuffle product

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

$S * T$ as a linear combination of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,** or **root** B**,** or **root** $A \cup B$**.**

Heuristics for a shuffle product

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

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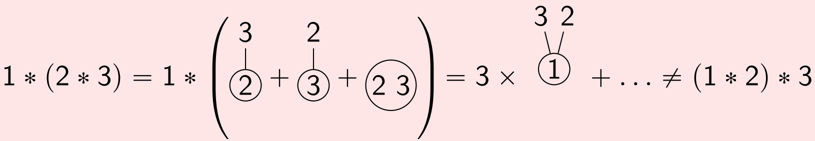
First problem in hypercubes



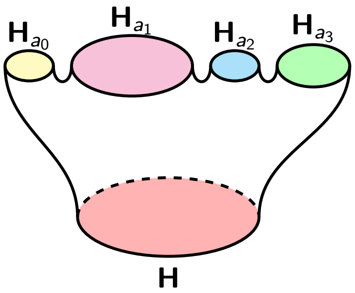
Universe and preteam

The considered hypergraphs belong to a set of hypergraphs $\mathfrak{U}$, called universe.

A preteam (=domain of definition) is a pair $\tau = (\{\mathbf{H}_a | a \in A\}, \mathbf{H})$ where

- $\{\mathbf{H}_a | a \in A, \mathbf{H}_a \in \mathfrak{U}\}$ is a set of pairwise disjoint hypergraphs, called participating hypergraphs
- $\mathbf{H} \in \mathfrak{U}$ is a hypergraph such that $H = \bigcup_{a \in A} H_a$, called supporting hypergraph.

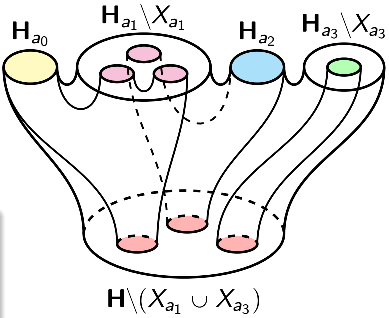
$$* : \prod_{a \in A} \mathcal{C}(\mathbf{H}_a) \to \mathcal{C}(\mathbf{H})$$



Strict and quasi-strict teams [Curien–D.O.–Obradovic, 25]

A preteam is a (resp. quasi-strict) strict team if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are

- in $\mathfrak{U}$
- and included in the connected components of $\mathbf{H} \setminus (\bigcup_{a \in A} X_a)$ (resp. or totally disconnected).



$(X_{a_0} = X_{a_2} = \emptyset)$

Examples:

- Strict teams : Associahedra, Permutohedra, Restrictohedra, ...
- Quasi-strict teams : Simplices, Hypercubes, Erosohedra, ...

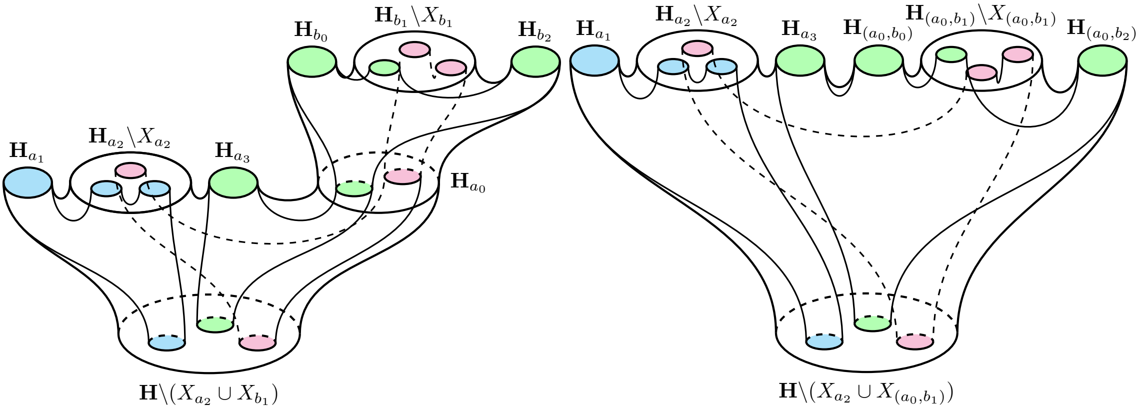
Shuffle product

Considering a team E and denoting by δ a tuple of constructs of the team's participating hypergraphs, we inductively associate to δ a sum of constructs of the supporting hypergraph:

$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} (\bigcup_{b \in B} X_b)(*(\delta_1^B), \dots, *(\delta_{n_B}^B)),$$

Associative clan

A set of (resp. quasi-strict) strict team with "good" closure properties is called strict clan (each connected component obtained from the supporting hypergraph is itself a supporting hypergraph of a team).



Associativity of $*$

Theorem (Curien-D.O.-Obradović, 25)

Consider an associative clan $\mathcal{C}$. The product $*$ is associative if

- *$\mathcal{C}$ is strict,*
- *or $\mathcal{C}$ is quasi-strict and $q = -1$.*

- Strict clans: Associahedra, Permutohedra, Restrictohedra, ...
- quasi-strict clans: Simplices, Hypercubes, ...

Order on faces of nestohedra : generalised flip order [Curien–Laplante-Anfossi, Curien–D.O.–Obradovic]

- Hypergraphs on $\mathbb{Z}$
- For $X_1, X_2 \subseteq \mathbb{Z}$, we write $X_1 < X_2$ if $\max(X_1) < \min(X_2)$.
- Every considered set of hypergraphs (preteam, decomposition) is ordered

Definition

S and T two constructs of $\mathbf{H}$. $S \lessdot T$ if and only if there exists

- U parent of V in S such that $\min(U) > \max(V)$ and T obtained from S by merging U and V, (contraction)
- or V parent of U in T such that $\min(U) > \max(V)$, and S obtained from T by merging U and V. (split)

Proposition (Ronco 2012 ; Curien–D.O.–Obradovic (new cases))

For any strict ~~(or quasi-strict)~~ ordered associative clan, the shuffle product is the sum of elements in an interval in the generalised flip order.

What about the coproduct ? [Work in progress]

Question: Would the following coproduct work ?

$$\Delta(S) = \sum_{\text{c cut}} R_c(S) \otimes *(F_c(S)) + 1 \otimes S$$

Answer: not on hypercubes...

Possible for :

- associahedra
- permutohedra
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Thank you very much for your attention !