A minicourse on Permutation Flows

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joint work with González D'León and Hanusa

Overview.

Part I. Flow polytopes (netflow $\vec{a} = \vec{e}_0 - \vec{e}_n$)

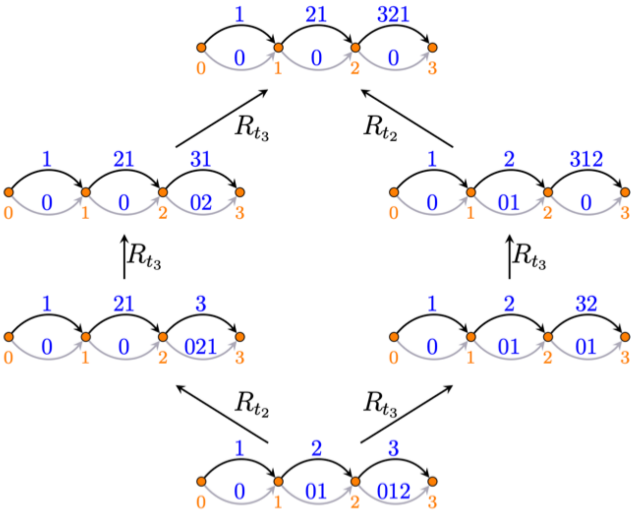
- faces, volume, framed triangulations, dual lattice structure
- a new combinatorial model for computing: permutation flows
- obtain the h^*-polynomial of the flow polytope $\mathcal{F}_G$

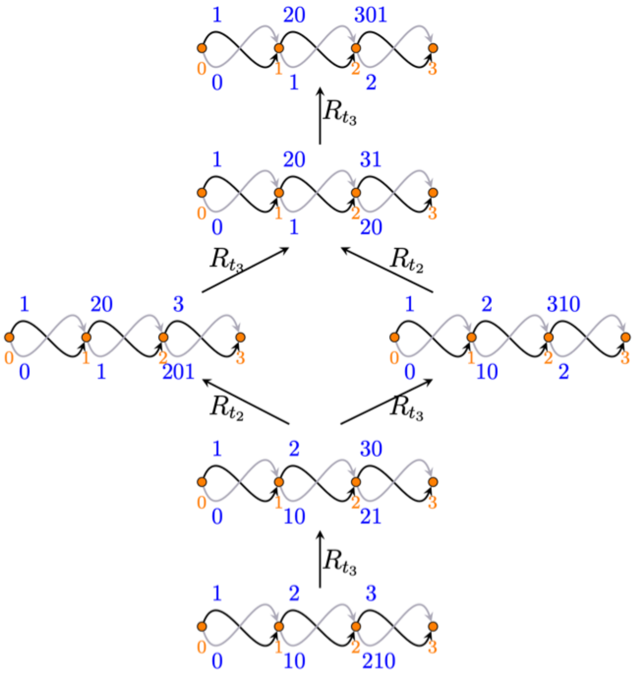
$$h^*_{\mathcal{F}_G} = A_G \quad \text{is the } G\text{-Eulerian polynomial.}$$

Part II. Flow polytopes (netflow $\vec{a} = (a_0, \ldots, a_{n-1}, -\Sigma a_i)$)

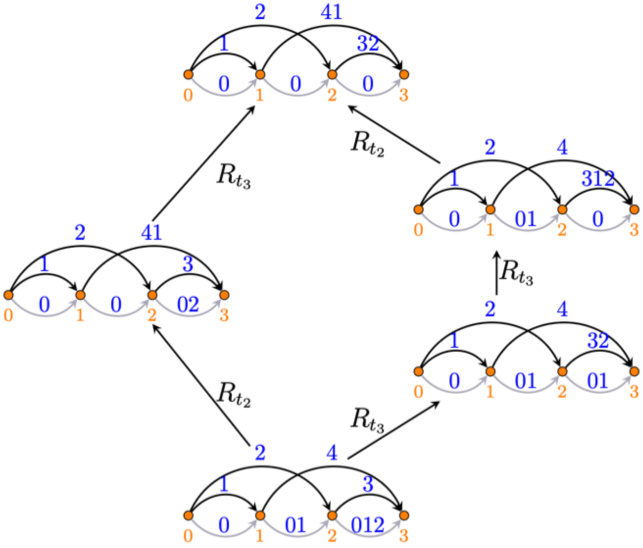
- permutation flow shuffles, and groves and vines etc.
- a new triangulation of $\mathcal{F}_G(\vec{a})$

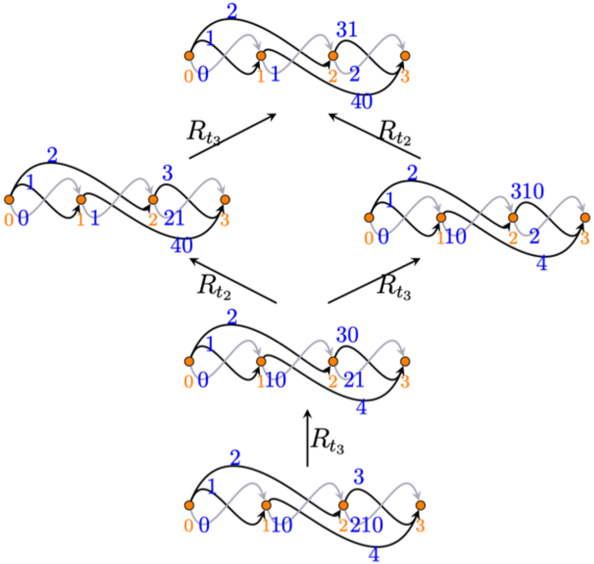
Permutahedra ,



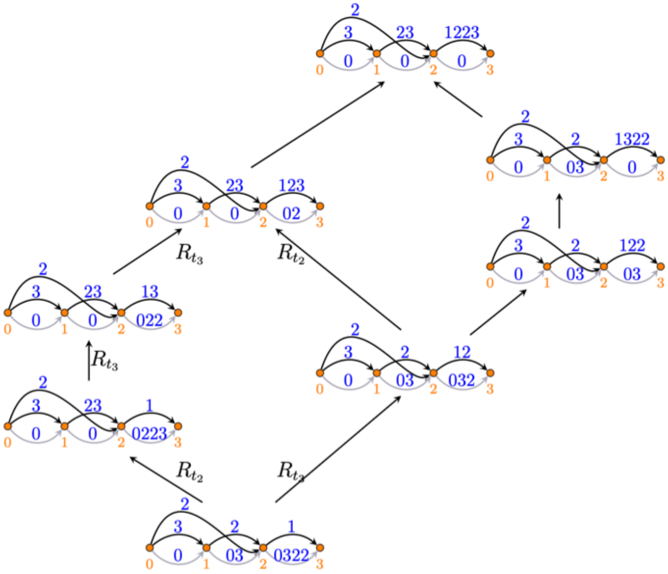


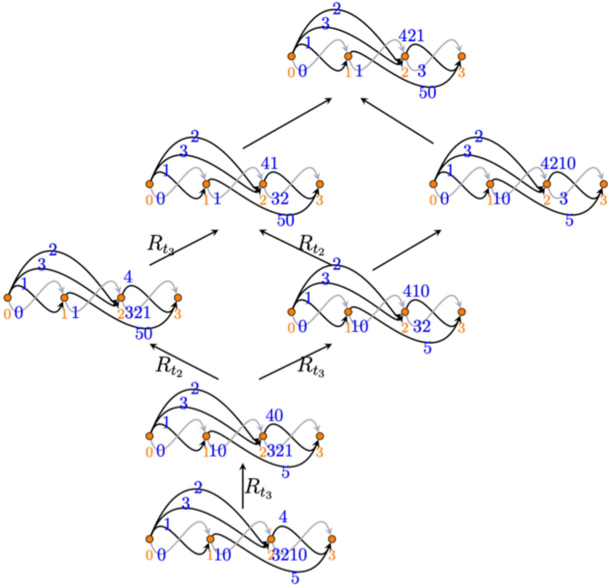
Associahedra ,





Beyond ,


2 3 23 1223 0 0 0
2 3 2 1322 0 03 0
2 3 23 123 0 0 02
2 3 2 122 0 03 03
2 3 23 13 0 0 022
2 3 2 12 0 03 032
2 3 23 1 0 0 0223
2 3 2 1 0 03 0322
0 1 2 3
R_{t_2}
R_{t_3}


2 3 1 421 0 1 3 50
2 3 1 41 0 1 32 50
2 3 1 4210 0 10 3 5
2 3 1 4 0 1 321 50
2 3 1 410 0 10 32 5
2 3 1 40 0 10 321 5
2 3 1 4 0 10 3210 5
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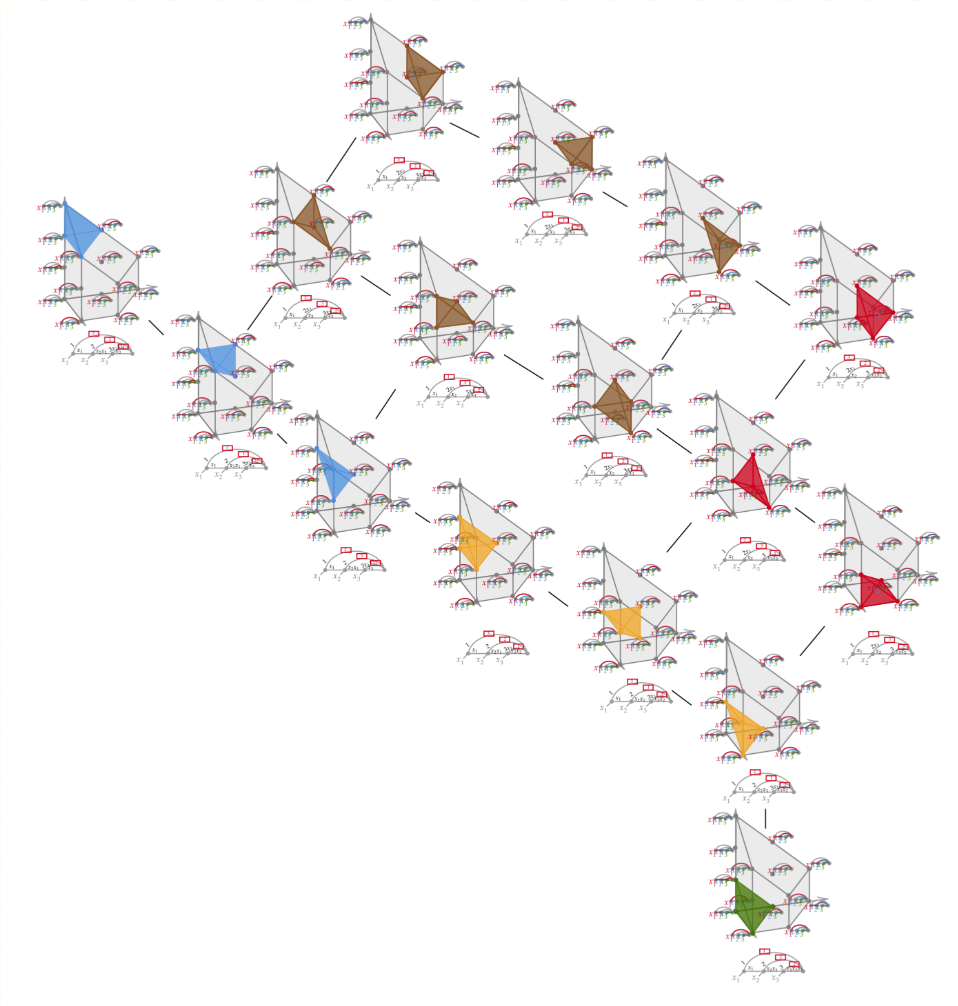
R_{t_2}
R_{t_3}

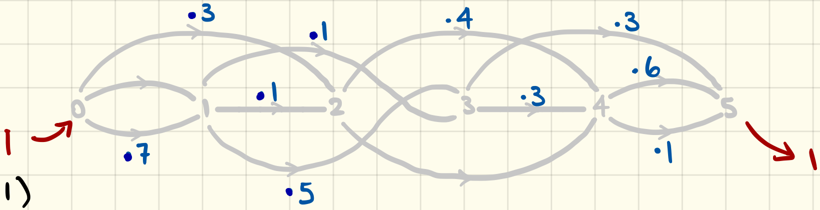
and Beyonder.



Flow polytopes.

Defn. G is an acyclic directed graph, $V = \{0, \dots, n\}$, m edges.

$\vec{a} = (a_0, \dots, a_{n-1}, -\Sigma a_i) \in \mathbb{Z}^{n+1}$, $a_v \geq 0$ for $v = 0, \dots, n-1$.



$\vec{a} = (1, 0, 0, 0, 0, -1)$

The flow polytope $\mathcal{F}_G$ is the set of all flows on G.

$$\mathcal{F}_G = \{ \vec{x} \in \mathbb{R}^{E}_{\geq 0} \mid N_G \vec{x} = \vec{a} \}.$$

signed incidence matrix of G.

$d = \dim \mathcal{F}_G = m - n$.

Faces.

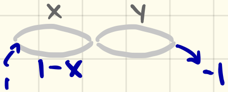
Theorem [Hille]. Faces of $\mathcal{F}_G(\vec{a})$ are of the form $\mathcal{F}_H(\vec{a})$ where H is a subgraph of G.

Corollary. Vertices of $\mathcal{F}_G(\vec{a})$ are $\mathcal{F}_T(\vec{a})$, T is a subtree.

Corollary. Vertices of $\mathcal{F}_G$ are **routes** (directed paths from 0 to n).

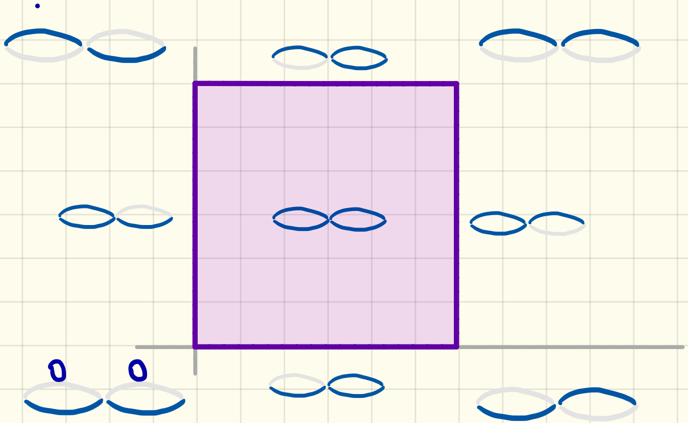
when $\vec{a} = \vec{e}_0 - \vec{e}_n$

$\vec{a} = \vec{e}_0 - \vec{e}_n$



$0 \leq x \leq 1$

$0 \leq y \leq 1$

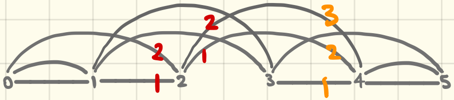


Framed Triangulations. [Danilov, Karzanov, Koshevoy].

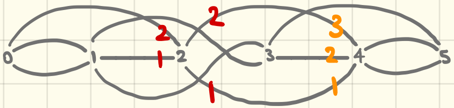
Defn. A framing at v is an ordering of the incoming and of the outgoing edges at v.

A framing F of G is the set of framings at every vertex (we include framings at 0 and n).

(G, F')



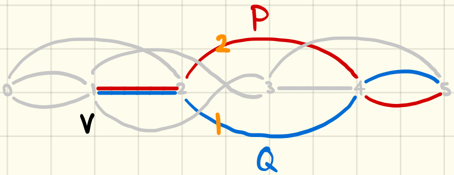
(G, F)



The framing at v induces total orderings on Prefixes(v) and on Suffixes(v).

paths from v to n

Is $\boxed{P > Q}$ or $P < Q$?



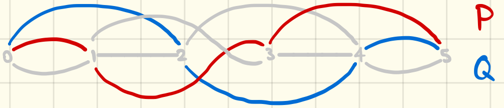
Framed Triangulations.

Defn. Two routes P and Q are in conflict at v if prefixes Pv, Qv have the opposite ordering from suffixes vP, vQ.

P and Q are coherent if they are not in conflict at any v.

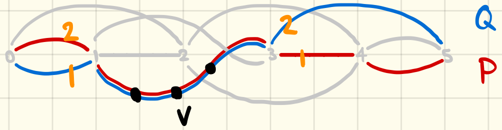
Question. Are P and Q coherent ?

(a)



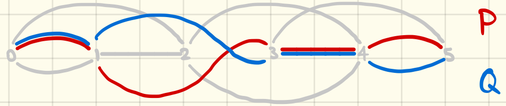
Yes! No intersections

(b)



Not coherent

(c)



Yes.

Framed Triangulations.

Defn. A clique is a set of pairwise coherent routes.
The set Cliques(G, F) is partially ordered by containment.
Given a clique $\mathcal{C}$, its rank or dimension is

$$\operatorname{rank} \mathcal{C} = |\mathcal{C}| - 1 .$$

Let $\Delta_{\mathcal{C}} := \operatorname{conv} \{ \underset{\nwarrow}{\vec{z}(R)} \in \mathbb{R}^E_{\geq 0} \mid R \in \mathcal{C} \}$.

indicator vector of the route

Theorem [Danilov, Karzanov, Koshevoy].
Let (G,F) be a framed graph. Then

$$\operatorname{DKK}(G, F) := \{ \Delta_{\mathcal{C}} \mid \mathcal{C} \text{ is a maximal clique of } (G, F) \}$$

is the set of top-dimensional simplices in a regular unimodular triangulation of $\mathcal{F}_G$. Moreover, $\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{C}'}$ share a facet if $\mathcal{C}$ and $\mathcal{C}'$ differ in exactly one route.

Framed Triangulations.

(G, F)

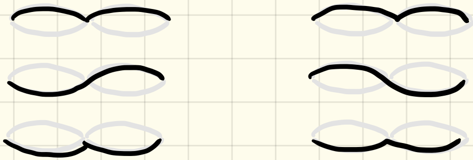
$\vec{a} = \vec{e}_0 - \vec{e}_n$

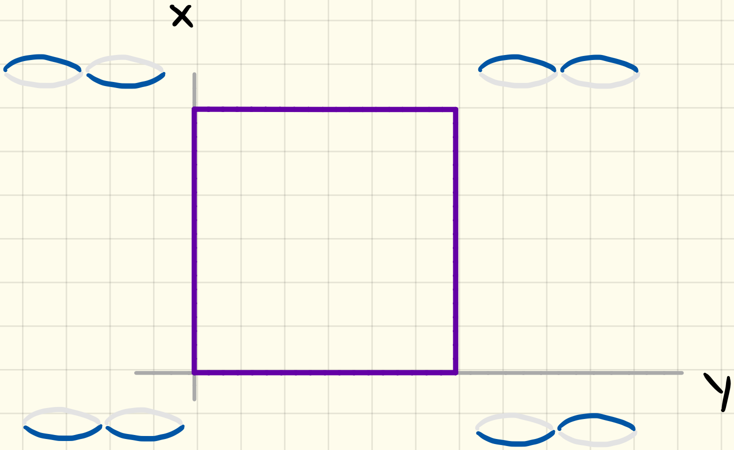
$m = 4$

$n = 2$

$d = 2$

There are two maximal cliques.





Framed Triangulations.

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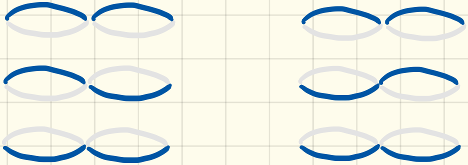
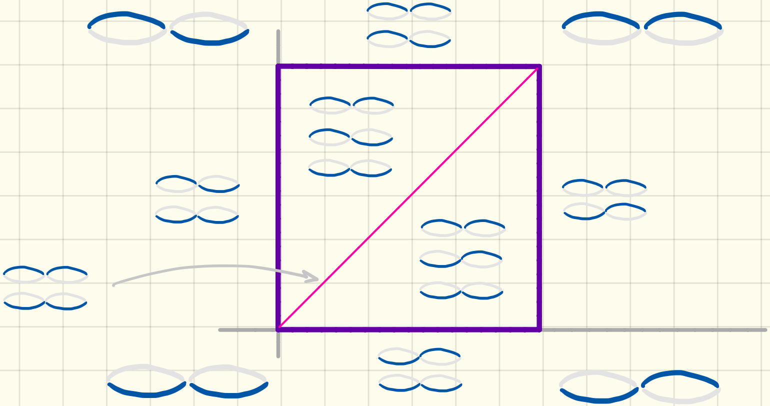
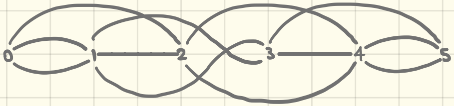




Framed Triangulations.

Question. (G, F) 

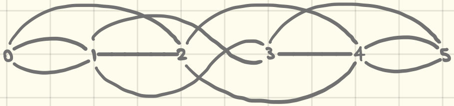
has $m = 12$
$n = 5$
$d = 7$

(a) Can you compute a maximal clique ?

(b) Can you then compute another maximal clique (by hand ?!) so that these simplices share a facet ?

Framed Triangulations.

Question. (G, F)



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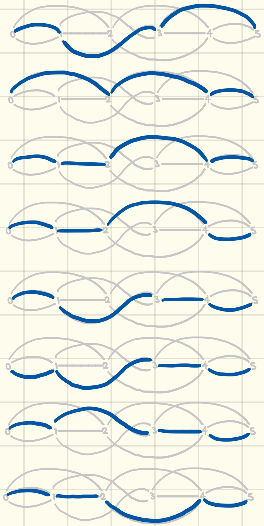
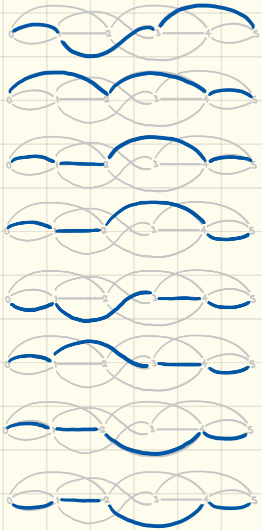
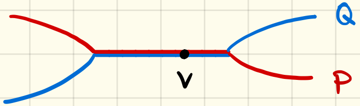




Dual graph has a lattice structure.

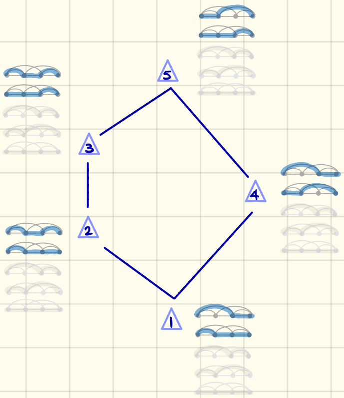
The set MaxCliques(G, F) has a poset structure.

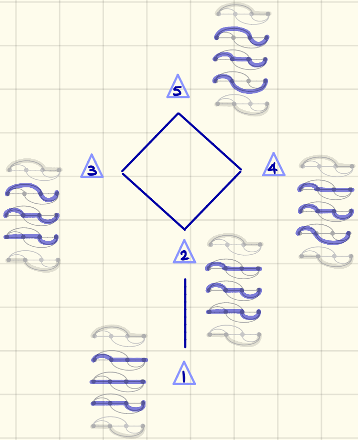
Suppose $\Delta_{\mathcal{C}} \sim \Delta_{\mathcal{C}'}$ in the dual graph, so $\mathcal{C}' = \mathcal{C} \setminus P \cup Q$.
with P and Q in conflict at some v.



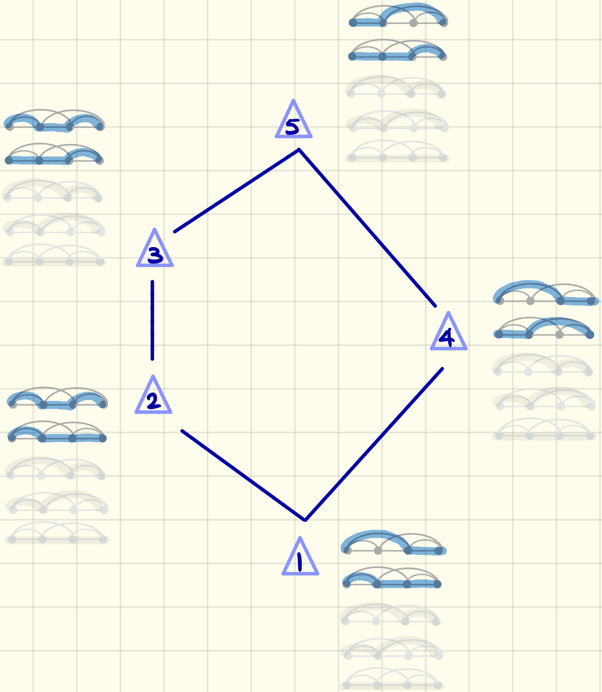
If $vP <_{out(v)} vQ$,
let $\mathcal{C} \lessdot \mathcal{C}'$ in MaxCliques(G, F)

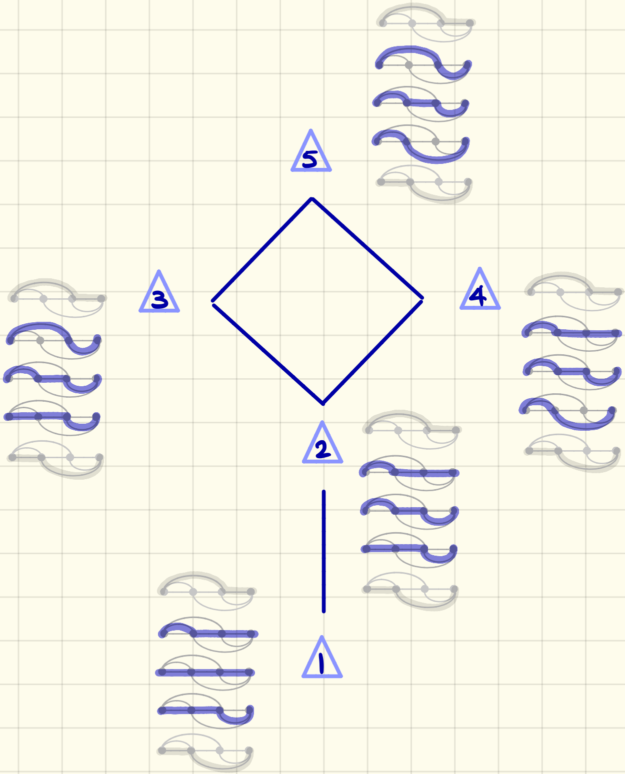
The transitive closure of these $\lessdot$ defines a partial order.





Dual graph has a lattice structure.



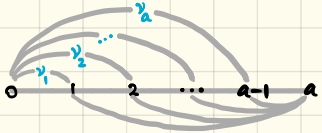


ν-Tamari lattice, principal order ideals $I(\nu)$ in Young's lattice.

Theorem. [Bell, González, Mayorga, Y. 23]

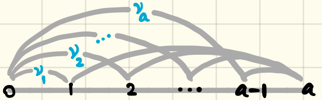
Let $\nu = (\nu_1, \dots, \nu_a) \in \mathbb{Z}^a_{\geq 0}$.

car(ν)



1. planar-framed triangulation of $\mathcal{F}_{car(\nu)}$:

   The dual graph is the Hasse diagram of $I(\nu)$.



2. length-framed triangulation of $\mathcal{F}_{car(\nu)}$:

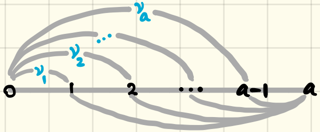
   The dual graph is the Hasse diagram of $Tam(\nu)$.

3. $\operatorname{vol} \mathcal{F}_{car(\nu)} = Cat(\nu) = \det \left[\binom{1+\sum_{k=1}^{a-j} \nu_k}{1+j-i} \right]_{1 \leq i \leq j \leq a-1}$

4. $h^*_{\mathcal{F}_{car(\nu)}} = \nu$-Narayana polynomial.

ν-Tamari lattice, principal order ideals $I(\nu)$ in Young's lattice.

Theorem. [Bell, González, Mayorga, Y. 23]



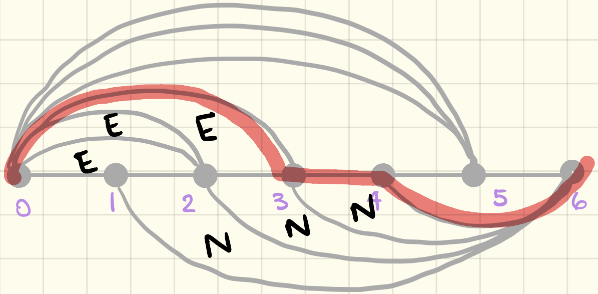
ν planar-framed triangulation of $\mathcal{F}_{car(\nu)}$:

The dual graph is the Hasse diagram of $I(\nu)$.

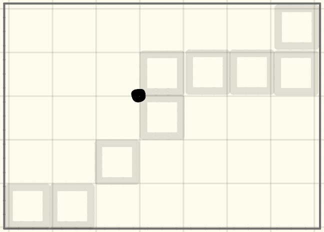
$$\{\text{routes in } car(\nu)\} \xleftrightarrow{\psi} \{\text{lattice points above } \nu\}.$$

$(2, 1, 0, 3, 0)$

$\nu = N\,E^2N\;EN\;\;N\,E^3N\;E$



$\xleftrightarrow{\psi}$



ν-Tamari lattice, principal order ideals $I(\nu)$ in Young's lattice.

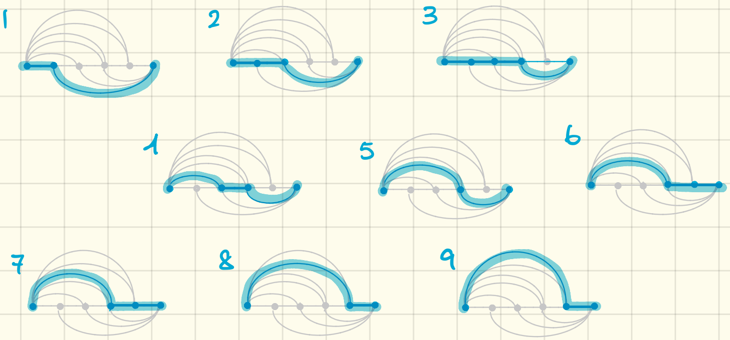
Theorem. [Bell, González, Mayorga, Y. 23]

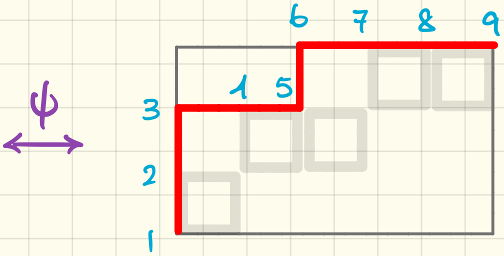
1 planar-framed triangulation of $\mathcal{F}_{car(\nu)}$:

The dual graph is the Hasse diagram of $I(\nu)$.

$$\{\text{routes in } G(\nu)\} \overset{\psi}{\longleftrightarrow} \{\text{lattice points above } \nu\}.$$

$$\{\text{maximal cliques in } car(\nu)\} \overset{\varphi}{\longleftrightarrow} \{\nu\text{- Dyck paths}\}$$





ν-Tamari lattice, principal order ideals $I(\nu)$ in Young's lattice.

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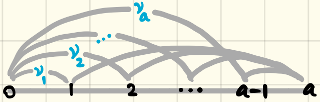
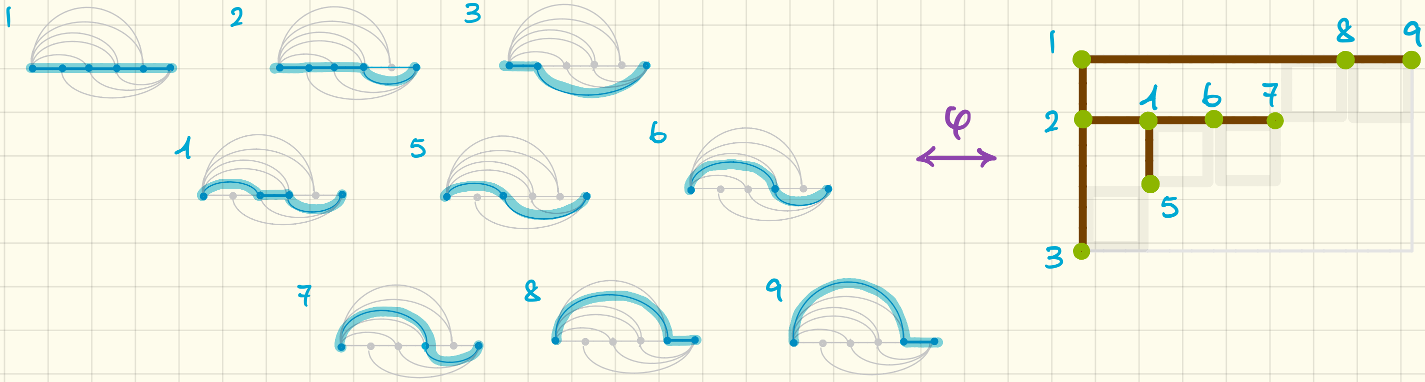
2, length-framed triangulation of $\mathcal{F}_{car(\nu)}$:



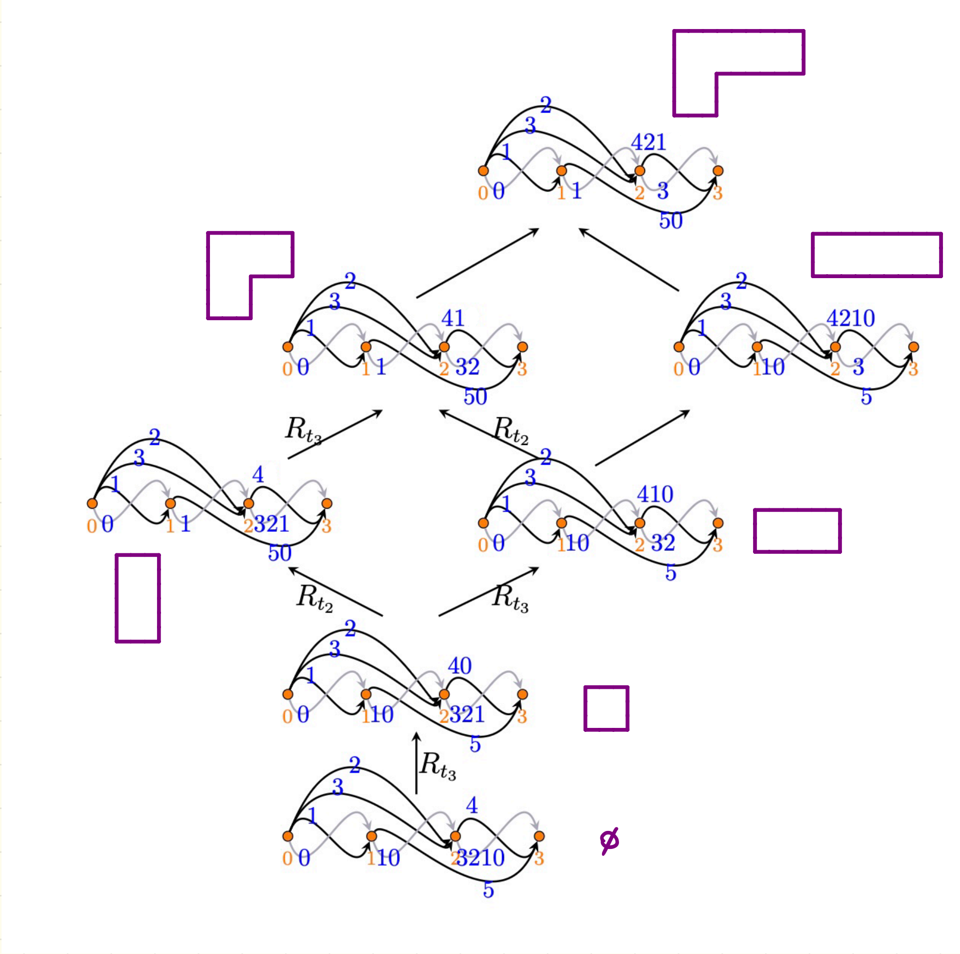
The dual graph is the Hasse diagram of $Tam(\nu)$.

$\{$routes in $car(\nu)\}$ $\overset{\varphi}{\longleftrightarrow}$ $\{$lattice points above $\nu\}$

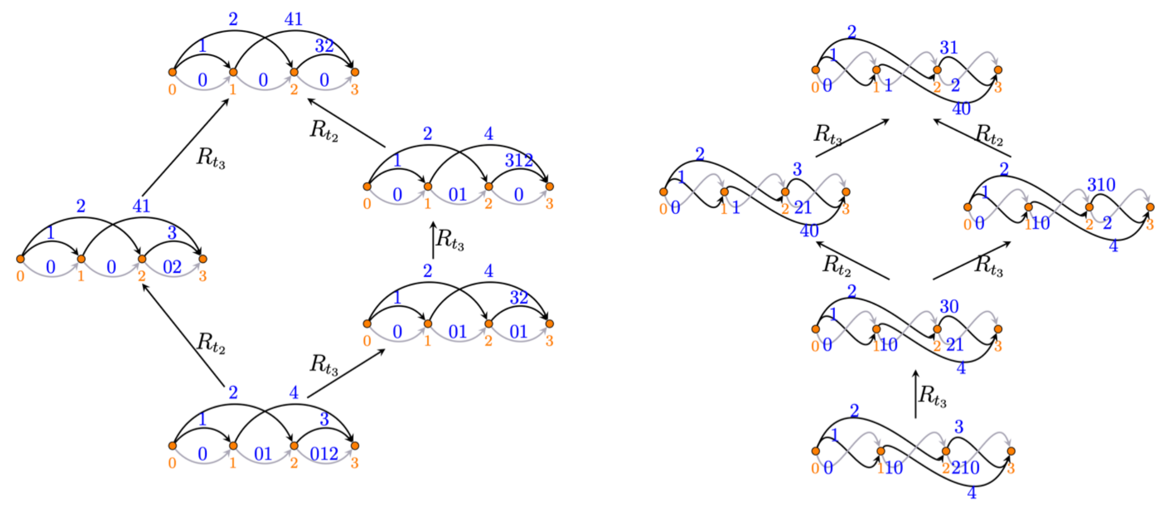
$\{$maximal cliques in $car(\nu)\}$ $\overset{\varphi}{\longleftrightarrow}$ $\{\nu$-binary trees$\}$



$I(\nu)$, $\nu = (1,2,1)$.



DKK triangulations unify the study of the Tamari lattice and the poset of order ideals of the Type A root lattice.



But these are lattices!

Conjecture. All dual graphs of DKK(G, F) have a lattice structure.

More evidence : $\vec{s}$- weak order.

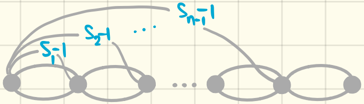
Introduced by Ceballos and Pons, it is a lattice defined on s-decreasing trees.

They conjectured that the $\vec{s}$- weak order can be realized as the 1- skeleton of a polyhedral subdivision of a polytope.

Theorem. [González, Morales, Phillipe, Tamayo, Y. 25]

Let $\vec{s} = (s_1, \ldots, s_{n-1}) \in \mathbb{Z}_{\geq 1}^{n-1}$.

oru$(\vec{s})$:



1. The dual graph of DKK$(\mathcal{F}_{\text{oru}(\vec{s})}, F)$ is the Hasse diagram of the $\vec{s}$- weak order.

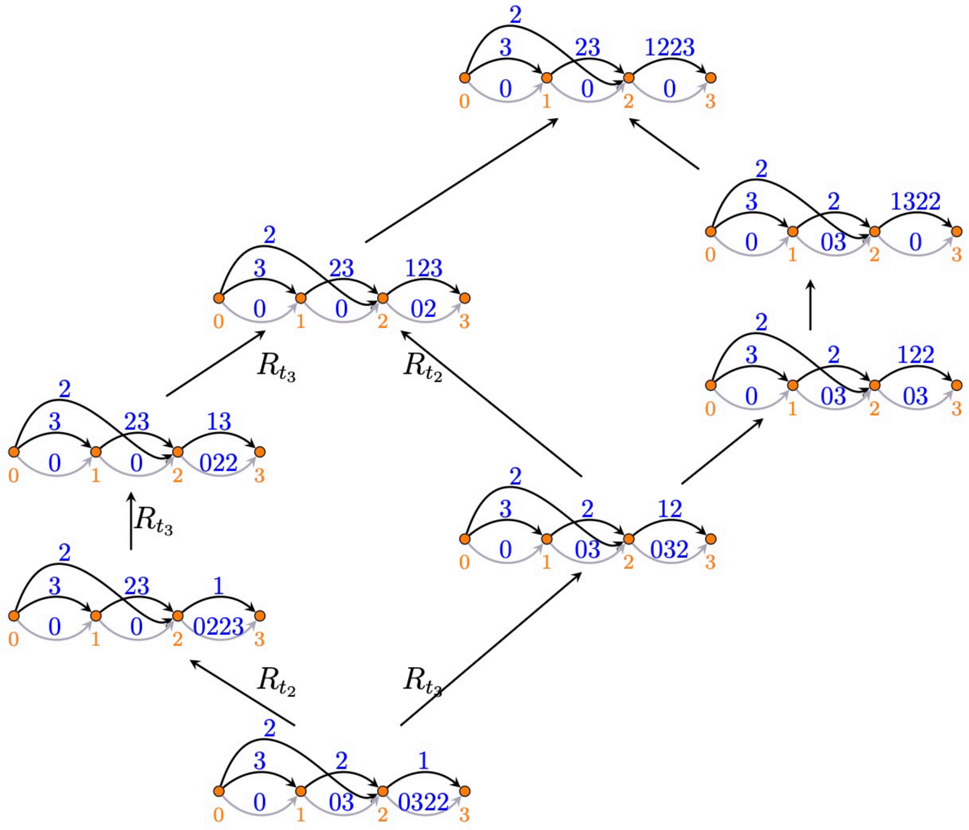
2. The $\vec{s}$- weak order can be realized as the dual graph of a fine mixed subdivision of a sum of hypercubes.

3. The $\vec{s}$- weak order can be realized as the 1- skeleton of a polyhedral complex.

oru (1,2,1)

These maximal cliques are in bijection with :

- $\vec{s}$- decreasing trees
- $\vec{s}$- Stirling permutations



Dual graph has a lattice structure.

Theorem. [Bell, Ceballos], [Berggren, Serhiyenko]

The poset on $\mathrm{MaxCliques}(G, F)$ is a lattice!

Other notable examples of framing lattices include:

• Boolean lattice	
• c-Cambrian	[Reading]
• alt ν-Tamari	[Ceballos, Chenevière]
• cross Tamari	[Bell, Ceballos]
• Grassmann Tamari	[Santos, Stump, Welker]
• grid Tamari	[McConville]
• (ε, I, J)-Cambrian	[Pilaud]

Overview.

Part I. Flow polytopes (netflow $\vec{a} = \vec{e}_0 - \vec{e}_n$)

- faces, volume, framed triangulations, dual lattice structure
- a new combinatorial model for computing : permutation flows
- obtain the h^*-polynomial of the flow polytope $\mathcal{F}_G$

$$h^*_{\mathcal{F}_G} = A_G \quad \text{is the } G\text{-Eulerian polynomial.}$$

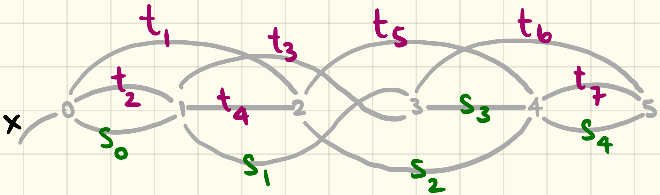
Permutation Flows.

Cliques are difficult to work with.
We'll encode its information in a more compact manner.

Defn.

slack edges :

split edges :



It will be helpful to add an inflow half edge x to G.
Note # split edges = d.

The set of partial permutations of a set A is

$$\mathfrak{P}_A = \bigsqcup_{B \subseteq A} \mathfrak{S}_B .$$

Given $\pi : \underbrace{E \cup \{x\}}_{\hat{E}} \longrightarrow \mathfrak{P}_{\{0,1,\dots,d\}}$, its support is

$$\text{supp}(\pi) = \{ e \in \hat{E} \mid \pi(e) \neq \emptyset \}.$$

Permutation Flows.

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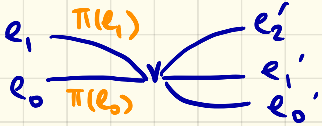
$$\mathrm{supp}(\pi) = \{ e \in \hat{E} \mid \pi(e) \neq \emptyset \}.$$

At a vertex v with

$$in(v) = \{ e_0, \dots, e_i \}$$
$$out(v) = \{ e_0', \dots, e_j' \}$$



the **v^{th}-incoming summary** and **v^{th}-outgoing summary** are

$$InPerm(v) = \pi(e_0) \cdots \pi(e_i), \quad OutPerm(v) = \pi(e_0') \cdots \pi(e_j').$$

Defn. A **permutation flow** on (G, F) is $\pi : \hat{E} \longrightarrow \mathcal{P}_{[0,d]}$ s.t.

- if π is nonempty, then $\pi(x) = 0$.
- at v with $in(v) = \{ e_0, \dots, e_i \}$, $out(v) = \{ e_0', \dots, e_j' \}$ OutPerm(v) is an unshuffle of InPerm(v) and a (possibly empty) subword of $e_i' \cdots e_j'$.
- if $e_h' \in$ OutPerm(v), then e_h' is the first letter of $\pi(e_h')$.

Permutation Flows.

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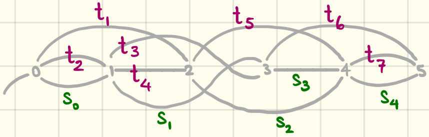
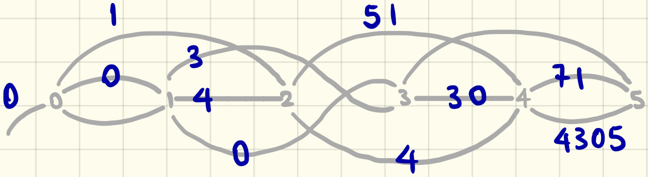




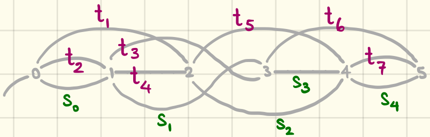
0 ⧢ 34 41 ⧢ 5 43051 ⧢ 7

30 ⧢ ∅

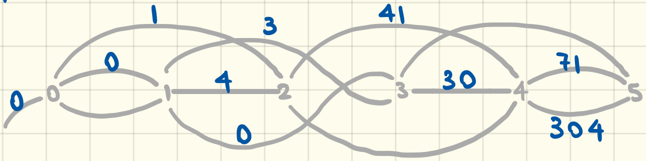
Permutation Flows.

Defn. A permutation flow on (G, F) is $\pi : \hat{E} \longrightarrow \mathfrak{P}_{[0,d]}$ s.t.

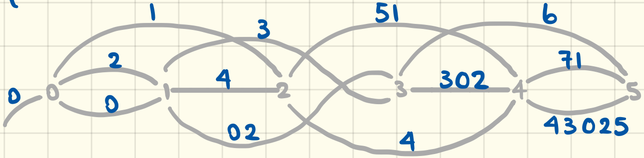
- if π is nonempty, then $\pi(0) = 0$.
- at v with $\mathrm{in}(v) = \{ e_0, \dots, e_i \}$, $\mathrm{out}(v) = \{ e'_0, \dots, e'_j \}$ OutPerm (v) is an unshuffle of InPerm(v) and a (possibly empty) subword of $e'_0 e'_1 \cdots e'_j$.
- if $e'_h \in$ OutPerm (v), then e'_h is the first letter of $\pi(e'_h)$, and e'_h is a split edge of supp(π).



π



ρ



Permutation Flows.

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  OutPerm(v) is an unshuffle of InPerm(v) and a (possibly empty) subword of $e_0' e_1' \cdots e_j'$.
- if $e_h' \in$ OutPerm(v), then e_h' is the first letter of $\pi(e_h')$,
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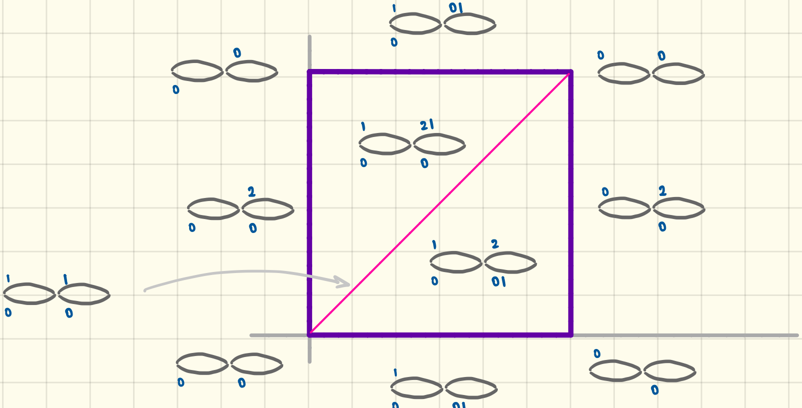
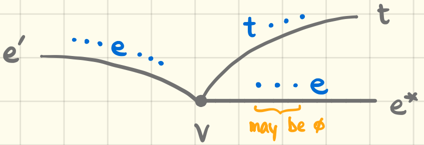


Permutation Flow splits.

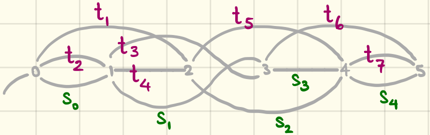
Define a partial order on the set PermuFlows (G, F) via split reductions.

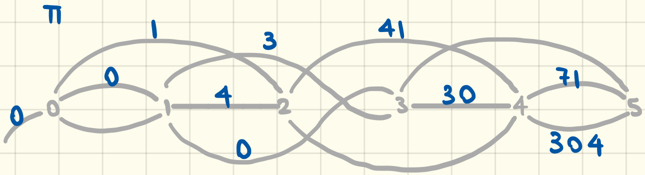
Defn. An edge $e \in E$ is a split in π if it is the first letter in $\pi(e)$.

An edge $t \in E$ is a direct split of e at v if



e^* is largest in out(v) satisfying this





Splits $(\pi) = t_1 \ t_3 \ t_4 \ t_7$

A direct split of the letter 0 at $v = 1$ is: 4

The letter 3 is a direct split of 0 at $v = 1$.

Permutation Flow poset.

Defn. Let π, ρ be permuflows of (G, F). $\pi \leq \rho$

π is a split reduction of ρ if π can be obtained from ρ via:

- keeping a letter e,
- if $\nexists$ t which is a direct split of e, then deleting all e, or
- if $\exists$ t which is a direct split of e at v (with t a min.), then deleting all e, after v, and replace all t by e.

Proposition. [González, Hanusa, Y. 25^+]

Split reduction defines a partial order on PermuFlows(G, F)

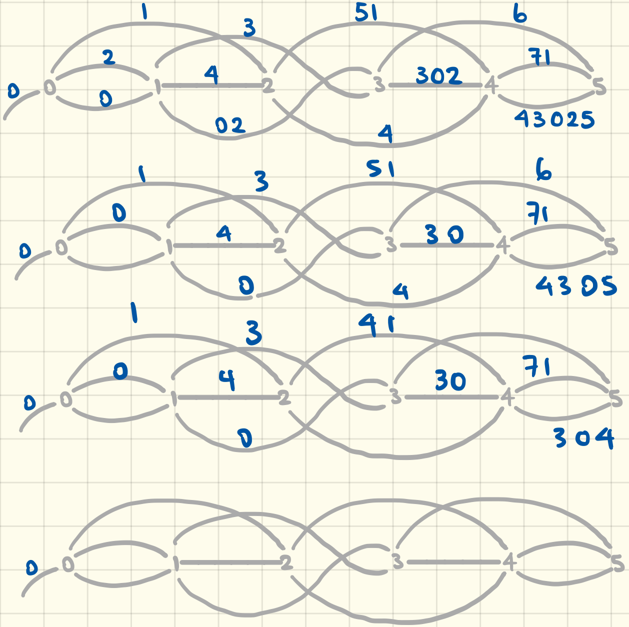
We write $\pi \leq \rho$ if π is a split reduction of ρ.

Permutation Flow poset.

Defn. π is a split reduction of ρ if π can be obtained from ρ via :

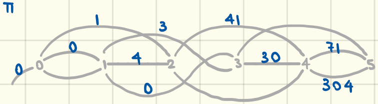
- keeping a letter e,
- if $\nexists$ t which is a direct split of e, then deleting all e, or
- if $\exists$ t which is a direct split of e at v (with t a min.), then deleting all e, after v, and replace all t by e.

ρ

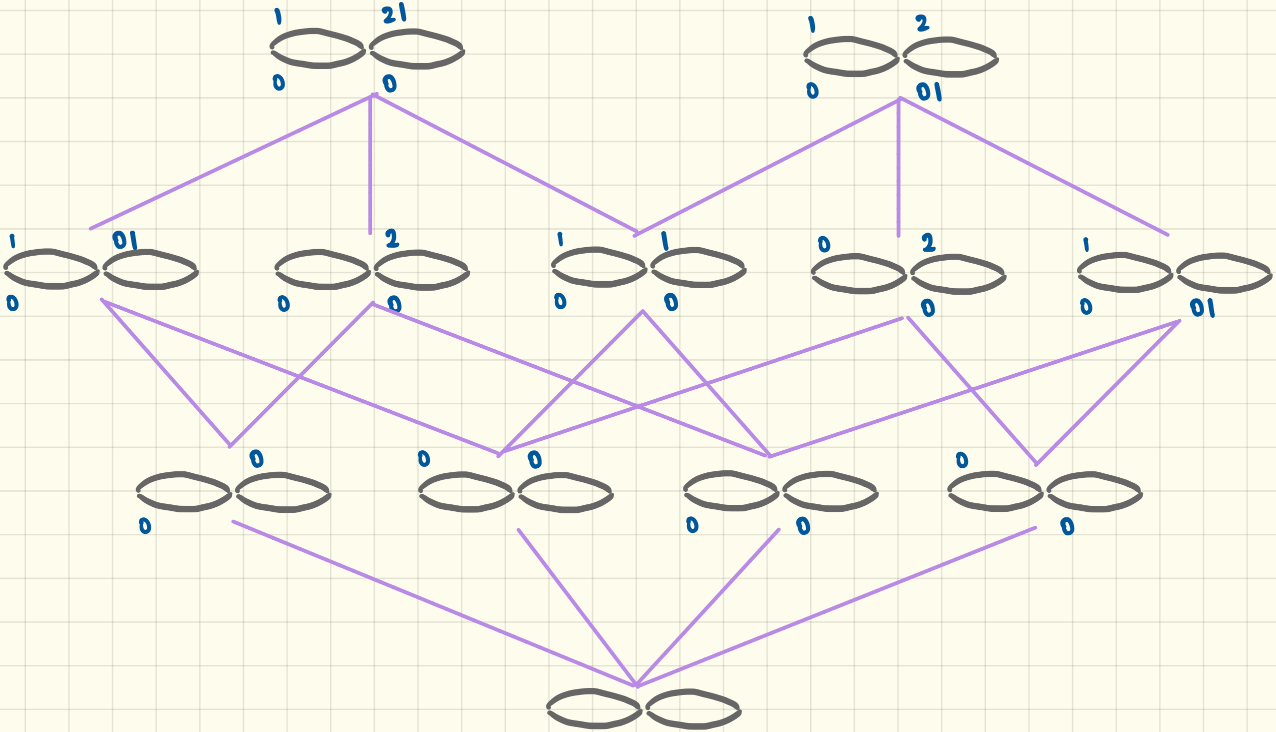


Target :

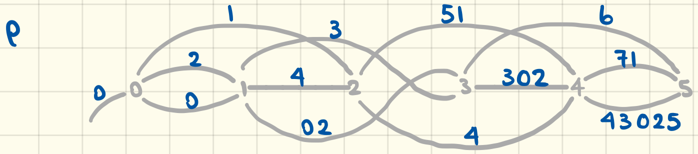
π



Face poset of the triangulated square.



Maximum Permutation Flows.



OutPerm(n) =

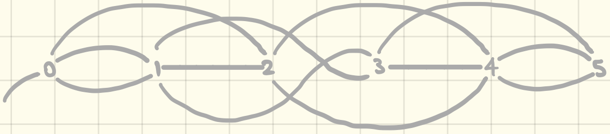
Maximum permuflows have the following properties :

•

•

Proposition. [González, Hanusa, Y. 25*]

$$\{\ \mathrm{OutPerm}_{\rho}(n)\ \} \longleftrightarrow \mathrm{MaxPermuFlows}\,(G,F)\,.$$



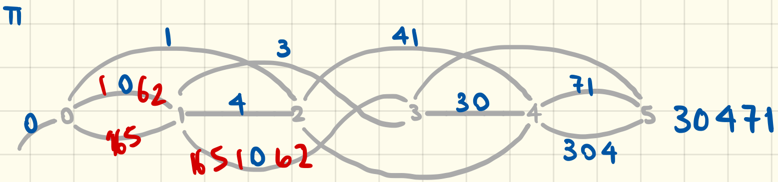
Permutation Flows encode DKK cliques.

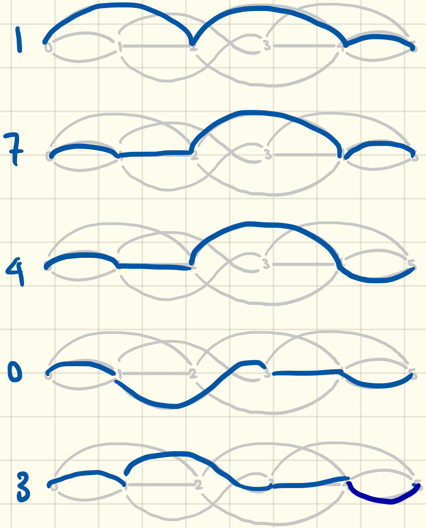
Theorem. [González, Hanusa, Y. 25^+]

$$\mathrm{PermuFlows}(G, F) \longrightarrow \mathrm{Cliques}(G, F)$$
$$\pi \longmapsto \mathrm{Routes}(\pi)$$

is an order-preserving bijection.

Recovering routes from π.





OutPerm(1) = ~~6~~ 51062 · 4 · 3

Permutation Flows encode DKK cliques.

Theorem. [González, Hanusa, Y. 25+]

$$\begin{tikzcd} \mathrm{PermuFlows}(G,F) \arrow[r] & \mathrm{Cliques}(G,F) \\ \pi \arrow[r, mapsto] & \mathrm{Routes}(\pi) \end{tikzcd}$$

is an order-preserving bijection.

Proof idea: Construct inverse map.

Do this tomorrow. Need some other objects.