February 24, 25 and 27, 2025

Enumerative combinatorics and effective aspects of differential equations, CIRM, Marseille, France

# Summation theory of difference rings and applications


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# Outline

1. Part 1: The basic summation tools in difference rings

2. Part 2: The underlying difference ring theory

3. Part 3: Applications in combinatorics, number theory and particle physics

# Part 1: The basic summation tools in difference rings

# Part 1: The basic summation tools in difference rings

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals. 2006

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

↑ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

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$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

# A warm-up example: simplify

$$\color{red}{\sum_{k=0}^{\infty}}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\color{blue}{\sum_{j=0}^{\infty} f(j)} = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

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In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= $<<$ **Sigma.m**

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In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!}+\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= **SigmaLimit[res, {n}, a]**

Out[4]= $$\frac{1}{n!}\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)}$$

# A warm-up example: simplify

$$\textcolor{red}{\sum_{k=0}^{\infty}}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\textcolor{blue}{\sum_{j=0}^{\infty} f(j)} = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# A warm-up example: simplify

$$\color{red}{\sum_{k=0}^{\infty}}\color{black}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\color{red}{\sum_{k=1}^{\infty}}\color{black}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\color{red}{\sum_{k=1}^{\infty}}\color{black}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$ :

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $k \geq 1$.

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for all $k \geq 1$.

**no solution** ☹

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $k \geq 1$.

**no solution** ☹

# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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for all $k \geq 1$.

Sigma *computes*: $c_0(n) = -n,\ c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

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FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

# Zeilberger's creative telescoping paradigm

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for all $k \geq 1$.

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Summing this equation over $k$ from $1$ to $a$ gives:

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$$||\qquad\qquad\qquad\qquad ||$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \qquad - n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

## Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S}[1,\mathbf{k}] + \mathbf{S}[1,\mathbf{n}] - \mathbf{S}[1,\mathbf{k}+\mathbf{n}]}{\mathbf{kn}(\mathbf{k}+\mathbf{n}+1)};$$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

## Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

## Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)\,(S[1,a] + S[1,n] - S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S[1,n] + 1}{(n+1)^3}$

In[5]:= $\mathbf{mySum = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};}$

## Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

## Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

## Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

## Solve a recurrence

In[8]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

## Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$
$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

# A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Big)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

# The summation paradigms

## 1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
$n$: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
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FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

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Special cases:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases:

$$\sum_{k=1}^{n}\frac{2^k}{k}\sum_{i=1}^{k}\frac{2^{-i}}{i}\sum_{j=1}^{i}\frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{k=1}^{n}\frac{1}{(1+2k)^2}\sum_{j=1}^{k}\frac{1}{j^2}\sum_{i=1}^{j}\frac{1}{1+2i}$$

(cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases:

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}$$ (binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\sum_{k=1}^{n} \left( \prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \left( \sum_{j=1}^{k} \frac{1}{j\binom{4j}{3j}^2} \right) \left( \sum_{j=1}^{k} \begin{bmatrix} 2j \\ j \end{bmatrix}_q \right)$$

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FIND all solutions expressible by indefinite nested products/sums
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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$

$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$

$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

> Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

> HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

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In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

# Example: a challenging email

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
-Doron
```

[arose in the bounds on the run time of the simplex algorithm on a polytope]

# The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks
suspiciously like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \boxed{\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}}$$

with

$$S_1(j) := \sum_{i=1}^{j} \frac{1}{i}$$

# The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

`Sigma.m`

**Recurrence finder**

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}$$

# The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

Sigma.m **Recurrence solver** Sigma.m

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)} \in \left\{ c_1 \frac{S_1(k)}{k} + c_2 \frac{1}{k} + \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k)}{2k^2} \middle| c_1, c_2 \in \mathbb{R} \right\}$$

where

$$S_2(k) = \sum_{i=1}^{k} \frac{1}{i^2}$$

# The inner sum

$$k^2\mathbf{A}(k) - (k+1)(2k+1)\mathbf{A}(k+1) + (k+1)(k+2)\mathbf{A}(k+2) = \frac{1}{k+1}$$

`Sigma.m` **Recurrence solver** `Sigma.m`

$$\mathbf{A}(k) = \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)} = \boxed{0\frac{S_1(k)}{k} + \zeta(2)\frac{1}{k} + \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k)}{2k^2}}$$

where

$$S_2(k) = \sum_{i=1}^{k} \frac{1}{i^2} \qquad \zeta(z) = \sum_{i=1}^{\infty} \frac{1}{i^z}$$

Simplify

$$\sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\zeta(2)}{2k^2}$$

Simplify

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \sum_{j=1}^{\infty} \frac{S_1(j)}{j(j+k)}.$$

$$||$$

$$\sum_{k=1}^{\infty} \frac{S_1(k+1)-1}{k(k+1)} \times \frac{kS_1(k)^2 - 2S_1(k) + kS_2(k) + 2k\zeta(2)}{2k^2}$$

$||$telescoping + limit calculations

$$-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5)$$

$$||$$

$$0.999222... \neq 1$$

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HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{S_1[k](S_1[n+1]-1)}{kn(n+1)(k+n)}]$

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In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{S_1[k](S_1[n+1]-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

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In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{S_1[k]^2(S_1[n+1]-1)^2}{k\,(k+n)\,n}]$

Out[5]= $-10\zeta_3+\zeta_2^2(\frac{58\zeta_3}{5}-\frac{29}{5})-10\zeta_5+\zeta_2(-\zeta_3+13\zeta_5-4)+\frac{457\zeta_7}{8}$

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Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{S_1[k](S_1[n+1]-1)}{k\,(k+n)^2n^2}]$

Out[5]= $2\zeta_3+\zeta_2^2(\frac{17\zeta_3}{10}+\frac{17}{10})+\zeta_2(2\zeta_3-3\zeta_5-4)-\frac{9\zeta_5}{2}+\frac{3\zeta_7}{16}$

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In[3]:= $<<$ **EvaluateMultiSums.m**

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In[4]:= $$\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{S_1[k](S_1[n+1]-1)}{kn(n+1)(k+n)}]$$

Out[4]= $-4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5$

In[5]:= $$\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{S_1[k]S_1[n]S_1[n+l+k]}{k\,(k+n)\,(k+n+l+1)^2}]$$

Out[5]= $$3\zeta_3^2 - \frac{15\zeta_5}{2} + \zeta_2(9\zeta_5 - 6\zeta_3) + \frac{149\zeta_7}{16} + \frac{114}{35}\zeta_2^3$$

# Part 1: The basic summation tools in difference rings

# Telescoping in difference rings

Simplify

$$\sum_{k=1}^{n} S_1(k)$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$

# Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

# Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

# Telescoping

GIVEN $f(k) = S_1(k)$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over $k$ from $1$ to $n$ gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$
$$= (S_1(n+1) - 1)(n+1).$$

# Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

# Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

$\boxed{\mathrm{const}_\sigma \mathbb{A} = \mathbb{Q}}$

Take the ring

$$\mathbb{A} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

# Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand**

$$\boxed{\text{const}_\sigma \mathbb{A} = \mathbb{Q}}$$

Take the ring

$$\mathbb{A} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1$$

# Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

**A difference ring for the summand** $\boxed{\mathrm{const}_\sigma \mathbb{A} = \mathbb{Q}}$

Take the ring

$$\mathbb{A} := \mathbb{Q}(k)[s]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1$$

$$\sigma(s) = s + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}$$

# Telescoping in the given difference field

FIND $g \in \mathbb{A}$:

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with

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Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

# Creative telescoping in difference rings

Simplify

$$\sum_{k=1}^{n} \binom{n}{k} S_1(k)$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$

Simplify

$$A(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

**A difference ring for the summand**

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Simplify

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**A difference ring for the summand**

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Simplify

$$A(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

**A difference ring for the summand**

$$\boxed{\mathrm{const}_\sigma \mathbb{A} = \mathbb{Q}(n)}$$

Take the ring

$$\mathbb{A} := \mathbb{Q}(n)(k)[p, p^{-1}]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

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Simplify

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FIND $g\in\mathbb{A}$ and $c_0,c_1\in\mathbb{Q}(n)$:

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$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)}$$

FIND $g \in \mathbb{A}$ and $c_0, c_1 \in \mathbb{Q}(n)$:

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FIND $g \in \mathbb{A}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

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FIND $g \in \mathbb{A}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2}$$

We compute

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)s)\,p)}{(1-k+n)\,(2-k+n)}.$$

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This gives

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

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Summing over $k$ from $0$ to $n$ gives

$$\boxed{g(n,n+1) - g(n,0)} = \boxed{\begin{array}{l} c_0(n)\,\mathsf{A}(n) + \\ c_1(n)\,\big[\mathsf{A}(n+1) - f(n+1,n+1)\big] \\ c_2(n)\,\big[\mathsf{A}(n+2) - f(n+2,n+1) - f(n+2,n+2)\big]. \end{array}}$$

for $A(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k)$

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Summing over $k$ from $0$ to $n$ gives

$$\boxed{1=4(1+n)A(n)-2(3+2n)A(n+1)+(2+n)A(n+2)}$$

for $A(n)=\sum_{k=0}^{n}\binom{n}{k}S_1(k)$

## 1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
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FIND a recurrence for $A(n)$

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GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

# Recurrence solving (d'Alembertian solutions)

**Special case: homogeneous recurrences with** $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

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$$\prod_{j=\lambda}^{n} b_1(j-1)$$

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$$||$$

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$$\prod_{j=\lambda}^{n} b_2(j-1)$$

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$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[\Big(\tilde{a}_{d-2}(n)S^{d-2} + \tilde{a}_{d-3}(n)S^{d-3} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

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**Special case: homogeneous recurrences with** $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$||$$
$$c(n)\Big(S - b_d(n)\Big) \dots \Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

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$$\parallel$$

$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

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$$\|$$

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$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

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$d$ linearly independent solutions

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$$\vdots$$

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# Recurrence solving (d'Alembertian solutions)

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Example

$$\vdots$$

$$L_d(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i_1+1} b_1(j-1)} \dots \sum_{i_{d-1}=\lambda}^{i_{d-2}-1} \frac{\prod_{j=\lambda}^{i_{d-1}} b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1} b_{d-1}(j-1)}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

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FIND a recurrence for $A(n)$

2. Recurrence solving

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$a_0(n), \dots, a_d(n), h(n)$:
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$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

**NOTE: By construction, the solutions are highly nested.**

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2'. Indefinite summation for simplification

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

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2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

In[6]:= mySum = SumName[SigmaBinomial[n, k]S[1, k], {k, 0, n}]

Out[6]= $\sum_{k=0}^{n} \binom{n}{k} S[1,k]$

## Compute a recurrence

In[7]:= rec = GenerateRecurrence[mySum, n][[1]]

Out[7]= $4(1+n)SUM[n] - 2(3+2n)SUM[n+1] + (2+n)SUM[n+2] == 1$

## Solve a recurrence

In[8]:= recSol = SolveRecurrence[rec, SUM[n]]

Out[8]= $\{\{0, -2^n\}, \{0, -2^n \sum_{i_1=1}^{n} \frac{1}{i_1}\}, \{1, -2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}\}\}$

## Combine the solutions

In[9]:= FindLinearCombination[recSol, mySum, n, 2]

Out[9]= $2^n \sum_{i_1=1}^{n} \frac{1}{i_1} - 2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}$

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$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(x) = x + 1, \qquad \mathcal{S}\, n = n + 1,$$

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$$\mathbb{K} = \mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

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We have

| | |
|---|---|
| $\text{const}_\sigma \mathbb{A} = \mathbb{Q}$ | $(\mathbb{A}, \sigma)$ is a $\Pi\Sigma$-field (Karr, 1981) |

# Summary: The telescoping problem in fields

- A difference field $(\mathbb{A}, \sigma)$ is a field $\mathbb{A}$ with an automorphism $\sigma : \mathbb{A} \to \mathbb{A}$.
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Given $f \in \mathbb{A}$

Find all $g \in \mathbb{A}$ with

$$\sigma(g) - g = f$$

# Summary: The parameterized telescoping problem in fields

- A difference field $(\mathbb{A}, \sigma)$ is a field $\mathbb{A}$ with an automorphism $\sigma : \mathbb{A} \to \mathbb{A}$.
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Note: $\mathbb{K}$ is a subfield of $\mathbb{A}$.

Given $(f_1, \dots, f_d) \in \mathbb{A}^d$

Find all $g \in \mathbb{A}$ and $(c_1, \dots, c_d) \in \mathbb{K}^d$ with

$$\underbrace{\sigma(g) - g = c_1 f_1 + \dots + c_d f_d}_{\textbf{Parameterized}\text{ telescoping}}$$

# Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow s\,p =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,s\,p}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,s\,p}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $g \in \mathbb{A}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2}$$

We compute

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)s)\,p)}{(1-k+n)\,(2-k+n)}.$$

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$$\underbrace{\mathbf{a_0\, g + a_1\, \sigma(g) + \cdots + a_m\, \sigma^m(g)} = c_1 f_1 + \cdots + c_d f_d}_{\text{Parameterized Linear Difference Equation (PLDE)}}$$

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is a $\mathbb{K}$ subspace of $\mathbb{A} \times \mathbb{K}^d$ with dimension $\leq d + m$.

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Given $(f_1, \dots, f_d) \in \mathbb{A}^d$, $\mathbf{0} \neq (a_0, \dots, a_m) \in \mathbb{A}^{m+1}$

Find a basis of

$$\Big\{(g, c_1, \dots, c_d) \in \mathbb{E} \times \mathbb{K}^d \mid a_0\, g + a_1\, \sigma(g) + \dots + a_m\, \sigma^m(g) = c_1\, f_1 + \dots + c_d\, f_d\Big\}$$

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Given $(f_1, \dots, f_d) \in \mathbb{A}^d$, $\mathbf{0} \neq (a_0, \dots, a_m) \in \mathbb{A}^{m+1}$

Find an appropriate $\mathbb{E} \geq \mathbb{A}$ and a basis of

$$\Big\{(g, c_1, \dots, c_d) \in \mathbb{E} \times \mathbb{K}^d \mid a_0\, g + a_1\, \sigma(g) + \dots + a_m\, \sigma^m(g) = c_1\, f_1 + \dots + c_d\, f_d\Big\}$$

One can find d'Alembertian solutions of parameterized linear difference equations defined in $\Pi\Sigma$-fields

$$\begin{aligned}&\big(1+S_1(n)+nS_1(n)\big)^2\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n)\\-&(1+n)(3+2n)S_1(n)\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2A(n+1)\\&+(1+n)^2(2+n)^3S_1(n)\big(1+S_1(n)+nS_1(n)\big)A(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,S_1(n)\prod_{l=1}^{n}S_1(l)+c_2S_1(n)^2\prod_{l=1}^{n}S_1(l)\mid c_1,c_2\in\mathbb{K}\Big\}$$

[S. Abramov, M. Bronstein, M. Petkovsek, CS, JSC 2021]

One can find d'Alembertian solutions of parameterized linear difference equations defined in $\Pi\Sigma$-fields

$$-2(1+n)^3(3+n)n!^2A(n) + (1+n)\left(8+9n+2n^2\right)n!A(n+1) - A(n+2) = 0$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\left\{c_1\prod_{i=1}^{n} i! + c_2\left(-2^n n!\prod_{i=1}^{n} i! + \tfrac{3}{2}\prod_{i=1}^{n} i!\sum_{i=1}^{n} 2^i i!\right) \mid c_1, c_2 \in \mathbb{K}\right\}$$

[S. Abramov, M. Bronstein, M. Petkovsek, CS, JSC 2021]

# Problem: The PLDE problem in rings

- A **difference ring** $(\mathbb{A}, \sigma)$ **is a ring** $\mathbb{A}$ with an autom. $\sigma : \mathbb{A} \to \mathbb{A}$.

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Note: $\mathbb{K}$ is a **subring** of $\mathbb{A}$.

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Will be a field by construction

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is a $\mathbb{K}$ subspace of $\mathbb{A} \times \mathbb{K}^d$ ~~with dimension $\leq d + m$~~.

One can find d'Alembertian solutions of parameterized linear difference equations defined in $R\Pi\Sigma$-rings (later!)

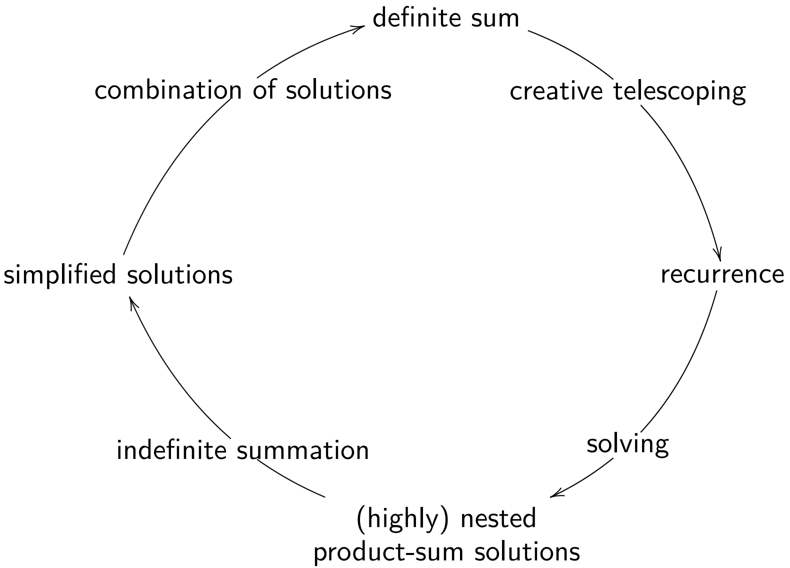
$$\Big[(1+n)(2+n)\Big(\Big(2+n+(1+n)\sum_{i=1}^{n}\frac{1}{i}\Big)(-1)^n-(1+n)^2\sum_{i=1}^{n}\frac{(-1)^i}{i}\Big)\Big]\,A(n)$$
$$+\Big[(1+n)(2+n)\Big(\Big(2+n+2(1+n)\sum_{i=1}^{n}\frac{1}{i}\Big)(-1)^n-(1+n)\sum_{i=1}^{n}\frac{(-1)^i}{i}\Big)\Big]\,A(n+1)$$
$$+\Big[(1+n)^2(2+n)\Big((-1)^n\sum_{i=1}^{n}\frac{1}{i}+n\sum_{i=1}^{n}\frac{(-1)^i}{i}\Big)\Big]A(n+2)$$
$$=(2+n)^2+(1+n)\sum_{i=1}^{n}\frac{1}{i}-2(1+n)^3(-1)^n\sum_{i=1}^{n}\frac{(-1)^i}{i}$$

$$\Big\downarrow \texttt{PLDESolver.m}$$

$$-\sum_{i=1}^{n}\frac{1}{i}(-1)^n-\kappa_1(-1)^n$$
$$+\kappa_2\Big(-2(-1)^n n-(1+4n)(-1)^n\sum_{i=1}^{n}\frac{1}{i}+\sum_{i=1}^{n}\frac{(-1)^i}{i}(1-2n)\Big)$$

[J. Ablinger, CS in Proc. ISSAC'21 and extensions by J. Obrovsky (PhD work in progress)]

# The Sigma-summation spiral



# Outline

1. Part 1: The basic summation tools
   in difference rings

2. Part 2: The underlying difference ring theory

3. Part 3: Applications in combinatorics,
   number theory and particle physics

# Sigma.m is based on difference ring/field theory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Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}}\quad [s]}_{\text{polynomial ring}}$

Simplify
$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{array}{llll} \mathrm{ev}': & \mathbb{Q}(x) \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\frac{p(x)}{q(x)}, n\right) & \mapsto & \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

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$$\text{ev}: \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \to \quad \mathbb{Q}$$

$$\mathbf{ev(s, n) = S_1(n)}$$

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$$\begin{aligned}\mathrm{ev}' : \quad & \mathbb{Q}(x) \times \mathbb{N} & \to \quad & \mathbb{Q} \\ & \left(\tfrac{p(x)}{q(x)}, n\right) & \mapsto \quad & \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

$$\begin{aligned}\mathrm{ev} : \quad & \mathbb{Q}(x)[s] \times \mathbb{N} & \to \quad & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) & \mapsto \quad & \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) S_1(n)^i \qquad \mathbf{ev(s, n) = S_1(n)}\end{aligned}$$

**Definition:** $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify
$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \qquad = \qquad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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$$||$$

$$\langle 0, 1, 1, 1, \ldots \rangle$$

Simplify
$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau: & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f,n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0,1,2,3,\dots \rangle \langle 0,1,\frac{1}{2},\frac{1}{3},\dots \rangle \\ & & \| \\ & & \langle 0,1,1,1,\dots \rangle \\ & & \nparallel \\ \tau(x\frac{1}{x}) = \tau(1) & = & \langle 1,1,1,1,\dots \rangle \end{array}$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) = \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \rangle$$

$$|| $$

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It is an injective ring homomorphism (ring embedding):

$$\tau(x)\tau(\tfrac{1}{x}) = \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

$$|| $$

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3. a ring automorphism

$$\begin{array}{rlcl} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

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$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

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$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

**Definition:** $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism $\sigma$ is called a difference ring; the set of constants is

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

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**In this example:**

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
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$\mathrm{ev}$ and $\sigma$ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

Simplify

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$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

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$\tau$ is an injective difference ring homomorphism:

$$\begin{array}{ccc}
\mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\
\downarrow \tau & = & \downarrow \tau \\
\mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim
\end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

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$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n) \rangle_{n \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}} / \sim, S)$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

$$\boxed{\begin{array}{ccccc} (\mathbb{A},\sigma) & \overset{\tau}{\simeq} & (\tau(\mathbb{A}),S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim, S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}] & & \end{array}}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k)\rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

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Output: $\bar{g} = x\,s - x$

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**Degree bound:** COMPUTE $b \geq 0$:

$$\forall \bar{g} \in \mathbb{Q}(x)[s] \quad \sigma(\bar{g}) - \bar{g} = s \quad \Rightarrow \quad \deg(\bar{g}) \leq b.$$

FIND $\bar{g} \in \mathbb{Q}(x)[s]$:

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**Degree bound:** COMPUTE $b \geq 0$: $b = 2$

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**Polynomial Solution:** FIND

$$\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s].$$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

$$\sigma(\bar{g}) - \bar{g} = s$$

ANSATZ $\bar{g} = g_2\,s^2 + g_1\,s + g_0 \in \mathbb{Q}(x)[s]$

$$\big[\sigma(g_2\,s^2 + g_1 s + g_0)\big] \\ -\big[g_2\,s^2 + g_1 s + g_0\big] = s$$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

$$\left[\sigma(g_2\, s^2) + \sigma(g_1 s + g_0)\right] - \left[g_2\, s^2 + g_1 s + g_0\right] = s$$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

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ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

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ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 s + g_0)\Big] \\ -\big[g_2\, s^2 + g_1 s + g_0\big] = s$$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\Big[\sigma(c)\big(s + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 s + g_0)\Big] - \big[c\, s^2 + g_1 s + g_0\big] = s$$

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$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\Big[c\big(s+\tfrac{1}{x+1}\big)^2+\sigma(g_1 s+g_0)\Big] - \big[c\, s^2+g_1 s+g_0\big] = s$$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\left[\tfrac{2s(x+1)+1}{(x+1)^2}\right]$$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

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coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}}$$

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coeff. comp.

$$\boxed{\sigma(g_2)-g_2=0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s+g_0)-(g_1\, s+g_0) = s - c\Big[\tfrac{2s(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1)-g_1 = 1 - c\,\tfrac{2}{x+1}}$$

$c = 0$, $g_1 = x + d$, $d \in \mathbb{Q}$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

$$\Big[\sigma(g_2)\big(s+\tfrac{1}{x+1}\big)^2 + \sigma(g_1 s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

coeff. comp.

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$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\Big[\tfrac{2s(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}}$$

$c = 0$, $g_1 = x + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{x+1}}$$

ANSATZ $\bar{g} = g_2\, s^2 + g_1\, s + g_0 \in \mathbb{Q}(x)[s]$

$$\Big[\sigma(g_2)\big(s + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 s + g_0)\Big] - \big[g_2\, s^2 + g_1 s + g_0\big] = s$$

g=sx-x

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, s + g_0) - (g_1\, s + g_0) = s - c\Big[\tfrac{2s(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}$$

$c = 0,\quad g_1 = x + d,\ d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{x+1}$$

$\leftarrow$ $g_0 = -x$, $d = 0$

# Summary: The PLDE problem in fields

- A difference field $(\mathbb{A}, \sigma)$ is a field $\mathbb{A}$ with an automorphism $\sigma : \mathbb{A} \to \mathbb{A}$.
- The set of constants is defined by

$$\mathbb{K} = \mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Note: $\mathbb{K}$ is a subfield of $\mathbb{A}$.

Given $(f_1, \dots, f_d) \in \mathbb{A}^d$, $\mathbf{0} \neq (a_0, \dots, a_m) \in \mathbb{A}^{m+1}$

Find an appropriate $\mathbb{E} \geq \mathbb{A}$ and a basis of

$$\Big\{(g, c_1, \dots, c_d) \in \mathbb{E} \times \mathbb{K}^d \mid a_0\, g + a_1\, \sigma(g) + \dots + a_m\, \sigma^m(g) = c_1\, f_1 + \dots + c_d\, f_d\Big\}$$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \quad \leq \quad (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$||$$

$$\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}]$$

# A general framework: $R\Pi\Sigma$-rings

1. Definition and telescoping

# Definition of an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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## Definition of an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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(k+1)!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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hypergeometric products $\quad \leftrightarrow \quad \sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$

## Definition of an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

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| | | | |
|---|---|---|---|
| (nested) hyperg. | $\leftrightarrow$ | $\sigma(p_1) = a_1\, p_1$ | $a_1 \in \mathbb{K}(x)^*$ |
| products | | $\sigma(p_2) = a_2 p_2$ | $a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^*$ |

# Definition of an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned}
\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\
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&\vdots\\
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$$\begin{aligned}\sigma(p_1)&=a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2)&=a_2p_2 && a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*\\ &\vdots\\ \sigma(p_e)&=a_ep_e && a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*\end{aligned}$$

$$(-\mathbf{1})^{\mathbf{k}} \quad\leftrightarrow\quad \sigma(\mathbf{z})=-\mathbf{z} \qquad \mathbf{z}^{\mathbf{2}}=\mathbf{1}$$

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$\alpha$ is a primitive $\lambda$th root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \quad \sigma(\mathbf{z}) = \alpha\,\mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$

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$$S_1(k+1) = S_1(k) + \frac{1}{k+1} \quad \leftrightarrow \quad \sigma(s_1) = s_1 + \frac{1}{x+1}$$

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(nested) sum $\leftrightarrow$

$$\sigma(s_1) = s_1 + f_1 \quad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z]$$

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$$\sigma(\mathbf{z})=\alpha\,\mathbf{z}\qquad \mathbf{z}^\lambda=\mathbf{1}$$

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## Definition of an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

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such that $\mathrm{const}_\sigma\mathbb{A} = \{c\in\mathbb{A}|\sigma(c)=c\} = \mathbb{K}$.

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$\alpha$ is a primitive $\lambda$th root of unity

**GIVEN** $f \in \mathbb{A}$;
**FIND**, in case of existence, a $g \in \mathbb{A}$ such that
$$\sigma(g) - g = f.$$

(nested) sum $\leftrightarrow$

$$\sigma(s_1) = s_1 + f_1 \qquad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z]$$
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# A general framework: $R\Pi\Sigma$-rings

1. Definition and telescoping
2. Crucial (equivalent) characterizations

## Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

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- with an automorphism as given in the previous slide.

**Theorem.** The following statements are equivalent:

1. $\text{const}_\sigma \mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

CS. Parameterized telescoping proves algebraic independence of sums. In: Proc. FPSAC'07. 2007.
CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

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**Theorem.** The following statements are equivalent:

1. $\text{const}_\sigma \mathbb{A} = \mathbb{K}$.

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2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under $\sigma$ except $\{0\}$ and $\mathbb{A}$)

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3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

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Remark 1: Related results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997) and (Hardouin/Singer, 2008)

## Galois theory for $R\Pi\Sigma$-extensions

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**Theorem.** The following statements are equivalent:

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Remark 2: Theory covers also the $q$-hypergeometric, mutli-basic and mixed cases

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of ΠΣ-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

$\mathrm{ev}$ and $\sigma$ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

$\tau$ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n)\rangle_{n\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

# Definition of an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

We get an embedding $\tau$ from $(\mathbb{A},\sigma)$ into the ring of sequences s.t.

$$\tau(\mathbb{A}) = \underbrace{\tau(\mathbb{K}(x))}_{\text{rational seq.}}\ [\langle\alpha^k\rangle_{k\geq 0}]\underbrace{[\tau(p_1),\tau(p_1^{-1})]\dots[\tau(p_e),\tau(p_e^{-1})]}_{\text{nested products}}\underbrace{[\tau(s_1)]\dots[\tau(s_r)]}_{\text{nested sums}}$$

$\alpha$ is a primitive $\lambda$th root of unity $\quad \alpha^{\mathbf{k}} \quad \leftrightarrow \quad \sigma(\mathbf{z})=\alpha\mathbf{z} \qquad \mathbf{z}^\lambda=\mathbf{1}$

(nested) sum $\leftrightarrow$

$$\begin{aligned}
\sigma(s_1) &= s_1+f_1 & f_1 &\in \mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]\\
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\end{aligned}$$

such that $\text{const}_\sigma\mathbb{A} = \{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

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$$\vdots$$

such that $\text{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

# Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

$$\sigma(s_i) = s_i + \frac{1}{(x+1)^i} \qquad i = 1, 2, 3, \dots$$

[J. Ablinger, CS. Algebraic independence of sequences generated by (cyclotomic) harmonic sums. Ann. Comb. 22(2), 2018.]

# Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

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2. There is an embedding of the polynomial ring $\mathbb{Q}(x)[s_1, s_2, \dots]$ into $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$s_1 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n \geq 0}, \qquad s_2 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n \geq 0} \qquad \dots$$

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$\Rightarrow$ The generalized harmonic numbers

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}, \quad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}, \quad S_3(n) = \sum_{i=1}^{n} \frac{1}{i^3}, \quad \dots$$

are algebraically independent among each other over the rational sequences.

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# A general framework: $R\Pi\Sigma$-rings

1. Definition and telescoping
2. Crucial (equivalent) characterizations
3. Algorithmic construction and simplification
   3.1 The sum case
   3.2 The product case

## Definition of an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2][s_3]\dots$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_{e-1}, p_{e-1}^{-1}]^* \end{aligned}$$

$\alpha$ is a primitive $\lambda$th root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \quad \sigma(\mathbf{z}) = \alpha\, \mathbf{z} \qquad \mathbf{z}^\lambda = \mathbf{1}$

(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

such that $\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} | \sigma(c) = c\} = \mathbb{K}$.

# Represent sums (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\text{const}_\sigma\mathbb{A} := \{k \in \mathbb{A} \mid \sigma(k) = k\}.$$

**Note 1:** $\text{const}_\sigma\mathbb{A}$ is a ring that contains $\mathbb{Q}$

**Note 2:** We always take care that $\text{const}_\sigma\mathbb{A}$ is a field

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There are 2 cases:

1. $\boxed{\nexists g \in \mathbb{A}: \ \sigma(g) = g + f}$: $(\mathbb{A}[s],\sigma)$ is a $\Sigma^*$-extension of $(\mathbb{A},\sigma)$
2. $\boxed{\exists g \in \mathbb{A}: \sigma(g) = g + f}$: No need for a $\Sigma^*$-extension!

## The naive Ansatz (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-ring/field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.
($\mathbb{A}$ an integral domain)

FIND, in case of existence, $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

## Still a naive Ansatz (in rings) for simplification

GIVEN an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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## Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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## Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

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appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{n}\Big(\frac{-2+k}{10(1+k^2)} + \frac{(1-4k-2k^2)S_1(k)}{10(1+k^2)(2+2k+k^2)} + \frac{(1-4k-2k^2)S_3(k)}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$= \frac{n^2+4n+5}{10(n^2+2n+2)}S_1(n) - \frac{(n-1)(n+1)}{5(n^2+2n+2)}S_3(n) - \frac{2}{5}\sum_{k=1}^{n}\frac{1}{k^2}$$

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appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1

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$$\sigma(g) - g = f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k S_1(k)^2 \binom{n}{k} = -\frac{1}{n} \sum_{i=1}^{a} \frac{(-1)^i}{i} \binom{n}{i} - (a-n)\left(n^2 S_1(a)^2 + 2nS_1(a) + 2\right) \frac{(-1)^a \binom{n}{a}}{n^3} - \frac{2}{n^2}$$

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3. $\boxed{\exists g \in \mathbb{A}\setminus\{0\} : \sigma(g) = a^n\, g \text{ only for } n \in \mathbb{Z} \setminus \{0, 1\}}$: ☹

# The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $a_1, \dots, a_e \in \mathbb{K}(x)^*$

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$$\mathbb{E}$$

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$$\mathbb{E} = \mathbb{K}(x) \underbrace{[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}]}_{\text{tower of } \Pi\text{-ext.}} \quad \underbrace{[z]}_{(-1)^k \text{ or } \gamma^k}$$

with

- $\frac{\sigma(p_i)}{p_i} \in \mathbb{K}(x)^*$ for $1 \leq i \leq e$
- $\sigma(z) = \gamma\, z$ and $z^\lambda = 1$ for some primitive $\lambda$th root of unity $\gamma \in \mathbb{K}^*$
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Note: There are similar results for the $q$-rational, multi-basic and mixed case and their nested versions

# Simplification to products of minimal size

Given

$$y_1 = \prod_{k=1}^{n} \frac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \frac{26244k^2(2+k)^2}{(3+k)^2},$$

$$y_3 = \prod_{k=1}^{n} \frac{\mathbb{i}k(2+k)^3}{729(5+k)}, \qquad y_4 = \prod_{k=1}^{n} -\frac{162k(2+k)}{5+k}$$

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we can compute

$$y_1 = \frac{216\,(\mathbb{i}^n)^2\,2^n\,(3^n)^8}{(n+1)^2(n+2)^3(n+3)^3 n!}, \quad y_2 = \frac{9\,(2^n)^2\,(3^n)^8\,(n!)^2}{(n+3)^2},$$

$$y_3 = \frac{15(n+1)^2(n+2)^2\mathbb{i}^n(n!)^3}{(n+3)(n+4)(n+5)\,(3^n)^6}, \quad y_4 = \frac{60\,(\mathbb{i}^n)^2\,2^n\,(3^n)^4\,n!}{(n+3)(n+4)(n+5)}.$$

in terms of

$$\underbrace{n! = \prod_{k=1}^{n} k, \quad 2^n, \quad 3^n}_{\text{transcendental sequences}} \qquad \text{and} \quad \mathbb{i}^n$$

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Given

$$y_1 = \prod_{k=1}^{n} \frac{-13122k(1+k)}{(3+k)^3}, \qquad y_2 = \prod_{k=1}^{n} \frac{26244k^2(2+k)^2}{(3+k)^2},$$

$$y_3 = \prod_{k=1}^{n} \frac{\mathbb{i}k(2+k)^3}{729(5+k)}, \qquad y_4 = \prod_{k=1}^{n} -\frac{162k(2+k)}{5+k}$$

we can compute

$$y_1 = \frac{5(1+n)^2(2+n)^5(3+n)^8}{52488(4+n)(5+n)}(-1)^n\Phi_1\Phi_2^{-2}, \quad y_2 = \frac{(4+n)^2(5+n)^2}{400}\Phi_1^2,$$

$$y_3 = \frac{2754990144(4+n)^2(5+n)^2}{25(1+n)^4(2+n)^{10}(3+n)^{16}}\Phi_2^3, \qquad y_4 = \Phi_1.$$

in terms of

$$\underbrace{\Phi_1 = \prod_{k=1}^{n} \frac{-162k(2+k)}{5+k}, \quad \Phi_2 = \prod_{k=1}^{n} \frac{-\mathbb{i}(3+k)^6}{9k(1+k)^2(2+k)(5+k)}}_{\textbf{minimal number}\text{ of transcendental sequences}} \quad \text{and} \quad \underbrace{(-1)^n}_{\textbf{minimal order}}$$

# A general framework: $R\Pi\Sigma$-rings

1. Definition and telescoping
2. Crucial (equivalent) characterizations
3. Algorithmic construction and simplification
   3.1 The sum case
   3.2 The product case
4. The full machinery: `SigmaReduce`

# Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$$\Big\downarrow \quad \texttt{SigmaReduce[A,n]}$$

$B(n)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

($\delta$ can be computed explicitly)

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- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

($\delta$ can be computed explicitly)

- and such that

the arising sums and products in $B(n)$ are

- algebraically independent
- simplified as illustrated above.

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**Application 1:** the expression $B(n)$ is usually much smaller

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**Application 2:** we solve the zero-recognition problem:

$$A(n) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B = 0$$

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**Application 2:** we solve the zero-recognition problem:

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**Application 3:** we get canonical form representations

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
$n$: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

# Outline

1. Part 1: The basic summation tools in difference rings

2. Part 2: The underlying difference ring theory

3. Part 3: Applications in combinatorics, number theory and particle physics

# Multi-summation approaches (no claim to completeness)

- Difference ring/field approach (Part 1 and 2)
- Sister Celine/WZ method
  - Sister Celine Fasenmyer. Some generalized hypergeometric polynomials. PhD-thesis. 1945.
  - H. Wilf, D. Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. 1992.
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  - R. Lyons, P. Paule, and A. Riese. A computer proof of a series evaluation in terms of harmonic numbers. Appl. Algebra Engrg. Comm. Comput., 13:327–333, 2002.
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- Holonomic system approach
  - D. Zeilberger. A holonomic systems approach to special functions identities. J. Comput. Appl. Math., 32:321–368, 1990.
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- Reduction based methods (only recent results)
  - A. Bostan, S. Chen, F. Chyzak, and Z. Li. Complexity of creative telescoping for bivariate rational functions. In ISSAC'10, pages 203–210, 2010.
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- Generating function approach
  - A. Bostan, P. Lairez, B. Salvy Multiple binomial sums J. Symb. Comput., 80 (2017), pp. 351-386

# Part 3: Applications

- combinatorics
- special functions
- number theory
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- numerics
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- elementary particle physics (QCD)

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# Highlights related to combinatorial problems

- Plane Partitions VI: Stembridge's TSPP Theorem
  (joint with G.E. Andrews, P. Paule; 2005)
- Unfair permutations
  (joint with H. Prodinger, S. Wagner, 2011)
- Asymptotic and exact results on the complexity of the Novelli-Pak-Stoyanovskii algorithm
  (joint with R. Sulzgruber; 2017)
- Evaluation of binomial double sums involving absolute values
  (joint with C. Krattenthaler; 2020)
- The Absent-Minded Passengers Problem
  (challenged by Doron, 2021)

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- We get a random permutation

$$\begin{matrix}\text{player} \\ \text{dices}\end{matrix} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

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$$\begin{matrix}\text{player} \\ \text{dices}\end{matrix} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

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$i < j$ and $a_i < a_j$

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$$\boxed{\sum_{1 \le i < j \le n} \frac{j}{i+j}}$$

**Theorem (Prodinger, Wagner).**

$A_n =$ no. of anti-inversions of a random unfair permutation of length $n$.

Then the mean of $A_n$ is

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$$= 0.3465735903n^2 - 0.4034264097n + O(\log n)$$

$$\text{fair case} = 0.25n^2 - 0.25n$$

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The variance of $A_n$ is

$$2\sum_{1\le i<j<k\le n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\le i<j<k\le n} \frac{k}{i+j+k}$$
$$+2\sum_{1\le i<j<k\le n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\le i<j<k\le n} \frac{kj}{(i+k)(i+j+k)}$$
$$-2\sum_{1\le i<j<k\le n} \frac{j}{i+j}\cdot\frac{k}{j+k} - 2\sum_{1\le i<j<k\le n} \frac{k}{i+k}\cdot\frac{k}{j+k}$$
$$-2\sum_{1\le i<j<k\le n} \frac{j}{i+j}\cdot\frac{k}{i+k} - \sum_{1\le i<j\le n} \frac{j^2}{(i+j)^2} + \sum_{1\le i<j\le n} \frac{j}{i+j}$$

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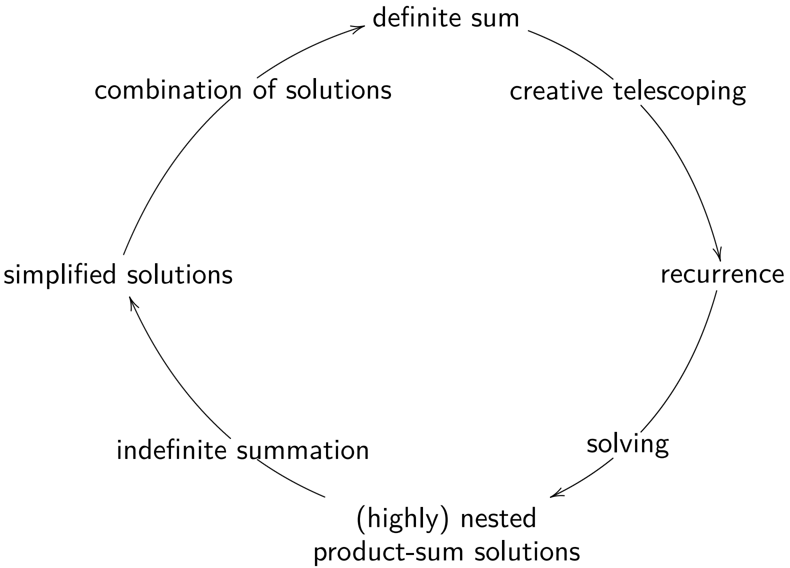
$$\sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{jk}{(i+j)(j+k)}$$

$$\sum_{k=3}^{n} \sum_{j=2}^{k-1} \boxed{\sum_{i=1}^{j-1} \frac{jk}{(i+j)(j+k)}}$$

# The Sigma-summation spiral



$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\boxed{\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}}$$

$$||$$

summation spiral

$$\boxed{\frac{1}{j+k}jk\sum_{r=1}^{j}\frac{1}{-1+2r}-\frac{jkS_1(j)}{2(j+k)}-\frac{k}{2(j+k)}}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\Big[\frac{1}{j+k}jk\sum_{r=1}^{j}\frac{1}{-1+2r}-\frac{jkS_1(j)}{2(j+k)}-\frac{k}{2(j+k)}\Big]$$

$$\sum_{k=3}^{n} \boxed{\sum_{j=2}^{k-1} \left[\frac{1}{j+k} jk \sum_{r=1}^{j} \frac{1}{-1+2r} - \frac{jkS_1(j)}{2(j+k)} - \frac{k}{2(j+k)}\right]}$$

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$$||$$

summation spiral

$$\boxed{\begin{aligned} & - k^2 \sum_{s=1}^{k} \frac{\sum_{r=1}^{s} \frac{1}{-1+2r}}{s} + \big((k-1)k + k^2 S_1(k)\big) \sum_{r=1}^{k} \frac{1}{-1+2r} \\ & - \frac{1}{4} k^2 S_1(k)^2 - \frac{1}{4} k^2 S_2(k) - \frac{1}{4} k(2k-3) S_1(k) + \frac{1}{4} \end{aligned}}$$

$$\sum_{k=3}^{n}\Big[\quad -k^2\sum_{s=1}^{k}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}+\big((k-1)k+k^2S_1(k)\big)\sum_{r=1}^{k}\frac{1}{-1+2r}$$

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$$||$$

summation spiral

$$n(n+1)(2n+1)\Big[-\frac{1}{6}\sum_{s=1}^{n}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}-\frac{1}{12})S_1(n)-\frac{1}{24}S_1(n)^2$$
$$+(\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}-\frac{1}{24}S_2(n)+\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}\Big]$$
$$-\frac{1}{8}(2n+1)^2\sum_{r=1}^{n}\frac{1}{-1+2r}+\frac{1}{12}(n+1)(4n+1)S_1(n)+\frac{7n}{24}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}$$

$$||$$

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$$-\frac{1}{8}(2n+1)^2\sum_{r=1}^{n}\frac{1}{-1+2r}+\frac{1}{12}(n+1)(4n+1)S_1(n)+\frac{7n}{24}$$

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$$\begin{aligned}
&\frac{n(29+126n+72n^2)}{216} + \frac{35+108n+81n^2-27n^3}{162}S_1(n) \\
&+ \frac{-3-16n-10n^2+8n^3}{12}S_1(2n) + \frac{-16+27n-54n^3}{108}S_1(3n) \\
&+ \frac{n(1+3n+2n^2)}{6}\left(3S_2(2n) - 2S_2(n) + 4\sum_{1\le i\le 2n}\frac{(-1)^i S_1(i)}{i}\right) \\
&+ \frac{8}{27}\sum_{i=1}^{n}\frac{1}{3i-2} + \frac{(-1)^n n}{4}\left(\sum_{i=1}^{n}\frac{(-1)^i}{i} - \sum_{i=1}^{3n}\frac{(-1)^i}{i}\right),
\end{aligned}$$

# Part 3: Applications

- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

# Highlights related to number theory

- Apéry's double sum is plain sailing indeed (2007)
- When is 0.999... equal to 1?
  (joint with R. Pemantle; 2007)
- Gaussian hypergeometric series and extensions of supercongruences
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- A case study for $\zeta(4)$
  (joint with W. Zudilin; 2021)
- Error bounds for the asymptotic expansion of the partition function
  [compare Hardy–Ramanujan, Wright, Rademacher, Lehmer, O'Sullivan]
  (joint with K. Banerjee, P. Paule, C.-S. Radu; 2023)
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- Reduction based methods (only recent results)
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  - A. Bostan, P. Lairez, and B. Salvy. Creative telescoping for rational functions using the Griffiths-Dwork method. In ISSAC'13, pages 93–100, 2013.
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- Generating function approach
  - A. Bostan, P. Lairez, B. Salvy Multiple binomial sums J. Symb. Comput., 80 (2017), pp. 351-386

# Multi-summation approaches (no claim to completeness)

- Difference ring/field approach (Part 1 and 2)
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  - Sister Celine Fasenmyer. Some generalized hypergeometric polynomials. PhD-thesis. 1945.
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# A simplified holonomic approach
# – parameterized telescoping

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=:\, P(k)(=\, P^{(1,-1)}(x))}.$$

FIND a closed form for $S(n)$.

Arose in joint cooperation with the JKU-Finite Element group.

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=:\, P(k)(=\, P^{(1,-1)}(x))}.$$

Intermediate step: FIND a recurrence for $P(k)$.

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=:\, P(k)(=\, P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND a closed form for S(n).

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j! k!} \left(\frac{1-x}{2}\right)^j}^{=: P(k)(= P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

FIND $g(k) := g_0(k)P(k) + g_1(k)P(k+1)$:

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=:\, P(k) (= P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

$$g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

# Telescoping

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$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=: P(k) (= P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

$$g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

VERIFICATION:

$$g(k) \xrightarrow{\text{depends on}} P(k), P(k+1)$$

$$g(k+1) \xrightarrow{\text{depends on}} P(k+1), P(k+2).$$

# Telescoping

GIVEN

$$S(n) := \sum_{k=1}^{n} \frac{2k+1}{k+1} \overbrace{\sum_{j=0}^{k} \frac{(-k)_j (k+1)_j (2)_{k-j}}{j!k!} \left(\frac{1-x}{2}\right)^j}^{=:\, P(k)(= P^{(1,-1)}(x))}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

Sigma computes $g(k) = \frac{1+k-x-2kx}{(x-1)(k+1)} P(k) + \frac{1}{x-1} P(k+1)$:

$$\boxed{g(k+1) - g(k) = \frac{2k+1}{k+1} P(k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$.

Summing this equation over $k$ from $1$ to $n$ gives:

$$\sum_{k=1}^{n} \frac{2k+1}{k+1} P(k) = -\frac{x+1}{x-1} - \frac{nP(n)}{(n+1)(x-1)} + \frac{P(n+1)}{x-1}.$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

$$\Updownarrow$$

$$\Big[g_0(k+1)P(k+1) + g_1(k+1)P(k+2)\Big] - \Big[g_0(k)P(k) + g_1(k)P(k+1)\Big] = \frac{2k+1}{k+1}P(k)$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

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$$\Big[g_0(k+1)P(k+1) + g_1(k+1)P(k+2)\Big] - \Big[g_0(k)P(k) + g_1(k)P(k+1)\Big] = \frac{2k+1}{k+1}P(k)$$

GIVEN

$$P(k+2)=\frac{(2k+3)x}{k+2}P(k+1)-\frac{k}{k+1}P(k).$$

ANSATZ: $g(k)=g_0(k)P(k)+g_1(k)P(k+1)$ such that

$$g(k+1)-g(k)=\frac{2k+1}{k+1}P(k).$$

$$\Updownarrow$$

$$\Big[g_0(k+1)P(k+1)+g_1(k+1)P(k+2)\Big]-\Big[g_0(k)P(k)+g_1(k)P(k+1)\Big]=\frac{2k+1}{k+1}P(k)$$

$$\Updownarrow$$

$$P(k)\big[-\frac{k}{k+1}g_1(k+1)-g_0(k)-\frac{2k+1}{k+1}\big]$$
$$+P(k+1)\big[g_0(k+1)+\frac{(2k+3)x}{k+2}g_1(k+1)-g_1(k)\big]=0$$

GIVEN

$$P(k+2)=\frac{(2k+3)x}{k+2}P(k+1)-\frac{k}{k+1}P(k).$$

ANSATZ: $g(k)=g_0(k)P(k)+g_1(k)P(k+1)$ such that

$$g(k+1)-g(k)=\frac{2k+1}{k+1}P(k).$$

$$\Updownarrow$$

$$\Big[g_0(k+1)P(k+1)+g_1(k+1)P(k+2)\Big]-\Big[g_0(k)P(k)+g_1(k)P(k+1)\Big]=\frac{2k+1}{k+1}P(k)$$

$$\Updownarrow$$

$$P(k)\Big[-\frac{k}{k+1}g_1(k+1)-g_0(k)-\frac{2k+1}{k+1}\Big]$$
$$+P(k+1)\Big[g_0(k+1)+\frac{(2k+3)x}{k+2}g_1(k+1)-g_1(k)\Big]=0$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2} P(k+1) - \frac{k}{k+1} P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1} P(k).$$

$$\Uparrow$$

$$g_0(k) = -\frac{k}{k+1} g_1(k+1) - \frac{2k+1}{k+1}$$

$$g_0(k+1) + \frac{(2k+3)x}{k+2} g_1(k+1) - g_1(k) = 0$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

$$\Uparrow$$

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$\downarrow \mathcal{S}_k g_0(k)$

$$g_0(k+1) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = 0$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

$$\Uparrow$$

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$-\frac{k+1}{k+2}g_1(k+2) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = \frac{2k+3}{k+2}$$

GIVEN

$$P(k+2)=\frac{(2k+3)x}{k+2}P(k+1)-\frac{k}{k+1}P(k).$$

ANSATZ: $g(k)=g_0(k)P(k)+g_1(k)P(k+1)$ such that

$$g(k+1)-g(k)=\frac{2k+1}{k+1}P(k).$$

$$\Uparrow$$

$$g_0(k)=-\frac{k}{k+1}g_1(k+1)-\frac{2k+1}{k+1}$$

$$-\frac{k+1}{k+2}g_1(k+2)+\frac{(2k+3)x}{k+2}g_1(k+1)-g_1(k)=\frac{2k+3}{k+2}$$

Sigma computes:

$$g_1(k)=\frac{1}{x-1},$$

GIVEN

$$P(k+2) = \frac{(2k+3)x}{k+2}P(k+1) - \frac{k}{k+1}P(k).$$

ANSATZ: $g(k) = g_0(k)P(k) + g_1(k)P(k+1)$ such that

$$g(k+1) - g(k) = \frac{2k+1}{k+1}P(k).$$

$$\Uparrow$$

$$g_0(k) = -\frac{k}{k+1}g_1(k+1) - \frac{2k+1}{k+1}$$

$$-\frac{k+1}{k+2}g_1(k+2) + \frac{(2k+3)x}{k+2}g_1(k+1) - g_1(k) = \frac{2k+3}{k+2}$$

Sigma computes:

$$g_1(k) = \frac{1}{x-1}, \qquad g_0(k) = \frac{1+k-x-2kx}{(x-1)(k+1)}.$$

# General case: telescoping

GIVEN $f(k) := h(k)P(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ :

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\boxed{\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = h(k+1).}$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - h(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s.$$

# General case: telescoping

GIVEN $f(k) := h_0(k)P(k) + \cdots + h_s(k)P(k+s)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ :

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j)$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - h_0(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - h_r(k), \qquad 0 < r < s.$$

# General case: parameterized telescoping

GIVEN $f_0(k) := h_0^{(0)}(k)P(k) + \cdots + h_s^{(0)}(k)P(k+s), \ldots,$
$f_d(k) := h_0^{(d)}(k)P(k) + \cdots + h_s^{(d)}(k)P(k+s)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ and $c_0, \ldots, c_d$:

$$g(k+1) - g(k) = c_0 f_0(k) + \cdots + c_d f(k).$$

1. FIND a solution $g_s(k)$ and $c_0, \ldots, c_d$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = c_0 \sum_{j=0}^{s} h_{s-j}^{(0)}(k+j) + \cdots + c_d \sum_{j=0}^{s} h_{s-j}^{(d)}(k+j)$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - \sum_{i=0}^{d} c_i h_0^{(i)}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - \sum_{i=0}^{d} c_i h_0^{(r)}(k), \qquad 0 < r < s.$$

# Summary: The PLDE problem in fields

- A difference field $(\mathbb{A},\sigma)$ is a field $\mathbb{A}$ with an automorphism $\sigma:\mathbb{A}\to\mathbb{A}$.
- The set of constants is defined by

$$\mathbb{K}=\text{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}\mid\sigma(c)=c\}$$

Note: $\mathbb{K}$ is a subfield of $\mathbb{A}$.

Given $(f_1,\dots,f_d)\in\mathbb{A}^d$, $\mathbf{0}\neq(a_0,\dots,a_m)\in\mathbb{A}^{m+1}$

Find an appropriate $\mathbb{E}\geq\mathbb{A}$ and a basis of

$$\Big\{(g,c_1,\dots,c_d)\in\mathbb{E}\times\mathbb{K}^d\mid a_0\,g+a_1\,\sigma(g)+\dots+a_m\,\sigma^m(g)=c_1\,f_1+\dots+c_d\,f_d\Big\}$$

# General case: parameterized telescoping

GIVEN $f_0(k) := h_0^{(0)}(k)P(k) + \cdots + h_s^{(0)}(k)P(k+s), \ldots,$
$f_d(k) := h_0^{(d)}(k)P(k) + \cdots + h_s^{(d)}(k)P(k+s)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ and $c_0, \ldots, c_d$:

$$g(k+1) - g(k) = c_0 f_0(k) + \cdots + c_d f(k).$$

1. FIND a solution $g_s(k)$ and $c_0, \ldots, c_d$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = c_0 \sum_{j=0}^{s} h_{s-j}^{(0)}(k+j) + \cdots + c_d \sum_{j=0}^{s} h_{s-j}^{(d)}(k+j)$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - \sum_{i=0}^{d} c_i h_0^{(i)}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - \sum_{i=0}^{d} c_i h_0^{(r)}(k), \qquad 0 < r < s.$$

# Application: multi-summation

# Definite summation

Show that

$$\underbrace{\sum_{k=0}^{n}\sum_{s=0}^{n}(-1)^{n+k+s}\binom{n}{k}\binom{n}{s}\binom{n+k}{k}\binom{n+s}{s}\binom{2n-s-k}{n}}_{=P(n,k)}=\sum_{k=0}^{n}\binom{n}{k}^4.$$

($A = B$, M. Petkovšek, H.S. Wilf, and D. Zeilberger)

# Definite summation

Show that

$$\sum_{k=0}^{n}\underbrace{\sum_{s=0}^{n}(-1)^{n+k+s}\binom{n}{k}\binom{n}{s}\binom{n+k}{k}\binom{n+s}{s}\binom{2n-s-k}{n}}_{=P(n,k)}=\sum_{k=0}^{n}\binom{n}{k}^4.$$

Proof. We COMPUTE for both sides the same recurrence

$$\begin{aligned}&-4(1+n)(3+4n)(5+4n)S(n)\\&\qquad -2(3+2n)(7+9n+3n^2)S(1+n)+(2+n)^3S(2+n)=0.\end{aligned}$$

Both sides agree at $n=0,1$. □

# Creative telescoping

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$

$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n)$:

$$g(n,k+1) - g(n,k) = c_0(n)P(n,k)$$

Sigma:

# Creative telescoping

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n)$:

$$g(n,k+1) - g(n,k) = c_0(n)P(n,k)$$

Sigma:

No solution

# Creative telescoping

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$

$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n)$:

$$g(n,k+1) - g(n,k) = c_0(n)P(n,k) + c_1(n)P(n+1,k)$$

Sigma:

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FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n)$:

$$\begin{aligned} g(n,k+1) - g(n,k) =& c_0(n)P(n,k) + \\ & c_1(n)\Big[b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1)\Big] \end{aligned}$$

Sigma:

# Creative telescoping

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$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$

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FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n)$:

$$g(n,k+1) - g(n,k) = c_0(n)P(n,k) + c_1(n)\Big[b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1)\Big]$$

Sigma:

No solution

# Creative telescoping

GIVEN

$$P(n,k+2)=a_0(n,k)P(n,k)+a_1(n,k)P(n,k+1)$$

$$P(n+1,k)=b_0(n,k)P(n,k)+b_1(n,k)P(n,k+1).$$

FIND $g(n,k)=g_0(n,k)P(n,k)+g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n),\ c_2(n)$:

$$\begin{aligned} g(n,k+1)-g(n,k)=&c_0(n)P(n,k)+\\ &c_1(n)P(n+1,k)+\\ &c_2(n)P(n+2,k) \end{aligned}$$

Sigma:

# Creative telescoping

GIVEN

$$P(n, k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$

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FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n),\ c_2(n)$:

$$\begin{aligned} g(n,k+1) - g(n,k) = & c_0(n)P(n,k) + \\ & c_1(n)\Big[b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1)\Big] + \\ & c_2(n)\Big[\beta_0(n,k)P(n,k) + \beta_1(n,k)P(n,k+1)\Big] \end{aligned}$$

Sigma:

# Creative telescoping

GIVEN

$$P(n,k+2) = a_0(n,k)P(n,k) + a_1(n,k)P(n,k+1)$$
$$P(n+1,k) = b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1).$$

FIND $g(n,k) = g_0(n,k)P(n,k) + g_1(n,k)P(n,k+1),\ c_0(n),\ c_1(n),\ c_2(n)$:

$$\begin{aligned} g(n,k+1) - g(n,k) =& c_0(n)P(n,k) + \\ & c_1(n)\Big[b_0(n,k)P(n,k) + b_1(n,k)P(n,k+1)\Big] + \\ & c_2(n)\Big[\beta_0(n,k)P(n,k) + \beta_1(n,k)P(n,k+1)\Big] \end{aligned}$$

Sigma: Solution

# The holonomic-difference ring approach – inhomogeneous contributions

# General case: inhomogeneous contributions

GIVEN $f(k) := h_0(k)P(k) + \cdots + h_s(k)P(k+s)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s) + a_{s+1}(k).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ :

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j).$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - h_0(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s.$$

# General case: inhomogeneous contributions

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FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s)$ :
$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for
$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j).$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:
$$g_0(k) = a_0(k)g_s(k+1) - h_0(k),$$
$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s.$$

# General case: inhomogeneous contributions

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and

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FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s) + g_{s+1}(k)$ :

$$g(k+1) - g(k) = f(k).$$

1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j).$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$\begin{aligned} g_0(k) &= a_0(k)g_s(k+1) - h_0(k), \\ g_r(k) &= a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s. \end{aligned}$$

# General case: inhomogeneous contributions

GIVEN $f(k) := h_0(k)P(k) + \cdots + h_s(k)P(k+s) + h_{s+1}(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s) + a_{s+1}(k).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s) + g_{s+1}(k)$ :

$$g(k+1) - g(k) = f(k).$$

1.1. FIND a solution $g_s(k)$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} h_{s-j}(k+j).$$

1.2. FIND a solution $g_{s+1}(k)$

$$\boxed{g_{s+1}(k+1) - g_{s+1}(k) = h_{s+1}(k) - a_{s+1}(k)g_s(k+1)}$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$\begin{aligned} g_0(k) &= a_0(k)g_s(k+1) - h_0(k), \\ g_r(k) &= a_r(k)g_s(k+1) + g_{r-1}(k+1), \qquad 0 < r < s. \end{aligned}$$

# General case: inhomogeneous contributions

GIVEN $f_0(k) := h_0^{(0)}(k)P(k) + \cdots + h_s^{(0)}(k)P(k+s) + h_{s+1}^{(0)}(k), \ldots,$

$f_d(k) := h_0^{(d)}(k)P(k) + \cdots + h_s^{(d)}(k)P(k+s) + h_{s+1}^{(d)}(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \cdots + a_s(k)P(k+s) + a_{s+1}(k).$$

FIND $g(k) = g_0(k)P(k) + \cdots + g_s(k)P(k+s) + g_{s+1}(k)$ and $c_0, \ldots, c_d$:

$$g(k+1) - g(k) = c_0 f_0(k) + \cdots + c_d f(k).$$

1.1. FIND a solution $g_s(k)$ and $c_0, \ldots, c_d$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = c_0 \sum_{j=0}^{s} h_{s-j}^{(0)}(k+j) + \cdots + c_d \sum_{j=0}^{s} h_{s-j}^{(d)}(k+j)$$

1.2. FIND a solution $g_{s+1}(k)$

$$\boxed{g_{s+1}(k+1) - g_{s+1}(k) = c_0 f_{s+1}^{(0)}(k) + \cdots + c_d f_{s+1}^{(d)}(k) - a_{s+1}(k)g_s(k+1)}$$

2. COMPUTE the remaining $g_0, \ldots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - \sum_{i=0}^{d} c_i h_0^{(i)}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - h_r(k) - \sum_{i=0}^{d} c_i h_0^{(r)}(k), \qquad 0 < r < s.$$

# Summary: The parameterized telescoping problem in fields

- A difference field $(\mathbb{A}, \sigma)$ is a field $\mathbb{A}$ with an automorphism $\sigma : \mathbb{A} \to \mathbb{A}$.

- The set of constants is defined by

$$\mathbb{K} = \text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Note: $\mathbb{K}$ is a subfield of $\mathbb{A}$.

Given $(f_1, \dots, f_d) \in \mathbb{A}^d$

Find all $g \in \mathbb{A}$ and $(c_1, \dots, c_d) \in \mathbb{K}^d$ with

$$\underbrace{\sigma(g) - g = c_1\, f_1 + \dots + c_d\, f_d}_{\textbf{Parameterized telescoping}}$$

# General case: inhomogeneous contributions

GIVEN $f_0(k) := h_0^{(0)}(k)P(k) + \dots + h_s^{(0)}(k)P(k+s) + h_{s+1}^{(0)}(k), \dots,$

$f_d(k) := h_0^{(d)}(k)P(k) + \dots + h_s^{(d)}(k)P(k+s) + h_{s+1}^{(d)}(k)$

and

$$P(k+s+1) = a_0(k)P(k) + \dots + a_s(k)P(k+s) + a_{s+1}(k).$$

FIND $g(k) = g_0(k)P(k) + \dots + g_s(k)P(k+s) + g_{s+1}(k)$ and $c_0, \dots, c_d$:

$$g(k+1) - g(k) = c_0 f_0(k) + \dots + c_d f(k).$$

1.1. FIND a solution $g_s(k)$ and $c_0, \dots, c_d$ for

$$\sum_{j=0}^{s} a_{s-j}(k+j)g_s(k+j+1) - g_s(k) = c_0 \sum_{j=0}^{s} h_{s-j}^{(0)}(k+j) + \dots + c_d \sum_{j=0}^{s} h_{s-j}^{(d)}(k+j)$$

1.2. FIND a solution $g_{s+1}(k)$

$$g_{s+1}(k+1) - g_{s+1}(k) = c_0 f_{s+1}^{(0)}(k) + \dots + c_d f_{s+1}^{(d)}(k) - a_{s+1}(k)g_s(k+1)$$

2. COMPUTE the remaining $g_0, \dots, g_{s-1}$:

$$g_0(k) = a_0(k)g_s(k+1) - \sum_{i=0}^{d} c_i h_0^{(i)}(k),$$

$$g_r(k) = a_r(k)g_s(k+1) + g_{r-1}(k+1) - h_r(k) - \sum_{i=0}^{d} c_i h_0^{(r)}(k), \qquad 0 < r < s.$$

# Strategy:

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k) P(n,k)$$

where $P(n,k)$ is a definite sum and $f(n,k)$ is an R$\Pi\Sigma^*$-term.

1. Try to compute  recurrences

$$\begin{aligned} P(n,k+s+1) &= a_0(n,k)P(n,k) + \cdots + a_s(n,k)P(n,k+s) + a_{s+1}(n,k) \\ P(n+1,k) &= b_0(n,k)P(n,k) + \cdots + b_s(n,k)P(n,k+s) + b_{s+1}(n,k) \end{aligned}$$

for some $s \geq 0$.

2. Try to compute a recurrence for $S(n)$ (as showed above).

# Strategy:

GIVEN

$$S(n) = \sum_{k=0}^{n} f(n,k) P(n,k)$$

where $P(n,k)$ is a multi-sum and $f(n,k)$ is an $\mathrm{R}\Pi\Sigma^*$-term.

1. Try to compute (recursively) recurrences

$$\begin{aligned} P(n,k+s+1) &= a_0(n,k)P(n,k) + \cdots + a_s(n,k)P(n,k+s) + a_{s+1}(n,k) \\ P(n+1,k) &= b_0(n,k)P(n,k) + \cdots + b_s(n,k)P(n,k+s) + b_{s+1}(n,k) \end{aligned}$$

for some $s \geq 0$.

2. Try to compute a recurrence for $S(n)$ (as showed above).

# Highlights related to number theory

- Apéry's double sum is plain sailing indeed (2007)
- When is 0.999... equal to 1?
  (joint with R. Pemantle; 2007)
- Gaussian hypergeometric series and extensions of supercongruences
  (joint with R. Osburn; 2009)
- A case study for $\zeta(4)$
  (joint with W. Zudilin; 2021)
- Error bounds for the asymptotic expansion of the partition function
  [compare Hardy–Ramanujan, Wright, Rademacher, Lehmer, O'Sullivan]
  (joint with K. Banerjee, P. Paule, C.-S. Radu; 2023)
- Asymptotics for the reciprocal and shifted quotient of the partition function (joint with K. Banerjee, P. Paule, C.-S. Radu; 2024)

[Arose in the context to explore rational approximations of $\zeta(4)$]

**Conjecture** (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!} \,.$$

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and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3} \sum_{\nu=n-m+1}^{\infty} \frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6} \sum_{\nu=1}^{\infty} \frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

**Theorem (CS, Sigma, Zudilin)** For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m\Big(t+\frac{n}{2}\Big)\frac{(t-n)_m}{m!}\,\frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}}\,\frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}}\left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}}\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-m+j}{n}\frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

**Proof tactic:** Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

with

$$\begin{aligned}\alpha_0(n,m)&=(2n-m)^5,\\ \alpha_1(n,m)&=-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3\\ &\qquad+30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m),\\ \alpha_2(n,m)&=-(2n-m-1)^3(4n-m)(m+2).\end{aligned}$$

**Proof tactic:** Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

$$
\begin{aligned}
S(n,m) = \sum_{j=0}^{n}\sum_{\nu=1}^{\infty}\Bigg(&\frac{\binom{n}{j}^2\binom{j-m+2n}{n}(1+\nu)_{-m+2n}(1-j+\nu+n)_{-1+n}}{(1+\nu+n)_n(1+\nu+n)_{-m+2n}(\nu+n)^4(\nu-m+2n)^3}\\
&\times\Bigg((\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad-\nu(j-\nu-n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)\\
&\quad-S_1(\nu-m+3n)-S_1(-j+\nu+n)+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+\nu(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)\\
&\quad-S_1(\nu-m+3n)-S_1(-j+\nu+n)+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad-(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)+2S_1(\nu+n)\\
&\quad-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&\quad+\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{(j-\nu-2n)^2}-S_2(\nu)+2S_2(\nu+n)\\
&\quad-S_2(\nu+2n)-S_2(\nu-m+3n)-S_2(-j+\nu+n)\\
&\quad+S_2(\nu-m+2n)+S_2(-j+\nu+2n)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4(j+n)(\nu+n) - 3(\nu+n)^2 + n(-m+n) - j(m+2n)\Big) \\
&- 2(\nu+n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} - S_1(\nu) \\
&\quad + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&\quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
&\quad + 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&- 3(\nu-m+2n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} - S_1(\nu) \\
&\quad + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&\quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&- (\nu+n)(\nu-m+2n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} \\
&- S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&\quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
&\quad + 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\qquad \times \big( - S_1(\nu+n) + S_1(\nu+2n)\big) \\
&+ (\nu+n)(\nu-m+2n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} \\
&\quad - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\quad \times \big( - S_1(\nu) + S_1(\nu - m + 2n)\big) \\
&- (\nu+n)(\nu-m+2n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} \\
&\quad - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&\quad + S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\quad \times \big( - S_1(\nu+n) + S_1(\nu - m + 3n)\big) \\
&+ (\nu+n)(\nu-m+2n)\Big( - \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big( - \frac{1}{-j+\nu+2n} \\
&\quad - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
&\quad + S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m-n) + 2(j+n)(\nu+n)^2 - (\nu+n)^3 \\
&\quad - (\nu+n)\big(n(m-n) + j(m+2n)\big)\Big) \\
&\quad \times \Big( - \frac{1}{-j+\nu+2n} - S_1(-j+\nu+n) + S_1(-j+\nu+2n)\Big)\Big)\Big)
\end{aligned}
$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \quad \text{Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+\,a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$

$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \quad \text{Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$

$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$\Big\downarrow \quad \text{Sigma.m with Holonomic-DR approach}$$

$$(2n-m)^5S(n,m)$$
$$-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3+30n^2m-14nm^2$$
$$+2m^3+8n^2-10nm+2m^2-4n+m)S(n,m+1)$$
$$-(2n-m-1)^3(4n-m)(m+2)S(n,m+2)=R(n,m)$$

**Proof tactic:** Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

SigmaReduce

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$

$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

**Proof tactic:** Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\,\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

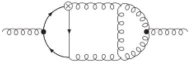
$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

Finally, check 2 initial values: another round of non-trivial summation...

# Part 3: Applications

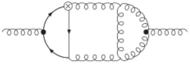
- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

# Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

# Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$\longrightarrow$

$$\int \Phi(n, \epsilon, x)\, dx$$

Feynman integrals

# Feynman integrals

$$\int_0^1 x^n \, dx$$

# Feynman integrals

$$\int_0^1 x^n (1+x)^n \, dx$$

# Feynman integrals

$$\int_0^1 \frac{x^n(1+x)^n}{(1-x)^{1+\varepsilon}}\,dx$$

# Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2$$

# Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1 + x_1)^n}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3$$

# Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

# Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

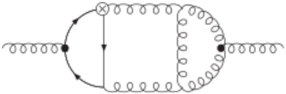
# Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^n (1+x_1)^n}{(1-x_1)^{1+\varepsilon}} \ldots \; dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

# Feynman integrals

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^n(1+x_1)^{n-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\; dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

# Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

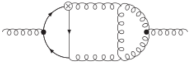
$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{n-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{n-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

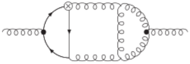
# Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles $\longrightarrow$ $\int \Phi(n, \epsilon, x)\,dx$

Feynman integrals

# Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$$\int \Phi(n,\epsilon,x)\,dx$$

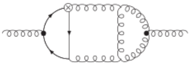
Feynman integrals

**DESY**

$$\sum f(n,\epsilon,k)$$

complicated multi-sums

# Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles $\longrightarrow$ $\int \Phi(n,\epsilon,x)dx$

Feynman integrals

$\downarrow$ **DESY**

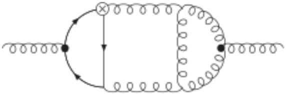
$\sum f(n,\epsilon,k)$

complicated multi-sums

**advanced difference ring theory** (Sigma-package) $\longleftarrow$

expression in special functions

# Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\binom{n-1}{j+2}\binom{j+1}{k+1} \qquad ||$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

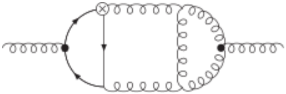
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

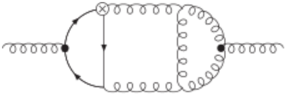
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{n-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{n-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

# Simplify

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-\left(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s)\right)$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big( - \frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big( - 22 + 6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big( - 6 + 5(-1)^n\big)S_{-4}(n)$$
$$+ \Big( - \frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big( - 17 + 13(-1)^n\big)S_{3,1}(n)$$
$$- \frac{8(-1)^n(2n+1) + 4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(1\cdots}{n^2\cdots}\cdots\frac{(2n+1)}{\cdots+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+\big(2+2(\cdots S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4}+(-1)^n\Big)S_2(n)^2$$
$$-2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+\Big(-\frac{4(1 \ldots}{n^2 \ldots} \text{[illegible]} \frac{\ldots(2n+1)}{\ldots+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{\ldots} \text{[illegible]} (-1)^n\Big)S_{-2}(n)\Big)$$

$$+\big(2+2(\text{[illegible]} S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big) \text{[illegible]} S_2(n)^2$$

$$-2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{\text{[illegible]}}{n+1}\Big)$$

$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$

$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$

$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$

$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$

$$+\Big(-\frac{4(1\ldots}{n^2\ldots}\ldots\frac{\ldots(2n+1)}{\ldots+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{\ldots}\ldots(-1)^n\Big)S_\ldots(n)$$

$$+\big(2+2(\ldots S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)\ldots S_2(n)^2$$

$$-2(-1)^nS_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{\ldots}{n+1}\Big)$$

$$+\Big(\frac{(-1)^n(5\ldots}{2n^2(n+\ldots}\ldots\frac{8(-1)^n(2n+1)}{n(n+1)}$$

$$+\frac{4(3n-1)}{n(n+1)}\Big)\ldots\Big)S_2(n) - \frac{16}{n(n+1)}\Big)$$

$$+\Big(\frac{(-1)^n(9n\ldots}{n(n+\ldots}\ldots\big(-6+5(-1)^n\big)S_{-4}(n)$$

$$+\Big(-\frac{2(-1)^n\ldots}{n(n+1)}\ldots S_{\ldots 2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$

$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$

$$+32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

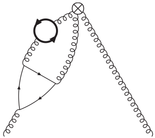
$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i}$$

$$S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

$$S_{-2,1,1}(n) = \sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

# Example (2): a 2-mass 3-loop Feynman integral

[arXiv:1804.02226]



All diagrams are produced with axodraw (J. Vermaseren).

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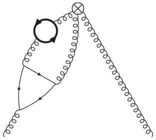

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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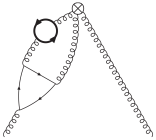

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Typical triple sum:

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+n)(-1+n)n\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+n)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+n)}$$

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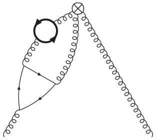

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$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+n)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+n)}$$

$6$ hours for this sum

$\sim$ 10 years of calculation time for full expression

# Example (2): a 2-mass 3-loop Feynman integral

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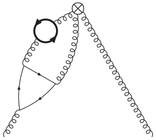

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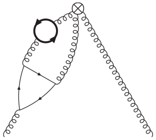
expression (377 MB) consisting of 8 multi-sums

# Example (2): a 2-mass 3-loop Feynman integral

[arXiv:1804.02226]



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$\downarrow$ EvaluateMultiSums.m

# Example (2): a 2-mass 3-loop Feynman integral

[arXiv:1804.02226]

| sum | size of sum (with $\varepsilon$) | summand size of constant term | time of calculation | | number of indef. sums |
|---|---|---|---|---|---|
| $\sum_{i_4=2}^{n-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$ | 17.7 MB | 266.3 MB | 177529 s | (2.1 days) | 1188 |
| $\sum_{i_3=3}^{n-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$ | 232 MB | 1646.4 MB | 980756 s | (11.4 days) | 747 |
| $\sum_{i_2=3}^{n-4}\sum_{i_1=0}^{\infty}$ | 67.7 MB | 458 MB | 524485 s | (6.1 days) | 557 |
| $\sum_{i_1=0}^{\infty}$ | 38.2 MB | 90.5 MB | 689100 s | (8.0 days) | 44 |
| $\sum_{i_4=2}^{n-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$ | 1.3 MB | 6.5 MB | 305718 s | (3.5 days) | 1933 |
| $\sum_{i_3=3}^{n-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$ | 11.6 MB | 32.4 MB | 710576 s | (8.2 days) | 621 |
| $\sum_{i_2=3}^{n-4}\sum_{i_1=0}^{i_2}$ | 4.5 MB | 5.5 MB | 435640 s | (5.0 days) | 536 |
| $\sum_{i_1=3}^{n-4}$ | 0.7 MB | 1.3 MB | 9017s | (2.5 hours) | 68 |

# Example (2): a 2-mass 3-loop Feynman integral

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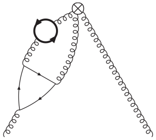

Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

# Example (2): a 2-mass 3-loop Feynman integral

[arXiv:1804.02226]

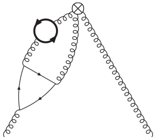
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left( \sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left( \sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left( \sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

# Example (2): a 2-mass 3-loop Feynman integral

[arXiv:1804.02226]



Mellin-Barnes- and ${}_pF_q$-technologies →

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

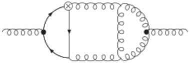
↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

← Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

# Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n, \epsilon, x)\, dx$$

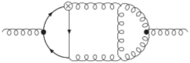
Feynman integrals

**DESY**

$$Dy = A\, y$$

coupled systems of linear DEs

# Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n, \epsilon, x)\, dx$$

Feynman integrals

**DESY**

$$Dy = A\, y$$

coupled systems of linear DEs

**RISC**

(guess & solve)

expression in special functions

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$

`SolveCoupledSystem.m` $\downarrow$

large no. of moments,
say $P(0), \ldots, P(10000)$

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$

`SolveCoupledSystem.m` ↓

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↙

numerics

coupled systems

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↙ numerics

↓ guessing (`ore_algebra` in Sage)

recurrence

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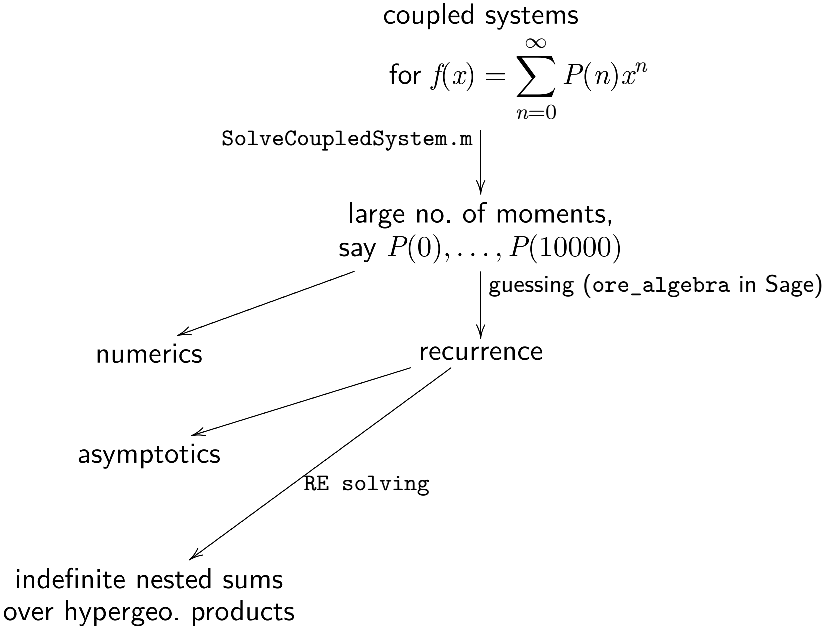
asymptotics


coupled systems
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SolveCoupledSystem.m
large no. of moments,
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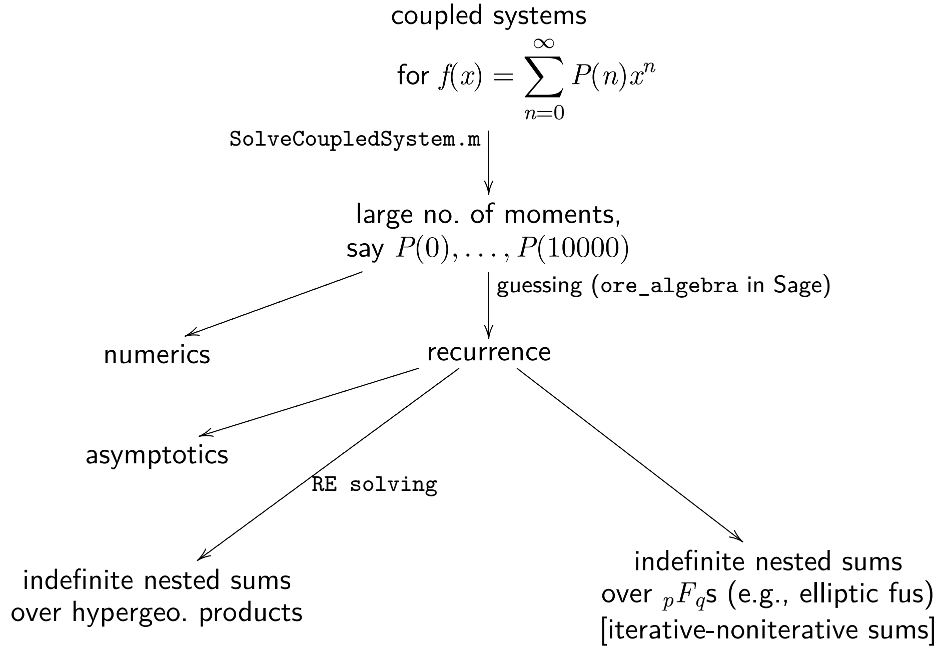
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
RE solving
indefinite nested sums
over hypergeo. products


coupled systems
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SolveCoupledSystem.m
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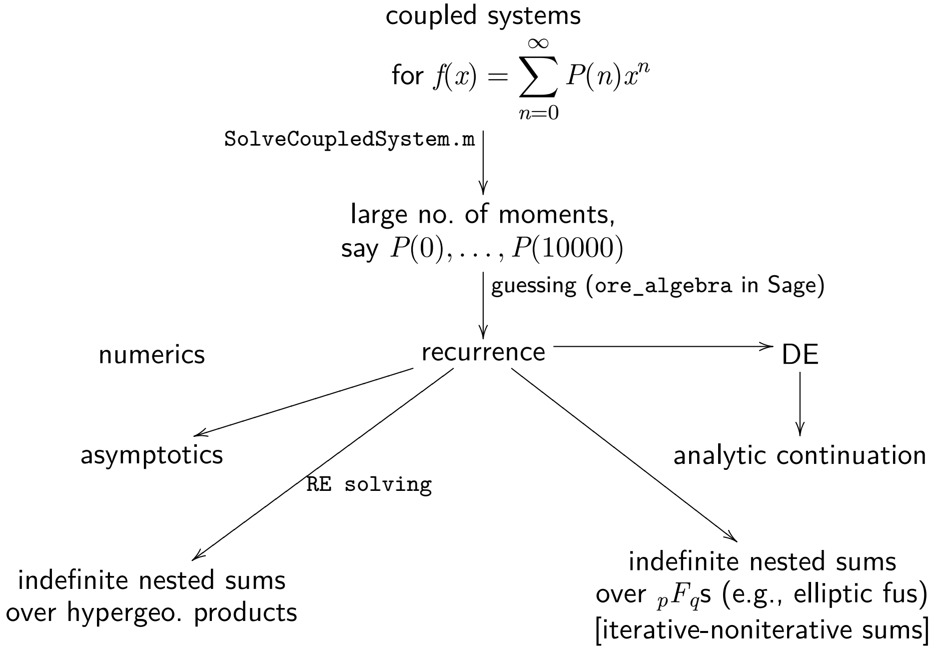
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]


coupled systems
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SolveCoupledSystem.m
large no. of moments,
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DE
asymptotics
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[iterative-noniterative sums]

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large no. of moments,
say $P(0), \ldots, P(10000)$

guessing (`ore_algebra` in Sage)

recurrence

`RE solving`

indefinite nested sums
over hypergeo. products

**Example** (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial =<< iFile16**

**Example** (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= **initial =<< iFile16**

Out[2]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,

475840076183/7501410000, 1432950323678333/21965628762000,

21648380901382517/328583783127600,

5286978432377857 6751/802218994536960000,

494228620940455239942 31/7537739922306161568 00,

3313187983290793592072 6113/5095579439852999697 60000,

5209274721836755168448777/8094998411185418045 9136,

5614371199734476902104 1145213/8825892663835864563843 53664,

45350043335384562819479002512 4807/7217228048879468556886950000000,

14061543374120479886110159898869387/226643167590350326435656036000000,

71558652266649190332490578517861993657116837030770022280781149589503 0000000,

162867290463592738928412712574188540568 36413/269396588055480390401343344736943104000000,

142872964263230246795142690584469183780 5299/239407595750341228278613159615736736 00000,

498938690219595294505102809199154550783080767/846888366785297981317126230405400272 0000000, In[3]:= **rec =<< rFile16**

Out[3]=

$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} + 2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} + 873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} + 1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} + 575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} + 69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} + 901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} + 7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} + 32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} + 86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} + 124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} + 87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} + 25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} + 1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 P[n]$$

$$-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} + 101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + 2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + 9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + 7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + 356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + 9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + 170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + 1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + 13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + 59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + 167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + 284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + 279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + 149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + 38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + 3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + 101701393436276172443717692853400n + 6204709909986751913151675960000\Big)P[n+1]$$

$$
\begin{aligned}
&+2(n+3)^{2}(2n+5)^{3}(2n+9)(2n+11)\Big(24739011624960n^{40} + 3317836466356224n^{39} + 215508170284466176n^{38} + 9032884062187945984n^{37} + \\
&274636134389959884800n^{36} + 6455501959255126179840n^{35} + 122094572934385260036096n^{34} + 1909387225793663151898624n^{33} + 25180108291969215434326016n^{32} + \\
&284171960705270647479074816n^{31} + 2775794400720227034854326272n^{30} + 23677622163992853854566219776n^{29} + 177624312783583749157935120384n^{28} + \\
&1178515602115604757944201871360n^{27} + 6947091965313419323781358354432n^{26} + 36515023100308314818702129258496n^{25} + 171621148571344894953594594017280n^{24} + \\
&722837793013976317556258102507520n^{23} + 2732534027077907914497042720534528n^{22} + 9281028665970648470895368668485120n^{21} + \\
&28337819215557708948254385336117248n^{20} + 77786125749274632150536464583130752n^{19} + 191877161455672780973502244537632256n^{18} + \\
&424953221702140663089937921965135648n^{17} + 843818276409975584824720931649555264n^{16} + 1499359936674956711935311062995422344n^{15} + \\
&2378007025570977662661938772843220240n^{14} + 3355671771434535852147325502571953770n^{13} + 4196375762867184563407432891655585484n^{12} + \\
&4627675779563752366067861596232781096n^{11} + 4473175960511956000526499430851993603n^{10} + 3761696365025837909581516781307249585n^{9} + \\
&2726553473467254373993685951699145492n^{8} + 1683383212304999468664293798012773485n^{7} + 871926653651504419744271839781064837n^{6} + \\
&371307437598003570058538796122994147n^{5} + 126427972742886389602285855482966072n^{4} + 33048762330145623969058704448697313n^{3} + \\
&6217924746857741077419160100404560n^{2} + 748298077423337427195946099994100n + 43181089548034246077698611794000\Big)P[n+2]
\end{aligned}
$$

$$-2(n+4)^{2}(2n+5)(2n+7)^{3}(2n+11)\Big(24739011624960n^{40} + 3322784268681216n^{39} + 216160919414112256n^{38} + 9074528155284275200n^{37} +$$
$$276348048819456311296n^{36} + 6506479077331107315712n^{35} + 123266585640616142569472n^{34} + 1931040885785102661976064n^{33} + 25510503383281445462081536n^{32} +$$
$$288418124175428279391485952n^{31} + 2822442799033603081019326464n^{30} + 24120717233320712351821332480n^{29} + 181295944719289040999116701696n^{28} +$$
$$1205246297785423925076555694080n^{27} + 7119049557560114436136213413888n^{26} + 37496933571993839665392189775872n^{25} + 176616172467048982234270428880896n^{24} +$$
$$745539218875020737621728364206080n^{23} + 2824909633156578132652259733712896n^{22} + 9618101958268071244680677589035520n^{21} +$$
$$29441860528446423517613263360742912n^{20} + 81033563306363873505877563416477312n^{19} + 200454769103641040142838133702338304n^{18} +$$
$$445286624972461749049425309485328992n^{17} + 887028447418790661018847407251573152n^{16} + 1581538101499869694224895701784875304n^{15} +$$
$$2517550244392724509968791166585362672n^{14} + 3566593026520465155504695877897282630n^{13} + 4479066125207404898722179511912639638n^{12} +$$
$$4962006990874351800791769650243464872n^{11} + 4819992643914265990647887896664485209n^{10} + 4074895386694182240941538222230233221n^{9} +$$
$$2970477229398746689186622534784613554n^{8} + 1845274131994015990683957902602775337n^{7} + 962091291302144537393228847830431614n^{6} +$$
$$412595107814836563208757757032740146n^{5} + 141540723940232563767779647013785485n^{4} + 37292931812630561528276365992452010n^{3} +$$
$$7074865777225416725452872895397100n^{2} + 858794112392644074221312049837000n + 49997386738260112603615104780000\Big)P[n+3]$$

$$+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} +$$
$$3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} +$$
$$38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} +$$
$$69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} +$$
$$32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} +$$
$$1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} +$$
$$20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} +$$
$$260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} +$$
$$2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} +$$
$$10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} +$$
$$34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} +$$
$$69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} +$$
$$82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} +$$
$$55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} +$$
$$19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} +$$
$$3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} +$$
$$168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n +$$
$$1274120732351744651125603886400\Big)P[n+4]$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} + 1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} + 136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} + 83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} + 12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} + 510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} + 6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} + 18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} + 29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} + 18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} + 2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} + 19313175036907229252501700n + 1248723341516324359641600\Big)P[n+5]==0$$

In[4]:= **recSol = SolveRecurrence[rec, P[n]]**

In[4]:= **recSol = SolveRecurrence[rec, P[n]]**

Out[4]=
$$\Big\{\Big\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\Big\}$$
$$\Big\{0, -\frac{(3+2n)\big(-8-9n+2n^2\big)}{(1+n)^2(1+2n)^2}\Big\}$$
$$\Big\{0, -\frac{(3+2n)\big(-5+8n^2\big)}{2(1+n)^2(1+2n)^2}+\frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)}+\frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big\}$$
$$\Big\{0, \frac{(3+2n)\big(-513-2184n-2416n^2+768n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\Big(-\frac{2(3+2n)\big(3+44n+48n^2\big)}{(1+n)^2(1+2n)^2}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+$$
$$\frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)}+\frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-$$
$$\frac{4(3+2n)\big(3+44n+48n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big\}$$

$$
\begin{aligned}
&\Big\{0,\ \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-\\
&49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+\\
&\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\\
&\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-\\
&\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+\\
&\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+\\
&\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-\\
&\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-\\
&\frac{8(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{-1+2i}}{(1+n)(1+2n)}\\
&\quad-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+\\
&\frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{-1+2i}}{(1+n)(1+2n)}\Big\},\{1,0\}\Big\}
\end{aligned}
$$

In[5]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[5]:= **sol = FindLinearCombination[recSol, {0, initial}, n, 7, MinInitialValue → 1]**

Out[5]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$
$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+\frac{44(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-$$
$$\frac{(3+2n)(23+139n+130n^2)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3}{9(1+n)(1+2n)}+\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+$$
$$\frac{16(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$
$$\frac{4(3+2n)(-41-53n+2n^2)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+\frac{32(3+2n)\left(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}\right)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-$$
$$\frac{4(3+2n)(23+139n+130n^2)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{3(1+n)^2(1+2n)^2}+\frac{32(3+2n)\left(\sum_{i=1}^{n}\frac{1}{i}\right)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^2}{3(1+n)(1+2n)}+\frac{64(3+2n)\left(\sum_{i=1}^{n}\frac{1}{-1+2i}\right)^3}{9(1+n)(1+2n)}-$$
$$\frac{16(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{i}}{3(1+n)(1+2n)}-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$
$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{j}\right)\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{i}}{3(1+n)(1+2n)}+\frac{128(3+2n)\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{i}\frac{1}{-1+2j}\right)^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[6]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol,**

**n, 2]//ToStandardForm, n]//CollectProdSum;**

In[6]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

Out[8]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\big(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\big)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\big(147+985n+1871n^2+1268n^3+212n^4\big)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\big(23+123n+114n^2\big)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\big(53+229n+190n^2\big)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\big(1+8n+8n^2\big)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[6]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

In[8]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[9]:= **SExpansion[sol, n, 2]**

Out[9]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

In[6]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[7]:= **sol = TransformToSSums[sol];**

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Out[9]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

## Special function algorithms

- `HarmonicSums` package
  Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
  Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]
- `RICA` package
  Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.

# Large recurrences that could be solved:

$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + \boxed{a_{44}(n)} F(n+44) = 0$$

see [J. Blümlein, P. Marquard, N. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. Nuclear Physics B 949(114751), pp. 1-97. 2019]

# Large recurrences that could be solved:

$$a_0(n)F(n) + a_1(n)F(n+1) + \cdots + \boxed{a_{44}(n)}\,F(n+44) = 0$$

$$a_{44}(n) = \boxed{A_0} + A_1\,n + A_2\,n^2 + \cdots + A_{1280}\,n^{1280} \in \mathbb{Z}[n]$$

see [J. Blümlein, P. Marquard, N. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. Nuclear Physics B 949(114751), pp. 1-97. 2019]

# Large recurrences that could be solved:

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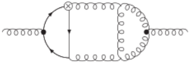
$$a_{44}(n) = \boxed{A_0} + A_1\,n + A_2\,n^2 + \cdots + A_{1280}\,n^{1280} \in \mathbb{Z}[n]$$

$A_0 =$
4304008147314341993338530140996136248565755323962853602010245518888042154135680724079551611 88
0407782953769826986938430875220157229036801680583719506271442796195795326505473754642759953 80
1941915418680522909821235645780857440976207934665943396292869018060656436842283823262737072 36
2 72263741023980001339814741586363122803454718941711425928638399272917785469657110169188973266
472928142262897812353354029052021281007242074393 582978630271549480525016244781790052083066906
8848230672430173098317534500146588881300441172641018506082448222703697040252835231399493777 22
05 28777614305224894592517564496567207234356842147076978964501563992025099650932206465152337 45
411377968633085602840009864899387676224905263 7474 88234472450650670550351585424260948093515810
4451713292809890896597634190353119742662377787352636977382379877057062282376973469423767820 84
597 1275603090924168185293338017425710805097111055845989796452616846948559035959487369550590 92
709205040031216976898102693387539500876306976 54664 381445062417130069224654407795926235050729 5
4354629514930638129546569377991896986210746844356967918343080871313800419648118754294383415 01
0225 9638186566503768053421837931888674543709557114942593141558766000835049446231176725061162 0
566498021242998120599744330215198719756120991045078 0834396714527976356045794938086615886104 06
5044512787958963995940902492744128165620136240019499503429149597190480772323497571197845402 12
84984 56703690862044901663749224372560685020881937717236954936033937917514072239068045345535 05
716133647753458991993436359110319748438098582746040 4 17996879959168452883677514819555007075059
25801083889626926535332053833991089517694235238096811767270017191095828480000000000000000000 0
000000 00000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000 0000000000000000000000000000000000000000
0000000000000000000000000000000000000

(1899 decimal digits)

see [J. Blümlein, P. Marquard, N. Rana, CS. The Heavy Fermion Contributions to the Massive Three Loop Form Factors. Nuclear Physics B 949(114751), pp. 1-97. 2019]

# Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n, \epsilon, x)\, dx$$

Feynman integrals

**DESY**

$$Dy = A\, y$$

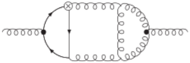
coupled systems of linear DEs

**RISC**

(guess & solve)

expression in special functions

# Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(n,\epsilon,x)\,dx$$

Feynman integrals

**DESY**

$$Dy = A\,y$$

coupled systems of linear DEs

**RISC**

(guess & solve)

expression in special functions

applicable

LHC at CERN

# Part 3: Applications

- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)