SEMIHOMOGENEOUS BUNDLES, FOURIER–MUKAI TRANSFORMS, AND TROPICALIZATION

Martin Ulirsch

based on joint work with

- Andreas Gross and Dmitry Zakharov
- Andreas Gross, Inder Kaur, and Annette Werner

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GAUS

combinatorial synergies

RIEMANN SURFACES AND METRIC GRAPHS: AN ANALOGY

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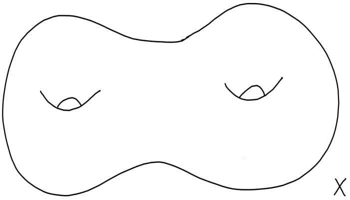
(= smooth projective alg. curve)



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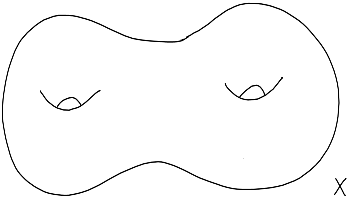

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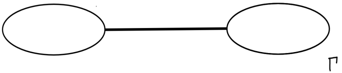

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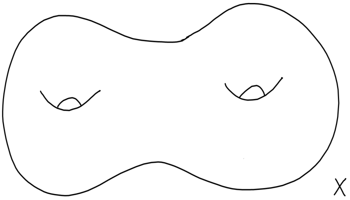

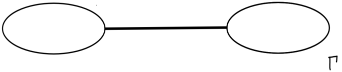

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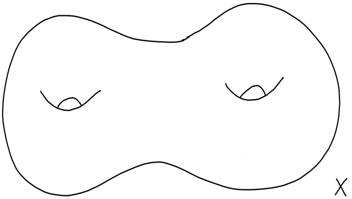

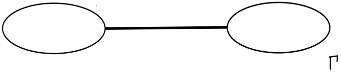

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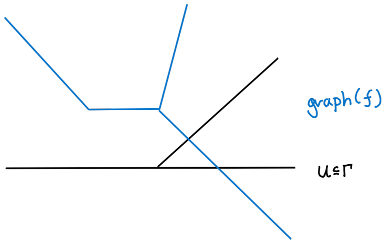

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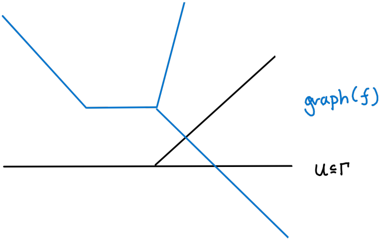

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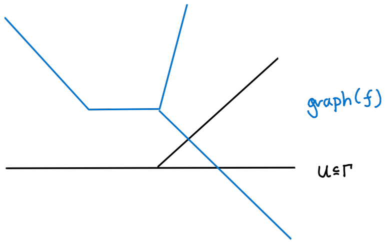

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TROPICAL LINE BUNDLES [À LA MIKHALKIN–ZHARKOV ’08, MOLCHO–WISE ’22]

Let Γ be a metric graph. Define a sheaf $\mathcal{H}_\Gamma$ of harmonic functions by

$$\mathcal{H}_\Gamma(U) = \left\{ f \colon U \to \mathbb{R} \;\middle|\; \mathrm{ord}_p(f) = 0 \text{ for all } p \in U \right\}$$

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- How to define a tropical analogue of vector bundles of higher rank?

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  1. $\nu(a) = \infty$ if and only if $a = 0$.
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- This motivates the definition of the tropical semifield $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \sqcup \{\infty\}, \min, +)$.
- Tropical geometry is often described as *(algebraic) geometry over* $\mathbb{T}$.

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- One can use this observation to define tropical vector bundles as principal $\mathrm{GL}_n(\mathbb{T})$-bundles.

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Definition (Allermann ’12, Gross–U.–Zakharov ’22)

Let Γ be a metric graph. A tropical vector bundle on Γ is an $S_n \ltimes \mathcal{H}_\Gamma^n$-torsor.

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Observation

For every tropical vector bundles E on Γ there is a topological cover $\phi\colon \widetilde{\Gamma} \to \Gamma$ as well as a line bundle L on $\widetilde{\Gamma}$ such that

$$\phi_* L \simeq E$$

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Let X be a complex elliptic curve. Choose a Lagrangian fibration $X \to \Gamma$, whose base is a metric cycle $\Gamma = \mathbb{R}/\ell\mathbb{Z}$.

Theorem (Gross–U.–Zakharov '22)

Let $M_{r,d}(X)$ be the moduli space of semistable vector bundles on X of degree d and rank r. There is a natural orbifold Lagrangian fibration

$$M_{r,d}(X) \to M_{r,d}^{main}(\Gamma),$$

whose Lagrangian base is the main component of a moduli space $M_{r,d}(\Gamma)$ of tropical semistable vector bundles on Γ of degree d and rank r.

Let X be a complex elliptic curve and $\mathcal{X} \to \mathbb{D}$ a semistable logarithmically smooth degeneration of X whose special fiber is a cycle of projective lines. Denote by Γ its metric dual graph.

Theorem (Gross–U.–Zakharov '22)

Let $M_{r,d}(X)$ be the moduli space of semistable vector bundles on X of degree d and rank r.

There is a logarithmically smooth degeneration of $M_{r,d}(X)$, for which the dual complex of its central fiber is the main component of the moduli space $M_{r,d}(\Gamma)$ of tropical semistable vector bundles on Γ of degree d and rank r.

A NON-ARCHIMEDEAN PERSPECTIVE ON TROPICALIZATION

Let X be an elliptic curve over a non-Archimedean field $K = \overline{K}$, whose reduction is maximally degenerate. Denote by X^{an} its Berkovich analytification. Then $X^{an} = \mathbb{G}_m/q^{\mathbb{Z}}$ and its minimal non-Archimedean skeleton is a metric cycle $\Gamma = \mathbb{R}/\ell\mathbb{Z}$.

Theorem (Gross–U.–Zakharov '22)

Let $M_{r,d}(X)$ be the moduli space of semistable vector bundles on X of degree d and rank r.

Then the essential non-Archimedean skeleton of $M_{r,d}^{an}(X)$ is naturally isomorphic to the main component of the moduli space $M_{r,d}(\Gamma)$ of semistable tropical vector bundles on Γ of degree d and rank r.

$\rightsquigarrow$ Think of the retraction from $M_{r,d}^{an}(X)$ onto the essential skeleton as a non-Archimedean SYZ-fibration in the sense of Kontsevich–Soibelman.

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SEMIHOMOGENEOUS VECTOR BUNDLES ON ABELIAN VARIETIES

Let X be an abelian variety over an algebraically closed field k. A vector bundle E on X is said to be semihomogeneous if, for every $x \in X$, we have

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where the U_i are suitable unipotent bundles and the E_i are simple semihomogeneous bundles.

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$$E = \bigoplus_i E_i \otimes U_i \ ,$$

where the U_i are suitable unipotent bundles and the E_i are simple semihomogeneous bundles. We have $E_i \simeq (\phi_i)_* L_i$ for a cover $\phi_i : \widetilde{X}_i \to X$ and a line bundle $L_i \in \mathrm{Pic}(\widetilde{X}_i)$.

Observation (Gross–Kaur–U.–Werner ’23)

Denote by $M_{H,1}(X)$ *the moduli space of simple semihomogeneous vector bundles* E *of fixed slope* $\delta(E) := \frac{\det E}{r(E)} = H \in \mathrm{NS}(X)_{\mathbb{Q}}$.

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There is a Fourier–Mukai equivalence

$$D^b(X) \xrightarrow{\sim} D^b\big(M_{H,1}(X)\big)$$

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Denote by $M_{H,k}(X)$ the moduli space of semihomogeneous vector bundles of slope H and rank $r = k \cdot r(H)$. Then $M_{H,1}(X)$ is a torsor over a suitable abelian variety and

$$M_{H,k}(X) \simeq \mathrm{Sym}^k M_{H,1}(X) .$$

Let X be a complex abelian variety of dimension g. Choose a Lagrangian fibration $X \to X^{\text{trop}}$, whose base is a real torus $X^{\text{trop}} = \mathbb{R}^g/\Lambda$.

Theorem (Gross–Kaur–U.–Werner ’23)

Let $M_{H,k}(X)$ be the moduli space of semihomogeneous vector bundles of slope H and rank $r = k \cdot r(H)$.

Then there is a natural orbifold Lagrangian fibration

$$M_{H,k}(X) \to M_{H,k}(X^{\text{trop}}),$$

whose Lagrangian base $M_{H,k}(X^{\text{trop}})$ is a moduli space of tropical semihomogeneous vector bundles on X^{trop} of slope H^{trop} and rank $r^{\text{trop}} = k \cdot r(H^{\text{trop}})$.

A LOGARITHMIC PERSPECTIVE ON TROPICALIZATION

Let X be an abelian variety of dimension g and $\mathcal{X} \to \mathbb{D}$ a logarithmically smooth degeneration of X, for which the dual complex of its central fiber is a (subdivision of) the real torus $X^{\mathrm{trop}} = \mathbb{R}^g/\Lambda$.

Theorem (Gross–Kaur–U.–Werner '23)
Let $M_{H,k}(X)$ be the moduli space of semihomogeneous vector bundles of slope H and rank $r = k \cdot r(H)$.

There is a logarithmically smooth degeneration of $M_{H,k}(X)$, for which the dual complex of its central fiber is (a subdivision of) a moduli space $M_{H,k}(X^{\mathrm{trop}})$ of tropical semihomogeneous vector bundles of slope H^{trop} and rank $r^{\mathrm{trop}} = k \cdot r(H^{\mathrm{trop}})$.

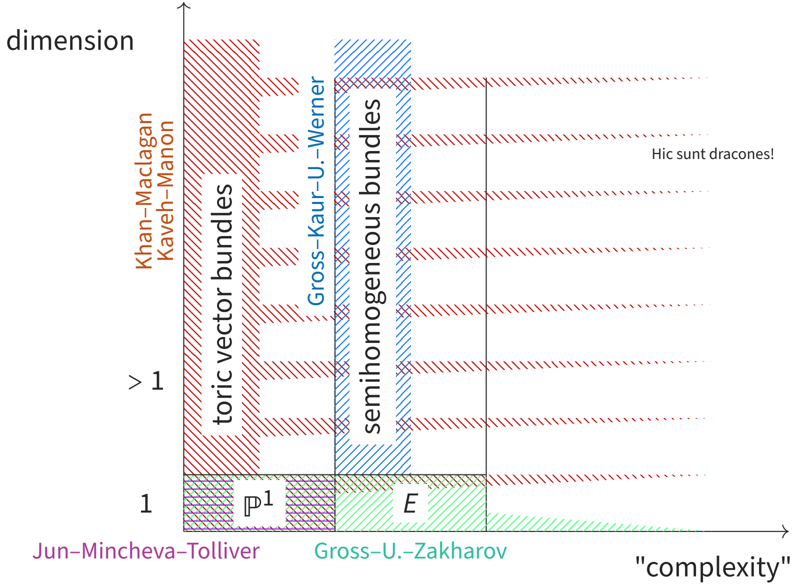
Let X be an abelian variety of dimension g over a non-Archimedean field K, whose reduction is maximally degenerate. Then we have $X^{\mathrm{an}} = \mathbb{G}_m^g/\Lambda$ and its non-Archimedean skeleton is the real torus $X^{\mathrm{trop}} = \mathbb{R}^g/\Lambda$.

Theorem (Gross–Kaur–U.–Werner ’23)

Let $M_{H,k}(X)$ be the moduli space of semihomogeneous vector bundles of slope H and rank $r = k \cdot r(H)$.

Then the essential non-Archimedean skeleton of $M_{H,k}^{\mathrm{an}}(X)$ is naturally isomorphic to a moduli space $M_{H,k}(X^{\mathrm{trop}})$ of semihomogeneous vector bundles on X^{trop} of slope H^{trop} and rank $r^{\mathrm{trop}} = k \cdot r(H^{\mathrm{trop}})$.

WHAT WE KNOW ABOUT TROPICAL VECTOR BUNDLES SO FAR



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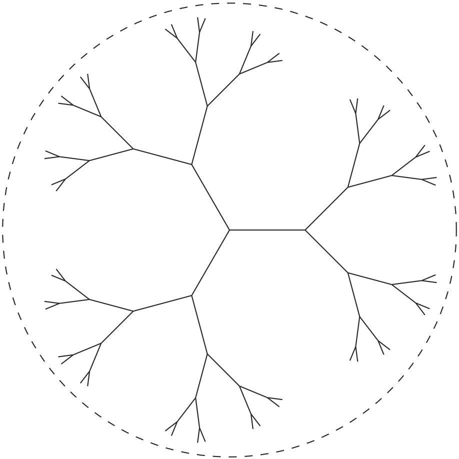
WHAT IS THE TROPICALIZATION OF A LINEAR MAP?



The affine Bruhat-Tits building 🏠$_1(\mathbb{Q}_2)$

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Definition

A seminorm on V is a map $||.|| \colon V \to \mathbb{R}$ such that

- $||v|| \geq 0$ for all $v \in V$ and $||0|| = 0$;
- $||\lambda v|| = |\lambda| \cdot ||v||$ for all $\lambda \in K$ and $v \in V$; and
- $||v + w|| \leq \max\left\{||v||, ||w||\right\}$ for all $v, w \in V$.

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Denote by $\overline{\mathcal{N}}(V)$ the space of seminorms on V^*, endowed with the coarsest topology making all

$$\overline{\mathcal{N}}(V) \ni ||.|| \longmapsto ||v^*|| \in \mathbb{R}$$

for all $v^* \in V^*$ continuous. The quotient $\overline{\mathcal{X}}(V) = \mathcal{N}(V)/\mathbb{R}_{>0}$ is called the Goldman–Iwahori space of V.

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Let $\#K = \infty$ with trivial valuation. We may identify $\overline{\mathcal{B}}_1(K)$ with the set

$$\left\{(0 \subsetneq V_1 \subsetneq (K^2)^*, c) \,\middle|\, c \in \overline{\mathbb{R}}_{>0}\right\} \cup \left\{(0 \subsetneq (K^2)^*)\right\}$$

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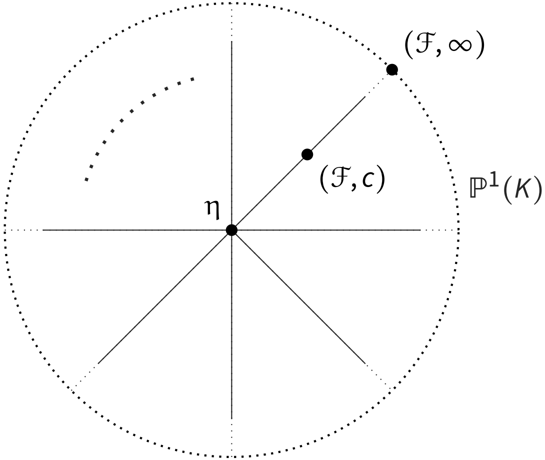

Theorem (Battistella–Kühn–Kuhrs–U.–Vargas '24)

There is a natural homeomorphism

$$\overline{\mathfrak{X}}(V) \xrightarrow{\sim} \varprojlim_{\iota \in I} \operatorname{Trop}\big(\mathbb{P}(V), \iota\big),$$

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Theorem (Dress–Terhalle ’98 + ϵ·BKKUV ’24)

The Goldman–Iwahori space $\overline{\mathfrak{X}}(V)$ *is the tropical linear space associated to the universal realizable valuated matroid.*

Observation (Heunen–Patta '17, Dress?, Kontsevich?)

- *A linear map* $f\colon V \to W$ *induces an (affine) morphism of (pointed) valuated matroids* $f^{\text{univ}}\colon M_V^{\text{univ}} \to M_W^{\text{univ}}$ *which, in turn, induces the continuous map* $\overline{\mathfrak{X}}(V) \to \overline{\mathfrak{X}}(W)$ *given by* $\|.\| \mapsto \|.\| \circ f^*$*.*

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$\rightsquigarrow \overline{\mathfrak{X}}(V)$ seems to be the better "tropical linear space".