Tame Geometry in Quantum Field Theory and Gravity

Thomas W. Grimm

Utrecht University

Based on work done in collaboration with

Mike Douglas, Damian van de Heisteeg, Arno Hoefnagels, Jeroen Monnee, David Prieto, Giovanni Ravazzini, Lorenz Schlechter, Mick van Vliet

Mathematicians: Benjamin Bakker, Christian Schnell, Jacob Tsimerman

Introduction

Tameness in physics

- Claim:

  All actual physical observables can be described using functions definable in a sharply o-minimal structure.

- Challenge for this talk

  What is an o-minimal structure?

  generalized finiteness principles preserved under finitely many logical operations [van den Dries][Knight,Pillay,Steinhorn]…

  What is an sharply o-minimal structure?

  refine o-minimality: quantitive way to assign finiteness measure — two integers (F,D), called (sharp) complexity — to definable sets
  $\to$ require polynomial behavior in D

  [Binyamini,Novikov '22][Binyamini,Novikov,Zack]… $\to$ Zack's talk

Intuition from polynomials

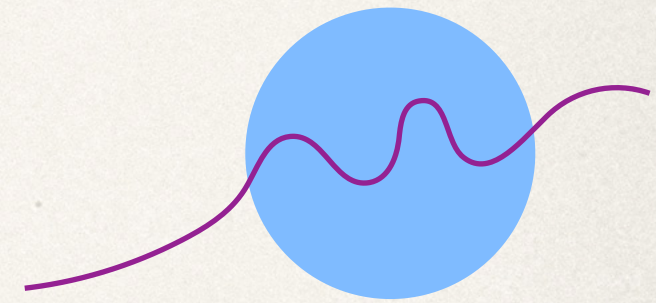
- Complexity for polynomials → D - degree of polynomial; F - number of variables

  $P(x) = a_1 x^2 + a_2 x + a_3$

  → amount of information needed to specify polynomial (real coefficients)

- Bounds from complexity:
  - Number of zeros of $P(x)$: $\#(P=0) \;\leq \mathcal{C}(F,D) \cong D^F$
  - Volume of an n-dimensional set $A = \{P(x) = y\}$

    $\mathrm{Vol}(B^{n+1}(r) \cap A) \leq c(n)\, \mathcal{C}(F,D)\, r^n$



see e.g. book [Yomdin,Comte]

- How to deal with exponential function?

  → new perspective: $\dfrac{d}{dx} e^{ax} = a\ e^{ax}$ → record information needed in differential equation

Example class of sharply o-minimal functions

- Pfaffian functions [Khovanskii '91][Gabrielov,Vorobjov '04]

Pfaffian chain: $f_1(x), \dots f_r(x)$

$$\partial_{x^i} f_1 = P_{1,i}(x, f_1)$$
$$\partial_{x^i} f_2 = P_{2,i}(x, f_1, f_2)$$
$$\vdots$$
$$\partial_{x^i} f_r = P_{r,i}(x, f_1, f_2, ..., f_r)$$

Pfaffian function: $g(x) = P(x_1, ..., x_n, f_1, f_2, ..., f_r)$ → $\mathbb{R}_{\text{Pfaff}} = \mathcal{P}(\mathbb{R}_{\text{alg}})$

- Key point: Pfaffian functions are o-minimal + have a notion of complexity

degree: $D = \deg(P) + \sum_{ij} \deg(P_{i,j})$

format: $F = n + r$ (number of variables + number of non-trivial functions)

→ bounds on zeros, number of poles, ….

Example class of sharply o-minimal functions

- Pfaffian functions [Khovanskii '91][Gabrielov,Vorobjov '04]

Pfaffian chain: $f_1(x), \ldots f_r(x)$

$$\partial_{x^i} f_1 = P_{1,i}(x, f_1)$$
$$\partial_{x^i} f_2 = P_{2,i}(x, f_1, f_2)$$
$$\vdots$$
$$\partial_{x^i} f_r = P_{r,i}(x, f_1, f_2, ..., f_r)$$

Pfaffian function: $g(x) = P(x_1, ..., x_n, f_1, f_2, ..., f_r)$ $\longrightarrow$ (F, D)

[Binyamni,Vorobjov]: $\mathbb{R}_{\text{rPfaff}}$ structure generated by restricted Pfaffian functions is sharply o-minimal

While sharp o-minimality of $\mathbb{R}_{\text{rPfaff}}$ is subtle and we need more general structures: Useful to think of (F, D) as in Pfaffian setting.

Tameness in physics

- Conjecture / Guess:

  All functions used to describe actual physical observables are definable in a sharply o-minimal structure.

- Challenge for this talk → But what is an actual physical observable?

  that's much harder and depends on the context

  - study trajectories of planets → classical gravity
    ignore quantum effects

  - study scattering of particles → ignore gravity
    take quantum phenomena into account

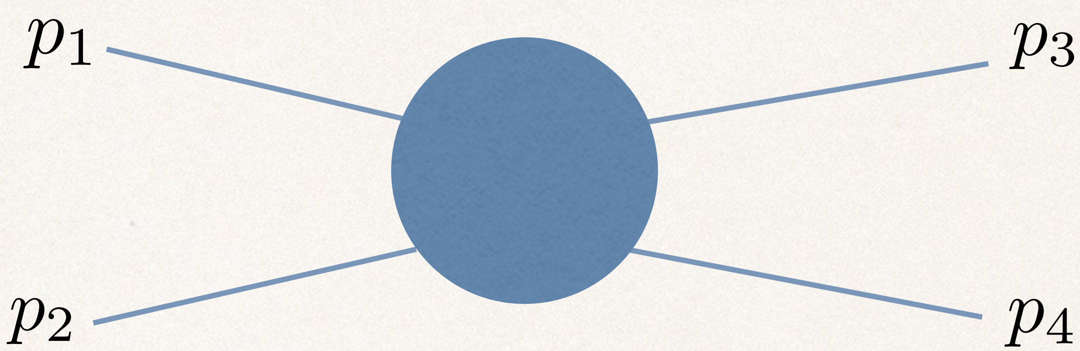
Notion of observable: depends on energy scale and considered forces

On Tame Geometry in Quantum Field Theories (QFTs)

Scattering amplitudes in QFTs

- Natural observables in QFTs: Scattering amplitudes

scattering particles



How likely is the process?

$$\mathcal{A}(p, m, \lambda)$$

- Physics: defined using path integrals - “sum over all possible processes”

- Taylor expansion: small coupling expansion $\lambda \ll 1$

$$\mathcal{A}(p,m,\lambda) = \lambda^2 \Big(\mathcal{A}_0(p,m) + \mathcal{A}_2(p,m)\lambda^2 + \mathcal{A}_4(p,m)\lambda^4 + \dots \Big)$$

$\rightarrow$ summing till fixed loop number: finite number of Feynman integrals

Scattering amplitudes in QFTs

- Result: For any QFT with finitely many particles and interactions all finite-loop amplitudes $\mathcal{A}_\ell$ are $\mathbb{R}_{\mathrm{an,exp}}$ - definable functions of the masses m, momenta p. [Douglas,TG,Schlechter]

Upshot: Many physicists study scattering amplitudes
→ very non-trivial definable functions: $\mathbb{R}_{\{\mathcal{A}_\ell\}} \subset \mathbb{R}_{\mathrm{an,exp}}$

properties of physical theory (QFT) → properties of such structures

- Question 1: Can tame geometry formalize the connection of algebraic relations and symmetries on the space of amplitudes $\mathcal{A}_\ell$.

likely yes: much recent progress on using tame geometry in Hodge theory
→ transcendental of amplitude vs. existence of algebraic relations
applying Ax-Schanuel for period integrals? [Bakker,Tsimerman '17]

Why is $\mathbb{R}_{\mathrm{an,exp}}$ -definability true?

- amplitudes are composed of finitely many Feynman integrals

$$\mathcal{I}_\nu(m,p) = \left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d k_j}{i\pi^{d/2}}\right) \prod_a \frac{1}{D_a(p,k,m)^{\nu_a}}$$

polynomials in p,k,m

- Idea: Feynman integrals can be related to period integrals of some auxiliary compact Kähler manifold Y_{graph}

review book by [Weinzierl] + many original works

e.g. $\Pi(z) = \int_C \Omega(z)$

p-form on Y_{graph} varying with its complex structure

- Use: all steps only involve definable maps, period integrals are definable in o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$

[Bakker,Klingler,Tsimerman '18]

[Bakker,Mullane '22] related integration results [Comte,Lion,Rolin]

A natural question

- Question 2: Can one assign complexity (*F*,*D*) to amplitudes $\mathcal{A}_\ell$?

likely yes: [Binyamini, Novikov '22] conjectured that period integrals are sharply o-minimal

recently: [Binyamini '24] period map is definable in $\mathbb{R}_{\mathrm{LN,exp}}$

LN - log-Noetherian functions → effectively o-minimal ✔
based on [Binyamini,Novikov '19]

e.g.

$$z^i \frac{\partial f_k}{\partial z^i} = P_{ki}(z, f)$$

f_k holomorphic bounded on punctured discs

conjectured to be sharply o-minimal

↓

What is (*F*,*D*)?

How does *(F,D)* change with properties of amplitude/QFT?

→ quantitative measure of algebraic relations ('transcendence degree')

An application: Cosmological correlators

- "Tree-level cosmological correlators" are scattering amplitudes described by differential equation:

$$dI = AI$$

differential equations (matrix A) are determined by 'kinematic flow algorithm' [Arkani-Hamed,Baumann, Hillman,Joyce,Lee, Pimentel '23]

defines Pfaffian chain: tree-level correlators are Pfaffian functions

graph with N_V vertices: $(F, D) = \left(2N_V + 4^{N_V-1} + N_L - 1, 3\right)$ [TG,Hoefnagels, van Vliet '24]

- Complexity gives bounds on number of poles of scattering amplitudes

[Khovanskii][Gabrielov,Vorobjov] - bound is exponentially overshooting physical expectation:

There should be a simpler representation! → alg. relations

to appear [TG,Hoefnagels,van Vliet]

- Question 3: Given a Pfaffian function is there a systematic way to determine the 'minimal' (D,F) representations?

An application: Cosmological correlators

- "Tree-level cosmological correlators" are scattering amplitudes described by differential equation:

$$dI = AI$$

differential equations (matrix A) are determined by 'kinematic flow algorithm' [Arkani-Hamed,Baumann, Hillman,Joyce,Lee, Pimentel '23]

defines Pfaffian chain: tree-level correlators are Pfaffian functions

graph with N_V vertices: $(F, D) = \left(2N_V + 4^{N_V - 1} + N_L - 1, 3\right)$ [TG,Hoefnagels, van Vliet '24]

- Complexity gives bounds on number of poles of scattering amplitudes

[Khovanskii][Gabrielov,Vorobjov] - bound is exponentially overshooting physical expectation:

There should be a simpler representation! → alg. relations

to appear [TG,Hoefnagels,van Vliet]

- Question 3: Are there reducts $\mathbb{R}_{\text{syms}} \subset \mathbb{R}_{\text{Pfaff}}$ with a new complexity (F,D) that take symmetries into account? → better bounds

Tameness of full amplitude

- Amplitudes $\mathcal{A}_\ell$ are part of full amplitude $\mathcal{A}(\lambda)$, but Taylor series is generally not convergent

- Toy example: ϕ^4 - theory on a point ('boring' 0d QFT):

$$\mathcal{A}^{(n)}(\lambda) = \int d\phi \, e^{-S(\phi,\lambda)} \, \phi^n \qquad S = \frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$ exponential period

Still: full integrals $\mathcal{A}^{(n)}(\lambda)$ Pfaffian functions: $(F, D)(\mathcal{A}^{(2n)}) = (4, 3 + \lceil n/4 \rceil)$

[TG,Schlechter, van Vliet '23]

- More general examples: [TG,Ravazzini,van Vliet '24]

One can show that λ - dependent amplitudes in several examples definable in $\mathbb{R}_{\mathscr{G}}$ - o-minimal structure generated by the Gevrey functions

[van den Dries,Speisegger] talk by Padgett

Note: did not yet include dependence on momenta p ! $\rightarrow$ need to combine both stories

Tameness of full amplitude

- Question 4: What are the o-minimal structures $\mathbb{R}_{\mathcal{A}}$ defining scattering amplitudes for some well-known QFTs? Sharply o-minimal?

$$\mathcal{A}(p,\lambda) = \Big(\; \mathcal{A}_0(p) \quad + \quad \mathcal{A}_1(p)\lambda \quad + \quad \mathcal{A}_2(p)\lambda^2 \quad + \dots \Big)$$

build $\mathbb{R}_{\mathcal{A}}$ expanding $\mathbb{R}_{\mathrm{LN,exp}}$, such that non-analytic expansions are allowed?

- Remark: It is expected that there are QFTs that have amplitudes not definable in an o-minimal structure!
  - $$\mathcal{A}^{(n)}(\lambda) = \int d\phi \, e^{-S(\phi,\lambda)} \, \phi^n$$ pick non-definable S
  - $$\mathcal{A}^{(n)}(g\cdot\tau) = \mathcal{A}^{(n)}(\tau)$$ non-trivial amplitude invariant under $g \in \mathrm{Sl}(2,\mathbb{Z})$

Tameness of full amplitude

- Question 4: What are the o-minimal structures $\mathbb{R}_{\mathcal{A}}$ defining scattering amplitudes for some well-known QFTs? Sharply o-minimal?

$$\mathcal{A}(p,\lambda) = \Big(\ \mathcal{A}_0(p) \quad + \quad \mathcal{A}_1(p)\lambda \quad + \quad \mathcal{A}_2(p)\lambda^2 \quad + \dots \Big)$$

build $\mathbb{R}_{\mathcal{A}}$ expanding $\mathbb{R}_{\mathrm{LN,exp}}$, such that non-analytic expansions are allowed?

- Two ways to proceed: (recall my claim about actual observables)

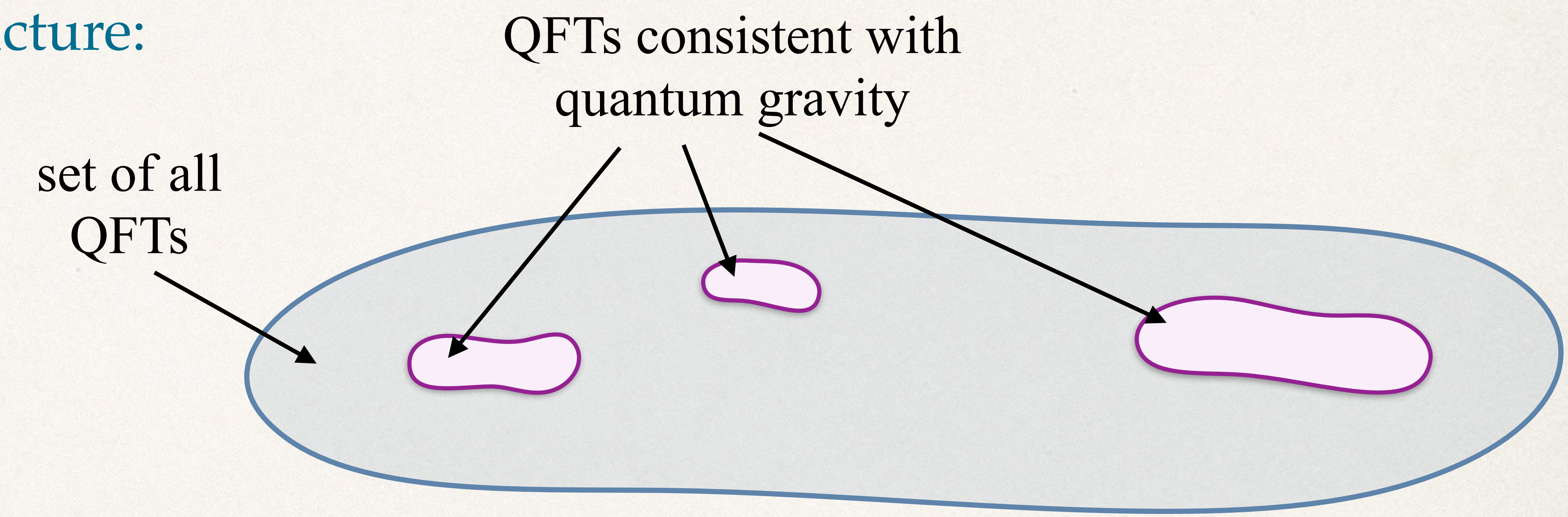
(1) restrict the class of QFTs: Conformal Field Theories [Douglas,TG, Schlechter '23]

(2) Are tameness properties inherent to QFTs that can be coupled to quantized gravity?

Connection with
Quantum Gravity Principles

Lessons from quantum gravity

- A popular picture:



- How to make this precise?
  - Work with a candidate theory of quantum gravity — string theory
  - Use ‘known’ quantum properties of black holes or other space-times

- Conjectures about the properties of effective QFTs consistent with quantum gravity - ‘swampland program’

Constraints from quantum gravity

- Best understood claims about quantum gravity:

  ‘No global symmetries’ $\rightarrow$ gauged or eventually broken

  [Banks,Dixon ’88][Banks,Seiberg]…

  - black hole arguments
  - confirmed in all string theory settings
  - proved within AdS/CFT for most global symmetries [Harlow,Ooguri]

  e.g. $\mathcal{A}^{(n)}(g \cdot \tau) = \mathcal{A}^{(n)}(\tau)$ gauged: images of fundamental domain are physically equivalent

- Compare: definability of the $j(\tau)$ -function when restricted to $\mathrm{Sl}(2,\mathbb{Z})$ fundamental domain [Peterzil,Starchenko]

$\rightarrow$ first glimpse at the importance of gravity to get tameness

Finiteness conjectures

- Conjectures about finiteness of effective QFTs compatible with Quantum Gravity

  [Douglas '05] [Vafa '05] [Acharya,Douglas '06]…[Hamada,Montero,Vafa,Valenzuela '21]…
  [Delgado,Heisteeg,Raman,Torres,Vafa '24]

  → central part of the program: studied by many physics groups

- Claims originated in string theory:
  - String theory has no continuous free parameters apart from ℓ_s
    → QFT couplings determined by quantum fields and
    discrete choices (topological data, fluxes,…)
    → finitely many

Stronger conjecture: Replace finiteness with tameness (o-minimality)

[TG '21][Douglas,TG,Schlechter '23]

Finiteness in string theory

- Finiteness conjectures about effective QFTs implies:
  - Number of distinct effective theories arising in string theory that are valid below a fixed cut-offs energy scale Λ_{fix} is finite

- String theory: Geometry-to-Physics map
  Gives precise mathematical statement in Hodge theory

complex d-dim. manifold Y_d: integral class $G \in H^d(Y_d, \mathbb{Z})$

$*G = G$ and $\int_Y G \wedge G = \ell$ $\rightarrow$ finitely many solutions even when changing complex structure

Proved tameness of locus of self-dual integral classes:

[Bakker,TG,Schnell,Tsimerman '21]

use (1) definability of period map [Bakker,Klingler,Tsimerman '18] and
(2) finiteness of orbits of symmetry groups of lattices

$\rightarrow$ non-trivial finiteness theorem generalizing finiteness theorem by [Cattani,Deligne,Kaplan '95] on Hodge classes

Finiteness in string theory

- Finiteness conjectures about effective QFTs implies:
  - Number of distinct effective theories arising in string theory that are valid below a fixed cut-offs energy scale Λ_{fix} is finite

- String theory: Geometry-to-Physics map
  Gives precise mathematical statement in Hodge theory

complex d-dim. manifold Y_d: integral class $G \in H^d(Y_d, \mathbb{Z})$

$*G = G$ and $\int_Y G \wedge G = \ell$ $\rightarrow$ finitely many solutions even when changing complex structure

- How many are there? $\rightarrow$ still open (use complexity of periods? lattice?)

10^{500} rough estimate from flux density [Ashok,Douglas '03] [Denef,Douglas '04]

Conjecture complexity from flux density:

$D = \text{poly}(\ell)$ $F = \mathcal{O}(h^{3,1}(Y))$ [TG,Monnee '23]

Finiteness and volume growth

- Another conjecture: Finiteness of amplitudes in quantum gravity

[Hamada,Montero,Vafa,Valenzuela '21]

Conjecture: Volume of any geodesic ball in moduli space $\mathcal{M}$ should grow maximally like Euclidean space [Delgado,Heisteeg,Raman,Torres,Vafa '24]

Riemannian manifold $\mathcal{M}$: $\mathcal{M}_D := \{x \in \mathcal{M} : \mathrm{dist}(x, x_0) \leq D\}$

$$\mathrm{vol}(\mathcal{M}_D) < C D^{\dim(\mathcal{M})} \qquad D \to \infty$$

Examples: Upper half-plane $\mathbb{H}$ with hyperbolic metric → not true

Fundamental domain of $\mathrm{Sl}(2, \mathbb{Z})$ in $\mathbb{H}$ with hyperbolic metric → true

Finiteness and volume growth

- Another conjecture: Finiteness of amplitudes in quantum gravity

  [Hamada,Montero,Vafa,Valenzuela '21]

Conjecture: Volume of any geodesic ball in moduli space $\mathcal{M}$ should grow maximally like Euclidean space [Delgado,Heisteeg,Raman,Torres,Vafa '24]

- Conjecture follows from tameness of embedding: in preparation [TG,Prieto]

  $\pi : \mathcal{M} \hookrightarrow \mathbb{R}^N$ isometrically [Nash], $\mathcal{M}^{\text{emb}} = \pi(\mathcal{M})$

  $\mathcal{M}^{\text{emb}}$ definable in o-minimal structure $\text{Vol}(B^N(r) \cap \mathcal{M}^{\text{emb}}) \leq \mathcal{C}\, r^{\dim(\mathcal{M})}$ [Yomdin,Comte]

  implies $\text{Vol}(\mathcal{M}_D^{\text{emb}}) \leq \mathcal{C}\, D^{\dim(\mathcal{M})}$

- Tameness of Riemannian manifold is weaker than o-minimality of isometric embedding

  quantum gravity: moduli spaces admit a tame isometric embedding

Thanks!

Some examples

- #complexity is minimal (*F*,*D*) needed to define the function
- exponential function: e^{ax} $(F, D) = (2, 2)$
- fewnomials: $ax^{2d} + bx^d$ $(F, D) = (1, 2d)$

alternative representation: $f_1 = x^d, \; f_2 = \dfrac{1}{x}$ $(F, D) = (3, 6)$

- trigonometric: $\cos(n\,x)$ on $[-\pi, \pi]$ $(F, D) = (3, 4 + n)$

Note: $x^2 = \sum_{n=0}^{\infty} a_n \cos(n\,x) \approx \sum_{n=0}^{N} a_n \cos(n\,x)$

N to infinity limit decreases complexity

References

- References

2410.23338	with Mick van Vliet
2404.12422	with Damian van de Heisteeg
2404.03716	with Arno Hoefnagels, Mick van Vliet
2407.08815	with Giovanni Ravazzini, Mick van Vliet
2311.09295	with Jeroen Monnee
2310.01484	with Lorenz Schlechter, Mick van Vliet
2302.04275	with Mike Douglas, Lorenz Schlechter
2210.10057	with Mike Douglas, Lorenz Schlechter
2112.08383	TG
2112.06995	with Benjamin Bakker, Christian Schnell, Jacob Tsimerman

\+ work in progress