Resolution of Foliated varieties by torus actions

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Weights vs smooth centers

- Weights are naturally associated with singularities and their resolution

$$Ex : x^2 + y^3 + z^4, \quad w(x) = \frac{1}{2} \quad w(y) = \frac{1}{3} \quad w(z) = \frac{1}{4}$$

Weights eliminate many difficulties associated with smooth centers:

- Smooth centers are not compatible with weights.
- They do not usually improve singularities,
- Extra logarithmic structure is needed when resolving by smooth centers

Resolution of singularites and foliations by weighted centers:

[McQ19] M. McQuillan, *Very fast, very functorial, and very easy resolution of singularities*, 2019
[ATW 19] -D. Abramovich, M. Temkin. J. Włodarczyk *Functorial embedded resolution via weighted blowings up* , 2019
[W22] J. Włodarczyk *Functorial resolution by torus actions*, March 2022, arXiv:2007.13846.
[ABTW 25] -D. Abramovich, A.Belotto, M. Temkin. J. Włodarczyk *Principalization on logarithmically foliated orbifolds*

Foliation:

- A foliation is an *involutive distribution*, closed under the Lie bracket.
- Represented by a coherent subsheaf $\mathcal{F} \subset \mathcal{D}_X^{\log}$, the sheaf of logarithmic derivations on a smooth variety X.

Principalization Over Foliated Manifolds

jointly with D.Abramovich, A.Belotto, M.Temkin

Theorem ((ABTW)Principalization in the presence of foliations)

Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety, and $\mathcal{I}$ a coherent ideal sheaf. Then, there exists a sequence of weighted (or cobordant) blow-ups:

$$(X_0, \mathcal{F}_0, E_0) \leftarrow (X_1, \mathcal{F}_1, E_1) \leftarrow \cdots \leftarrow (X_k, \mathcal{F}_k, E_k) = (X', \mathcal{F}', E'),$$

where each $\mathcal{F}_{i+1}$ is the controlled (or strict) transform of $\mathcal{F}_i$, satisfying:

1. *$\mathcal{I}'$ becomes locally principal and monomial.*
2. *Blow-up centers are $\mathcal{F}_i$-aligned and $\mathcal{I}_i$-admissible.*
3. *If $\mathcal{F}$ is nonsingular, $\mathcal{F}_{i+1}$ remains nonsingular under strict transform.*
4. *Functorial for field extensions, group actions, and smooth morphisms and derivations*

Theorem ((ABTW) Embedded Desingularization in the presence of foliations)

Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety, and let $Y \subset X$ be a subvariety. There exists a sequence of weighted (or cobordant) blow-ups:

$$(X_0, \mathcal{F}_0, E_0) \leftarrow (X_1, \mathcal{F}_1, E_1) \leftarrow \cdots \leftarrow (X_k, \mathcal{F}_k, E_k) = (X', \mathcal{F}', E'),$$

such that:

- *The strict transform Y' of Y is smooth, has normal crossings with E' is $\mathcal{F}'$-aligned and thus decomposes into $\mathcal{F}'$-transverse and $\mathcal{F}'$-tangential parts.*
- *Blow-up centers are $\mathcal{F}_i$-aligned and admissible for the strict transform of Y.*
- *If $\mathcal{I}^Y$ is $\mathcal{F}$-invariant, then all centers are also $\mathcal{F}_i$-invariant.*
- *Additional properties from the principalization theorem apply (functoriality, compatibility with smooth morphisms, etc.).*

Definition: Transverse Section

Definition (Transverse Section)

A subvariety $Y \subset X$ is said to be *transverse* to a foliation $\mathcal{F}$ at a point $p \in Y$ if:

- There exists a partial system of parameters $(x_1, \ldots, x_p)$ centered at p.
- There are derivations $\partial_{x_1}, \ldots, \partial_{x_p} \in \mathcal{F} \cdot \mathcal{O}_{X,p}$
- The ideal $\mathcal{I}^Y$ of Y is locally equal to $(x_1, \ldots, x_p)$.

Transverse Section: If the rank of $\mathcal{F}$ is equal to p, thus

$$\mathcal{F}_p = \mathrm{span}_{\mathcal{O}_{X,p}}(\partial_{x_1}, \ldots, \partial_{x_p})$$

then $Y \subset X$ is called the *transverse section* of $\mathcal{F}$ at p.

Resolution Preserving Transverse Locus

Theorem

(ABTW)[Resolution Preserving Transverse Locus] Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety. Let $Y \subset X$ be a closed subvariety generically transverse to $\mathcal{F}$.

There exists an embedded desingularization

$$(X_0, \mathcal{F}_0, E_0) \leftarrow (X_1, \mathcal{F}_1, E_1) \leftarrow \cdots \leftarrow (X_k, \mathcal{F}_k, E_k) = (X', \mathcal{F}', E'),$$

such that the strict transform Y' of Y is transverse to $\mathcal{F}'$.

- *If $\dim(Y) = \operatorname{rank}(\mathcal{F})$, then Y' is a transverse section to $\mathcal{F}'$.*
- *The sequence defines an isomorphism over the points of X where Y is $\mathcal{F}$-transverse.*

Rationally Totally Integrable Foliations

Definition (Totally Integrable Foliations)

Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety. We say that $\mathcal{F}$ is *globally rationally (or meromorphically) totally integrable* if there exists a rational (or meromorphic) morphism $\varphi : X \dashrightarrow B$ such that:

$$\mathcal{F} = \mathcal{D}^{\log}_{X/B}, \quad \mathcal{F}\cdot\mathcal{O}_{X,p} = \mathrm{span}_{\mathcal{O}_{X,p}} \left\{ \partial \in \mathcal{D}^{\log}_{X};\ \partial(f \circ \varphi) \equiv 0,\ \forall f \in \mathcal{O}_{B,\varphi(p)} \right\}.$$

$\mathcal{K}$-Monomial Foliations

Definition (Belotto)

Let $(X, \mathcal{F}, E)$ be a foliated logarithmic variety over $\mathbb{K}$, and let $\mathcal{K} \subset \mathbb{K}$ be a field.

A foliation $\mathcal{F}$ is said to be $\mathcal{K}$-*monomial* at a point $\mathfrak{a}$ if there exists a regular coordinate system (w, v) in a neighborhood U of $\mathfrak{a}$ such that $\mathcal{F}$ can be locally generated on U by:

$$\partial_{v_i} \quad \text{and} \quad \nabla_j = \sum_{k=1}^{n} a_{jk} w_k \, \partial_{w_k},$$

where $j = 1, \ldots, n$, $i = 1, \ldots, r$, and $a_{jk} \in \mathcal{K}$.

Alternatively, in terms of regular forms, $\mathcal{F}$ can also be expressed as:

$$\prod_{j=1}^{n} w_j \cdot \sum_{j=1}^{n} \beta_{ij} \frac{dw_j}{w_j}(\partial) \equiv 0, \quad i = 1, \ldots, r,$$

Globally $\mathcal{K}$-Darboux Totally Integrable Foliations

Definition (Globally $\mathcal{K}$-Darboux Totally Integrable Foliations)

Let $(X, \mathcal{F}, E)$ be a foliated logarithmic variety over $\mathbb{K}$ and $\mathcal{K} \subset \mathbb{K}$ be a field.

We say that the foliation $\mathcal{F}$ is *globally $\mathcal{K}$-Darboux totally integrable* if:

- There exists a rational (or meromorphic) morphism $\varphi : X \dashrightarrow B$ whose graph projects properly over X.
- There exists a $\mathcal{K}$-monomial foliation $\mathcal{G}$ over a smooth variety B.
- The foliation $\mathcal{F}$ is the inverse transform of $\mathcal{G}$ by φ.

Canonical Nonsingular Cobordant Resolution

Lemma (Resolution of $\mathcal{K}$-Monomial Foliations)

A canonical nonsingular cobordant resolution exists for a $\mathcal{K}$-monomial (or, in particular, $\mathcal{K}$-log-smooth) foliation. This resolution is compatible with smooth morphisms, group actions, and field extensions.

Resolution of Rationally Totally and $\mathcal{K}$-Darboux Integrable Foliations

Theorem

(ABTW)[Resolution of Totally Integrable Foliations] There exists a nonsingular cobordant resolution:

$$(X, \mathcal{F}, E) \to (X', \mathcal{F}', E')$$

of a Totally (resp. $\mathcal{K}$-Darboux) Integrable Foliation $(X, \mathcal{F}, E)$

- *X' admits a torus action by T, with a geometric quotient X'/T and a T-invariant projective birational morphism $\pi : X'/T \to X$.*
- *The strict transform $\mathcal{F}' := \pi^s(\mathcal{F})$ is a nonsingular T-invariant foliation on X'.*
- *Over the open subset $U = X \setminus \mathrm{Sing}(\mathcal{F})$, the foliation $\mathcal{F}$ descends to a nonsingular foliation $\mathcal{F}_{|U}$.*

Orbifold Resolution

- The process induces a $\mathbb{Q}$-monomial (resp. $\mathcal{K}$)-monomial orbifold resolution of aTotally (resp. $\mathcal{K}$-Darboux) Integrable Foliation$(X, \mathcal{F}, E)$:

$$\sigma : (X'', \mathcal{F}'', E'') \to (X, \mathcal{F}, E),$$

  where:
  - $X'' = [X'/T]$ is the stack-theoretic quotient of X' by T.
  - $\mathcal{F}''$ is the strict transform of $\mathcal{F}$.
  - E'' is a simple normal crossing (SNC) divisor descending from E'.
- $\mathcal{F}''$ on X'' becomes a $\mathcal{K}$-monomial foliation.

Functoriality

- The resolution process is functorial for:
  - Field extensions.
  - Smooth morphisms with respect to the pair $(\mathcal{F}, \varphi)$.

Cobordant resolution and rational Rees algebras (W)

Definition: The *order* of an ideal $\mathcal{I}$ at a point $p \in X$ is given by:

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Z}_{>0} \mid \mathcal{I} \subset m_p^a\},$$

where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal of p in the local ring $\mathcal{O}_{X,p}$.

Introducing Rees algebra with dummy variable t write:

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t^a \subset \mathcal{O}_X[m_p t]\}.$$

By rescaling $t \mapsto t^{1/a}$ using rational Rees algebras:

$$\mathrm{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a}]\}.$$

Definition of the Invariant via Rees Algebras (W)

Generalizing the Order of an Ideal:

$$\mathrm{inv}_p(\mathcal{I}) := \max\{(a_1, \ldots, a_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\},$$

where $a_1 \leq \ldots \leq a_k$ are rational numbers ordered lexicographically.

Extension to Rational Rees Algebras: For a rational Rees algebra $R = \bigoplus R_a t^a$:

$$\mathrm{inv}_p(R) := \max\{(a_1, \ldots, a_k) \mid R \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\}.$$

Extended Rees Algebra and Cobordant Blow-Up

Extended Rees Algebra:

$$\mathcal{A}^{\text{ext}} = \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

where $w_A := \operatorname{lcm}(a_1, \ldots, a_k)$ is the least common multiple of the rational numbers $a_1, \ldots, a_k$.

Rescaled Algebra:

$$\mathcal{O}_B = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}],$$

with $w_i = w_A / a_i$.

Definition: Full Cobordant Blow-Up:

$$B = \operatorname{Spec}_X\left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]\right) \to X,$$

at the center defined by $\mathcal{A}^{\text{ext}}$

Cobordant Blow-Up and Exceptional Divisor

Vertex of B:

$$V = \mathrm{Vert}(B) := V(x_1 t^{w_1}, \ldots, x_k t^{w_k}).$$

This is called the **vertex** of B, analogous to the vertex of an affine cone over a projective scheme.

Cobordant Blow-Up:

The T-invariant morphism:

$$\sigma_+ : B_+ = B \setminus \mathrm{Vert}(B) \to X.$$

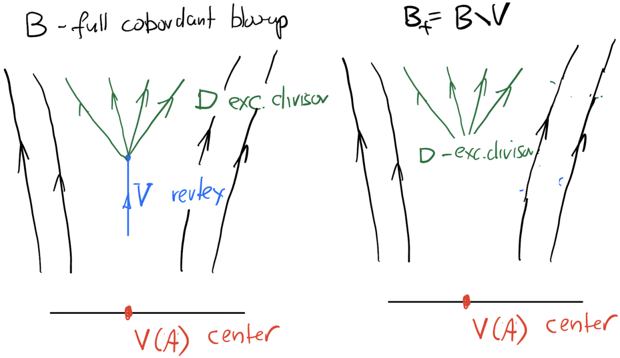
Trivial Cobordant Blow-Up:

$$B_- = B \setminus V(t^{-1}) = \mathrm{Spec}_X(\mathcal{O}_X[t, t^{-1}]) \to X,$$

where:

$$D := V_B(t^{-1})$$

is the exceptional divisor.


B - full cobordant blowup
D exc. divisor
V vertex
V(A) center
$B_+ = B \setminus V$
D - exc. divisor
V(A) center

Cobordant vs orbifold weighted blow-ups

The cobordant blow-ups is not a birational transformation. It introduces the action of torus T.

One can recover the standard definition of the *weighted blow-up* to be

$$B_+/T \to X,$$

where B_+/T is a *geometric quotient* (space of orbits) and *stack-theoretic weighted blow-up*

$$[B_+/T] \to X,$$

for the stack-theoretic quotient $[B_+/T]$.

Remark. *Weighted stack-theoretic blow-ups* were introduced in resolution context in by McQuilan Marzo and ATW. *Cobordant blow-ups* were considered first in W, and independently (as presentations of weighted blow-ups in Quek-Rydh.)

Admissibility and Controlled Transform

The **admissibility condition**:

$$\mathcal{I}t \subset \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

translates into:

$$\mathcal{O}_B \cdot \mathcal{I}t^{w_A} \subset \mathcal{O}_B.$$

This defines the **controlled transform** of $\mathcal{I}$ under $\sigma : B \to X$:

$$\sigma^c(\mathcal{I}) = \mathcal{O}_B \cdot \mathcal{I}t^{w_A}.$$

Resolution Principle

1. The invariant inv of the controlled transform:

$$\sigma^c(\mathcal{I}) := \mathcal{O}_B \cdot \mathcal{I} t^{w_A}$$

achieves its maximum at the vertex V in B, equal to its maximum along the center.

2. The invariant inv **drops** for the *cobordant blow-up* $B_+ := B \setminus V$ after removing V:

$$\max \mathrm{inv}_B(\sigma^c(\mathcal{I})) < \max \mathrm{inv}_X(\mathcal{I}),$$

leading to the **resolution of singularities**.

Example

Let $Y \subset X = \mathbb{A}^n$ be described as:

$$x_1^{b_1} + \ldots + x_n^{b_n}$$

$$\mathrm{inv}_0(Y) = \max\{(a_1, \ldots, a_n) \mid (x_1^{b_1} + \ldots + x_n^{b_n})t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\},$$

$= (b_1, \ldots, b_n)$ with the center

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/b_1}, \ldots, x_k t^{1/b_n}]^{\mathrm{int}}.$$

Rescaling gives $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_n}])$

$$\begin{aligned} \mathcal{O}_B \cdot \mathcal{I}_Y &= t^{-b_1 w_1} \left((x_1 t^{w_1})^{b_1} + \ldots + (x_n t^{w_n})^{b_n} \right) \\ &= t^{-b_1 w_1} \underbrace{\left(\left(x_1'\right)^{b_1} + \ldots + \left(x_n'\right)^{b_n} \right)}_{\sigma^s(\mathcal{I}_Y) \text{ - strict (controlled) transform}} . \end{aligned}$$

Resolution by Removing Singularities

Strict Transform:

$$\sigma^s(\mathcal{I}_Y) = \left(\left(x_1'\right)^{b_1} + \ldots + \left(x_n'\right)^{b_n} \right),$$

has exactly the same equation as $\mathcal{I}_Y$.
After **Cobordant Blow-up:**

$$\sigma_+ : B_+ \to Y.$$

The strict transform:

$$\sigma^s_+(Y) = \sigma^s(Y) \setminus \underbrace{V(x_1', \ldots, x_n')}_{\text{vertex } V}$$

on:

$$B_+ = B \setminus V,$$

becomes regular after removing the vertex V.

Resolution Process Defined by the Invariant $\mathrm{inv}_p(\mathcal{I})$

Process of Resolution:

- The maximum value of the invariant drops after each blow-up.
- The process continues until the invariant reaches the smooth point value:

$$\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1).$$

- At this stage, singularities on the strict transform of Y are resolved.

Foliated Varieties and the Invariant

Definition: Let $(X, \mathcal{F})$ be a smooth foliated variety, $p \in X$ be a point.

$$\mathrm{inv}_{p,\mathcal{F}}(\mathcal{I}) := \max\{(a_1, \ldots, a_k, \infty + c_1, \ldots, \infty + c_r) \mid$$

$$\mathcal{I}t \subset \widehat{\mathcal{O}}_{X,p}[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}, y_1 t^{1/c_1}, \ldots, y_r t^{1/c_r}]^{\mathrm{int}}\}$$

where:

$$\widehat{\mathcal{F}} = \widehat{\mathcal{O}}_{X,(x)} \cdot \mathcal{F} = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1(y), \ldots, \nabla_r(y))$$

- with **transversal part of the center** $\widehat{\mathcal{O}}_{X,p}[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$
- and **invariant part:** $\widehat{\mathcal{O}}_{X,p}[y_1 t^{1/c_1}, \ldots, y_r t^{1/c_r}]^{\mathrm{int}}$ such that $\mathcal{F}\left(\widehat{\mathcal{O}}_{X,p}[y_1 t^{1/c_1}, \ldots, y_r t^{1/c_r}]^{\mathrm{int}}\right) \subset \widehat{\mathcal{O}}_{X,p}[y_1 t^{1/c_1}, \ldots, y_r t^{1/c_r}]^{\mathrm{int}}$

Embedded Resolution of foliated varieties

- The maximum value of the invariant drops after each blow-up.
- The process continues until the invariant reaches the smooth point value:

$$\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1).$$

- At this stage, the singularities on the strict transform of Y are resolved, and it is $\mathcal{F}$-aligned. Thus locally in the coordinate system $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$,

$$Y = V(x_1, \ldots, x_k, y_1, \ldots, y_m) \quad m \leq n-k$$

and

$$\widehat{\mathcal{F}} = \widehat{\mathcal{O}}_{X,(x)} \cdot \mathcal{F} = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1(y), \ldots, \nabla_r(y))$$

where $V(x_1, \ldots, x_n)$ is $\mathcal{F}$- transverse and $V(y_1, \ldots, y_m)$ is $\mathcal{F}$-tangential so $\mathcal{F}(y_i) = 0$ for $i = 1, \ldots, m$.

Resolution of Darboux Integrable Foliations

Let $\phi : X \to B$ be a morphism, and $\mathcal{F} = \phi^{-1}(\mathcal{G})$ be the pull-back of the $\mathcal{K}$-monomial foliation $\mathcal{G}$ on a smooth B,

- Consider the product $Y = X \times B$ with projection $p_B : Y \to B$.
- Let $X := \Gamma(\varphi) \subset Y$ denote the graph of φ.

Define $\mathcal{H}$ on $Y = X \times B$ to be the inverse transform:

$$\mathcal{H} = p_B^{-1}(\mathcal{G})$$

Then $\mathcal{H}$ is $\mathcal{K}$-monomial and admits a *nonsingular cobordant resolution*.

Applying Embedded Desingularization

- The strict transform $X \subset Y$ becomes $\mathcal{H}$-aligned.
- The foliation $\mathcal{H}$ remains nonsingular.
- The restriction $\mathcal{F} = \mathcal{H}_{|Y}$ is nonsingular.

Example of cobordant resolution of singular foliation by a nonsingular one

Example

$\mathcal{F} = \mathrm{x}\partial x$ singular at 0 on $X = \mathbb{A}^1$,

$B = \mathrm{Spec}(\mathcal{O}_X[xt, t^{-1}] = \mathrm{Spec}(\mathcal{O}_X[x', t^{-1}], x' = xt.$

$B_+ = B \setminus V(x')$.

The strict transform of $\mathrm{x}\partial\mathrm{x}$ is nonsingular $\mathrm{x}'\partial\mathrm{x}'$, where $x' \neq 0$.