Specialization techniques and stable rationality

John Christian Ottem

Méthodes logarithmiques et non-archimédiennes en théorie des singularités
Logarithmic and non-archimedean methods in Singularity Theory
In honor of Bernard Teissier
27-31 January, 2025

Lecture 1:
Birational invariants and specialization

These talks will revolve around a paper written with **Johannes Nicaise**:

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The paper gives a quite general method for the (stable) rationality problem for hypersurfaces and complete intersections in toric varieties.

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X is *stably rational* if it is stably birational to $\mathbb{P}^n$.

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Theme in these lectures: Verify (2) by *specialization* to a highly singular variety.

Introduction

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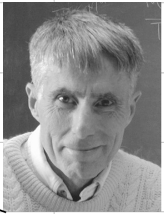

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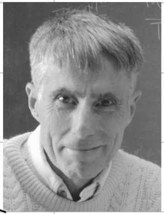

Obstruction to rationality: The intermediate jacobian $H^{1,2}(X)/H^3(X,\mathbb{Z})$.

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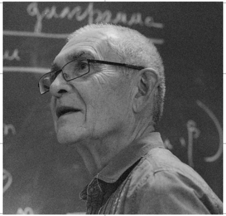

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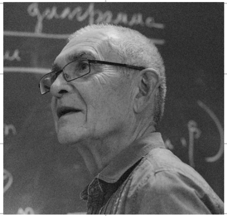

Obstruction to rationality: The birational automorphism group $\mathrm{Bir}(X)$ is finite.

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$\therefore$ $H^3(\widetilde{X}, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$ have the same torsion.

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- The invariant $H^3(X,\mathbb{Z})_{\text{tors}}$ is closely related to the Brauer group.

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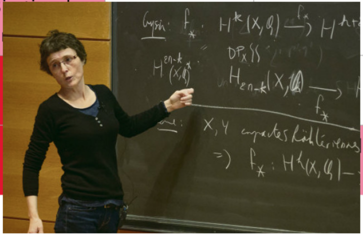

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9-folds	10-folds	11-folds	12-folds	13-folds	14-folds	15-folds	16-folds	17-folds	18-folds	19-folds
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where $Z \subset X \times X$ is a subvariety which does not dominate X via the first projection.

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Exercise involving Chow groups

Having a decomposition of Δ is a *stable birational invariant* for smooth projective varieties.

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$\rightsquigarrow$ $\Delta \neq [X \times x] + Z$, because $\Delta^* \omega = \omega$.

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Families of varieties and specializations

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In general, (i) and (ii) can vary drastically in a family.

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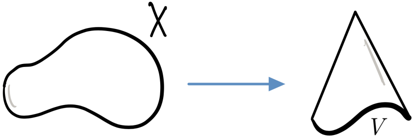

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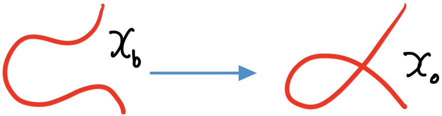

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$$H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}), \qquad H^0(\mathcal{X}_b, \mathcal{O}(2K_{\mathcal{X}_b}))$$

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Exercise involving Hilbert schemes

For a family $f : \mathcal{X} \to B$, the *Rational locus*

$$Rat(f) = \{b \in B \,|\, \mathcal{X}_b \text{ is rational}\}.$$

is a countable union of locally closed subsets of B.

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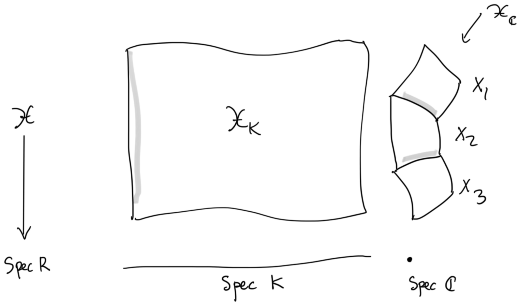

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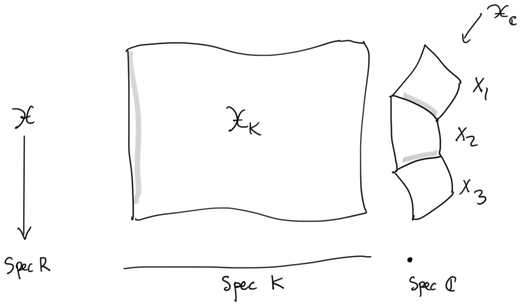

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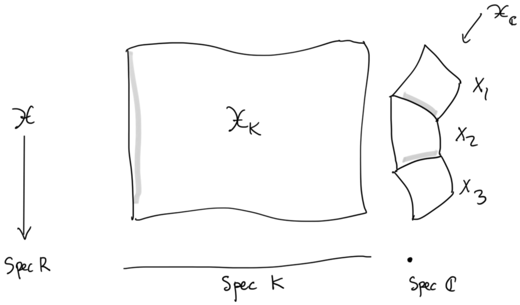

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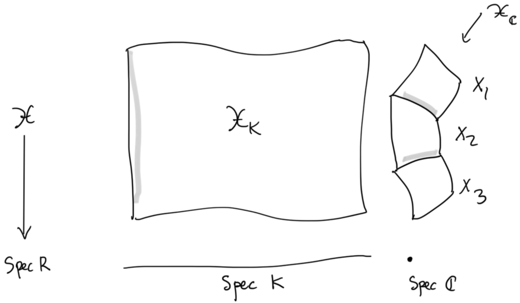

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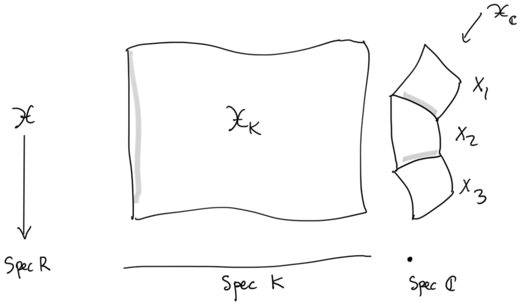

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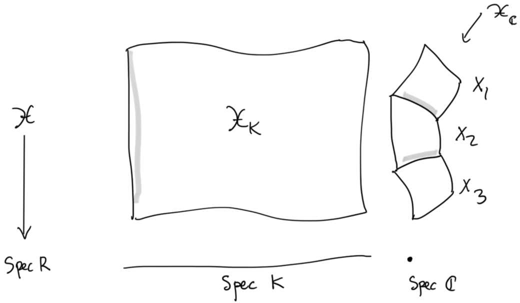

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$\rightsquigarrow$ *specialization map of Chow groups*

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$\rightsquigarrow$ the very general $\mathcal{X}_b$ is not stably rational.

Proposition

For $b \in B$ very general, the fiber $\mathcal{X}_b$ is isomorphic (as a scheme) to the geometric generic fiber $\mathcal{X}_{\overline{K}}$, where $K = k(B)$.

More precisely, there is a field isomorphism $\overline{K} \to k(b)$, and isomorphisms $\mathcal{X}_b \to \mathcal{X}_{\overline{K}}$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{X}_b \arrow[r] \arrow[d] & \mathcal{X}_{\overline{K}} \arrow[d] \\
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\end{tikzcd}$$

Therefore, if we only care about the very general member of some family of varieties (e.g., the very general hypersurface), this is the same thing as the geometric generic fiber.

Lecture 2:
The motivic volume formula of Nicaise–Shinder

Two varieties X and Y are *stably birational* if

$$X \times \mathbb{P}^m \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^l$$

for some $m, l \geq 0$.

X is *stably rational* if it is stably birational to $\mathbb{P}^n$.

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(i) A very general **degree 4** hypersurface

$$X \subset \mathbb{P}^n$$

is not stably rational for $n \leq 6$.

(ii) A very general **degree 5** hypersurface

$$X \subset \mathbb{P}^n$$

is not stably rational for $n \leq 13$.

(iii) A very general complete intersection of a **quadric** and a **cubic**

$$X = Q \cap C \subset \mathbb{P}^6$$

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The Motivic volume

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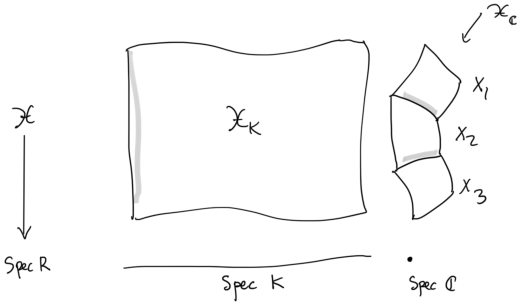

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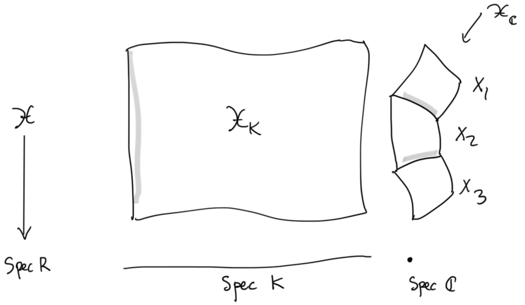

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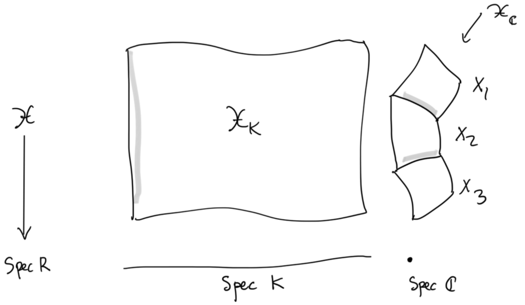

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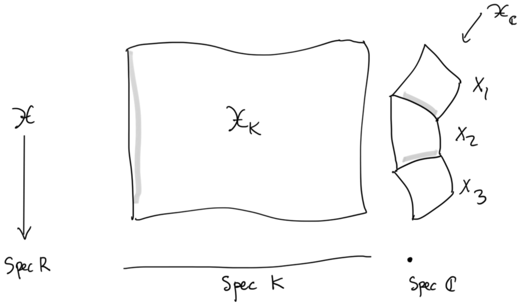

$\mathcal{X}_{\mathbb{C}}$ may have several components $\rightsquigarrow$ makes sense to do this comparison in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$.

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The method of Nicaise and Shinder was extended Kontsevich–Tschinkel with ‘stable rationality’ replaced by ‘rationality’.

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Moreover, for a family $f\colon \mathcal{X} \to B$ the rational locus

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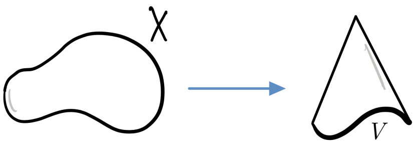

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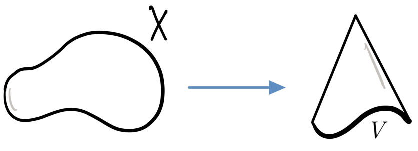

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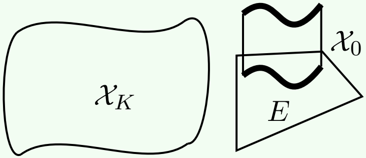

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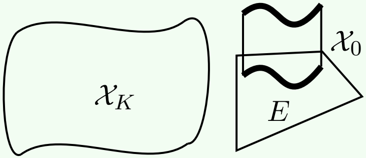

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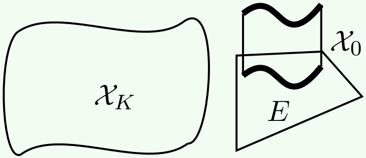

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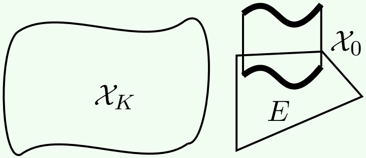

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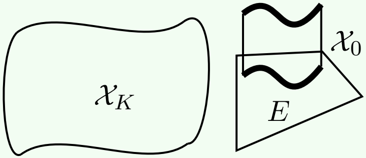

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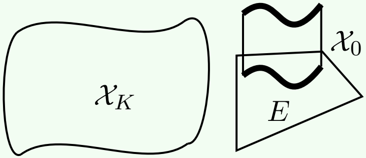

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$\mathcal{X}$ is *strictly toroidal* if, Zariski-locally on $\mathcal{X}$, we can find a smooth morphism

$$\mathcal{X} \to \operatorname{Spec} R[M]/(x^m - t^q)$$

for some toric monoid M, some positive rational number q, and some element m in M such that $k[M]/(x^m)$ is reduced.

Toroidal models

For our main applications, we need a more flexible notion than semi-stability:

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A monoid M is called *toric* if it is isomorphic to the monoid of lattice points in a strictly convex rational polyhedral cone.

Example

The scheme

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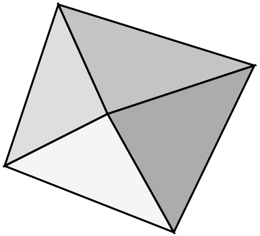
The special fiber has four irreducible components of dimension 2 intersecting at the origin, which never happens for strictly semi-stable schemes.



More generally:

Example

Let $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$. Then the following R-scheme is strictly toroidal

$$\mathcal{X} \quad = \quad \operatorname{Spec} R[x_{i,j} \mid i = 1, \ldots, r;\ j = 1, \ldots, a_i]/(t - \prod_{j=1}^{a_1} x_{1,j}, \ldots, t - \prod_{j=1}^{a_r} x_{r,j}).$$

These examples will be important when degenerating complete intersections.

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- The product of two strictly toroidal R-schemes is again strictly toroidal.

  This is not true for strictly semistable.
- Strictly toroidal degenerations arise naturally when we break up projective hypersurfaces into pieces of smaller degrees.

Example

Let $f_0, \ldots, f_r \in k[z_0, \ldots, z_{n+1}]$ be general homogeneous polynomials of degrees $d_0, \ldots, d_r$ such that $d_0 = d_1 + \ldots + d_r$.

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$\mathcal{X}$ is not strictly semi-stable at the points of $\mathcal{X}_k$ where f_0 and at least two among $f_1, \ldots, f_r$ vanish.

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There is a ring map

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \longrightarrow \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly toroidal proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(\mathcal{X})} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}. \tag{4}$$

$\mathcal{S}(\mathcal{X})$ = the set of strata of the special fiber $\mathcal{X}_k$.

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Sketch of the proof of the motivic volume formula

Weak factorization theorem $\rightsquigarrow$ reduce to showing that

$$\mathrm{Vol}(\mathcal{X}_{\mathbb{C}}) = \sum_{Z \in S(\mathcal{X}_{\mathbb{C}})} (-1)^{\mathrm{codim}(Z)} [Z]_{\mathrm{sb}} \in \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

is invariant under blow-ups

$$\beta \colon \widetilde{X} \to \mathcal{X}$$

in smooth subvarieties $Y \subset \mathcal{X}$.

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Hence the volume does not change:

$$\mathrm{Vol}(\widetilde{X}) = \sum_{Z \in S(\mathcal{X}_{\mathbb{C}})} (-1)^{\mathrm{codim}(Z)} [Z]_{\mathrm{sb}} = \mathrm{Vol}(\mathcal{X}) \tag{5}$$

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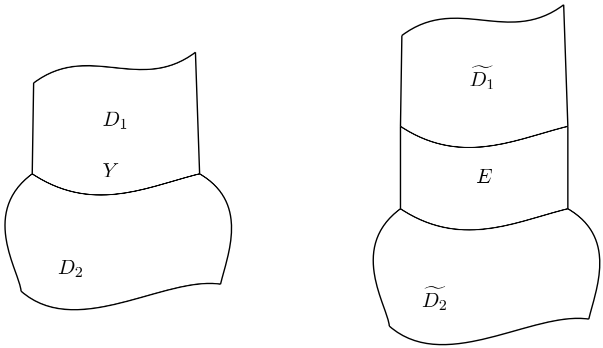

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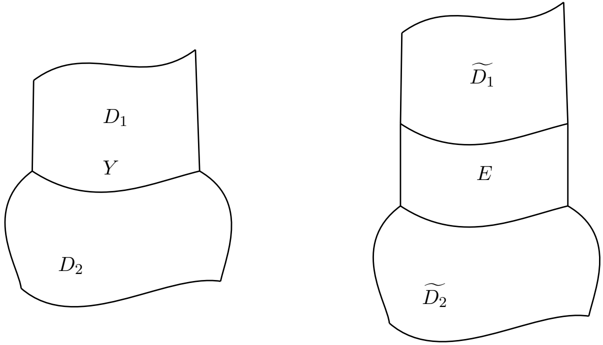

$$\mathrm{Vol}(\widetilde{X}) \quad = \quad [\widetilde{D_1}]_{\mathrm{sb}} + [\widetilde{D_2}] + [E]_{\mathrm{sb}} - [\widetilde{D_1} \cap E] - [\widetilde{D_2} \cap E]$$

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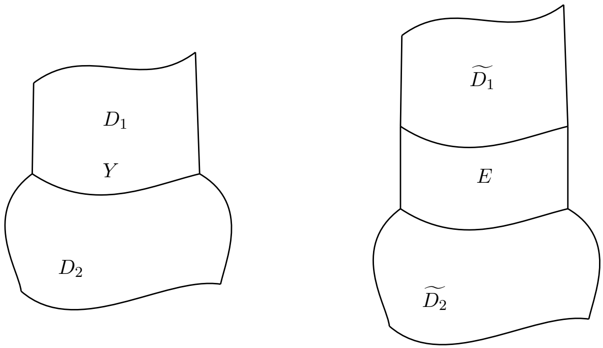

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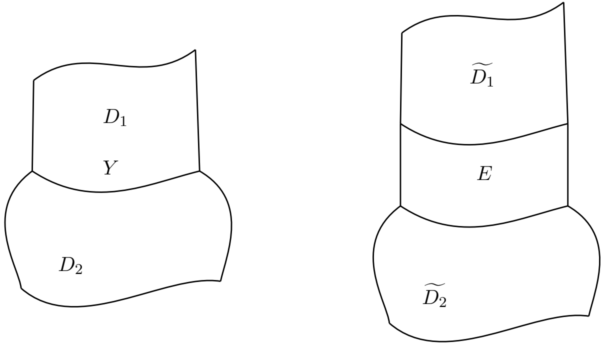

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&= \mathrm{Vol}(\mathcal{X}).
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Case (a)

Let $D_1, \dots, D_n$ be the components of $\mathcal{X}_0$, ordered so that

$$Y \subset D_1 \cap \ldots \cap D_a$$

and $Y \not\subseteq D_i$ for $i > a$.
By Weak Factorization, we may assume that Y intersects the divisors $D_{a+1}, \dots, D_b$ transversally, and for $i \geq a+1$, the divisors

$$Z_i = D_i \cap Z$$

are all distinct and form a normal crossing divisor on Z.

Case (a)

After blowing up Y, the new special fiber has the following divisors:

- The strict transforms $\widetilde{D_i}$ for $i = 1, \ldots, n$
- The exceptional divisor E which is a projective bundle over Y.

The special fiber is still a simple normal crossing divisor. As $\widetilde{D_1} \cap \ldots \cap \widetilde{D_a} = \emptyset$, the motivic volume is a sum of terms of the form

$$\widetilde{D}_{A \cup B} \text{ with sign } (-1)^{|A|+|B|-1}$$

and

$$\widetilde{D}_{A \cup B} \cap E \text{ with sign } (-1)^{|A|+|B|}$$

over all subsets $A \subset \{1, \ldots, a\}$ with $|A| < a$ and $B \subset \{a+1, \ldots, n\}$.
Note that $\widetilde{D}_{A \cup B} \cap E$ is stably birational to $D_{[a] \cup B}$ in $\mathcal{X}$. This, together with the identity

$$\sum_{A \subset [a], |A| < a} (-1)^{|A|} = (-1)^{a-1}$$

shows that the volume is equal to

$$\sum_{A,B} (-1)^{|A|+|B|-1} D_{A \cup B} + \sum_{B} (-1)^{|B|} (-1)^{a-1} D_{[a] \cup B} = \sum_{C \subset [b]} (-1)^{|C|-1} D_C$$

which is exactly the volume.

Case (b)

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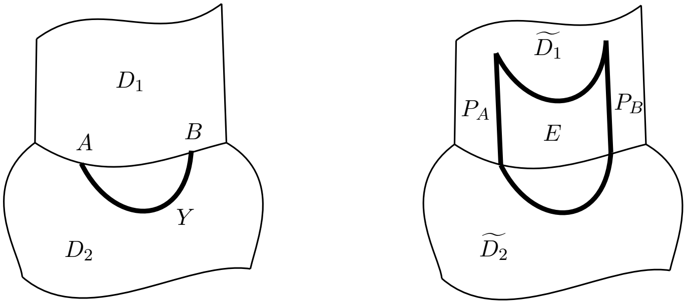

$$\mathrm{Vol}(\widetilde{\mathcal{X}}) \quad = \quad [\widetilde{D_1}] + [\widetilde{D_2}] + [E] - {\color{red}[\widetilde{D_1} \cap E]} - [\widetilde{D_2} \cap E] - [\widetilde{D_1} \cap \widetilde{D_2}] + {\color{blue}[\widetilde{D_1} \cap \widetilde{D_2} \cap E]}$$

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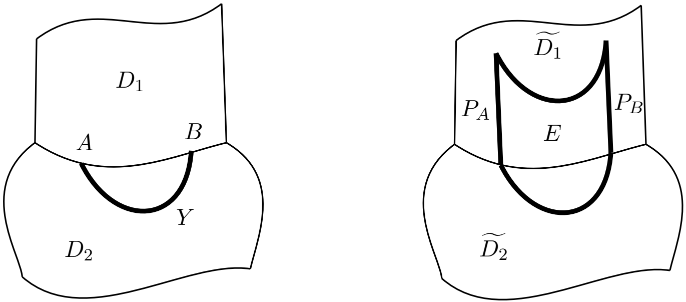

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&= [D_1] + [D_2] + [Y] - [P_A] - [P_B] - [Y] - [D_1 \cap D_2] + [A] + [B]
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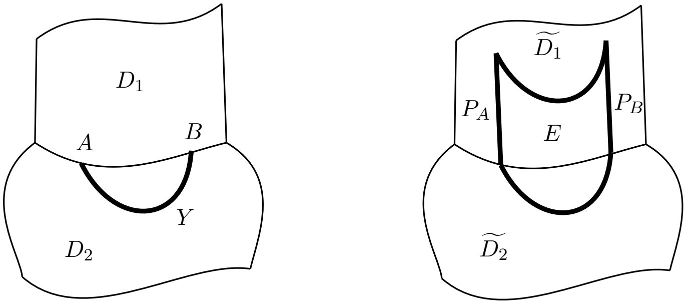

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&= [D_1] + [D_2] + [Y] - [P_A] - [P_B] - [Y] - [D_1 \cap D_2] + [A] + [B] \\
&= [D_1] + [D_2] - [D_1 \cap D_2] \\
&= \mathrm{Vol}(\mathcal{X}).
\end{aligned}$$

Case (b)

Assume that $Y \subset D_{[a]}$ is not a component of $D_{[a]}$.
On the blow-up $\widetilde{X}$, there are the divisors

- The strict transforms $\widetilde{D_i}$ for $i = 1, \ldots, b$
- The exceptional divisor E which is a projective bundle over Y.

We have two types of intersections:

$$\widetilde{D}_{A \cup B}$$

and

$$\widetilde{D}_{A \cup B} \cap E$$

over all subsets $A \subset \{1, \ldots, a\}$ and and $B \subset \{a+1, \ldots, b\}$.

For the intersection $\widetilde{D}_{A\cup B}$, the map $\widetilde{D}_{A\cup B} \to D_{A\cup B}$ is birational on each component, so every term in the volume of $\widetilde{X}$ matches a unique term in the volume of $\mathcal{X}$ with the same sign.
To conclude, we claim that the other terms cancel out.
To see this, note that the components of

$$\widetilde{D}_{A\cup B} \cap E$$

are in bijection with the components of

$$D_{A\cup B} \cap Y$$

As each component of $\widetilde{D}_{A\cup B} \cap E$ maps as a generic projective bundle over $D_B \cap Y$, they are stably birational.
For a fixed connected component of $D_B \cap Y$, the alternating sum of components of $\widetilde{D}_{A\cup B} \cap E$ that map to it is zero. Thus the extra terms cancel out.

Applications

Important observation: Vol maps $\operatorname{Spec} K$ to $\operatorname{Spec} \mathbb{C}$.

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Corollary

Let $\mathcal{X}$ be a strictly toroidal proper R-scheme with smooth generic fiber $\mathcal{X}_K$. If

$$\sum_{E \in \mathcal{S}(\mathcal{X})} (-1)^{\operatorname{codim}(E)} [E]_{\mathrm{sb}} \neq [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$, then X_K is not stably rational.

Example (Voisin)

A very general double quartic threefold $X \to \mathbb{P}^3$ is not rational.

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Sketch of proof.

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with $\mathcal{X}_K \subset \mathbb{P}^{n+1}_K$ smooth hypersurface, with the property that *stably irrational strata of low dimension do not cancel out in the alternating sum*

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

Main strategy

We often get better results using degenerations with several components.

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$\therefore$ We deduce irrationality of $\mathcal{X}_K$ from that of varieties of lower dimension.

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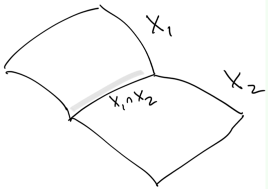

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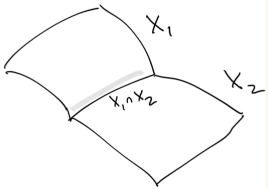

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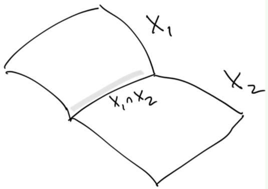

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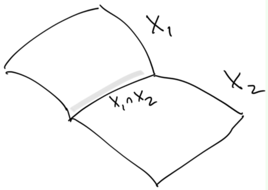

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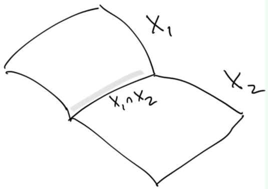

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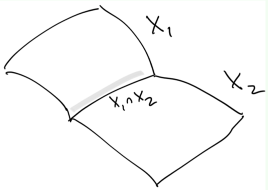

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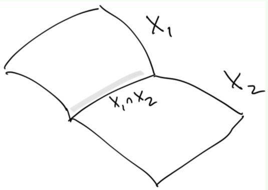

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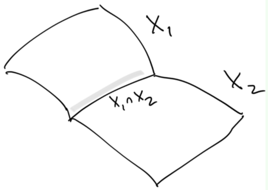

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The above result settles the rationality problem for all complete intersections of dimension ≤ 4 - except cubic fourfolds.

The proof for $(2,3)$-complete intersections

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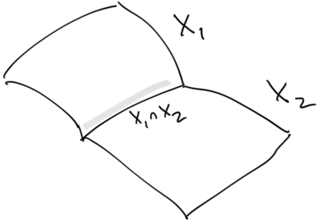

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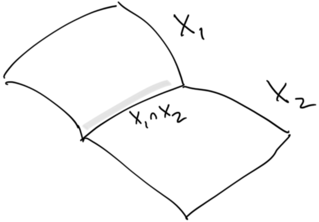

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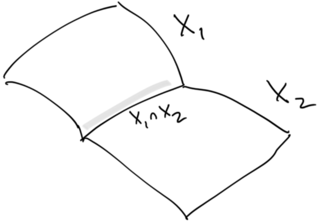

There are three strata:

- $X_1 = Q_0 \cap C$
- $X_2 = E \cap C$
- $X_{12} = Q_0 \cap E \cap C$

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$\rightsquigarrow$ X_1 is stably irrational by [Schreieder 2017].

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Lecture 3:
Toric degenerations and applications

Recap

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such that: $\forall$ strictly toroidal proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$,

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \text{ strata of } \mathcal{X}_{\mathbb{C}}} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}. \tag{7}$$

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- **Obstruction:** If one can write down a family $\mathcal{X}$ so that the alternating sum (7) does not cancel out to $[\operatorname{Spec} \mathbb{C}]$ in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$, then $\mathcal{X}_K$ not stably rational.

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Let $q, f \in k[x_0, \ldots, x_8]$ be very general of degrees $2, 4$.

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In any event,

$$\begin{aligned} \operatorname{Vol}(\mathcal{X}_K) &= [E_1]_{\mathrm{sb}} + [E_2]_{\mathrm{sb}} - [E_{12}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}. \end{aligned}$$

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By induction, this is $\neq [\operatorname{Spec} \mathbb{C}]$.

Projective toric varieties

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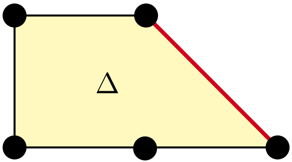
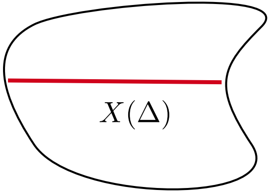
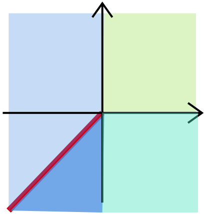
1-1 inclusion preserving correspondence between faces of Δ and toric strata of X:







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$\Delta \cap M = \{m_0, \ldots, m_r\}$.

$$\begin{array}{rcl} f\colon (\mathbb{C}^*)^n & \longrightarrow & \mathbb{P}^r \\ x & \mapsto & (x^{m_0} : \cdots : x^{m_r}). \end{array}$$

$$\begin{array}{rcl} X(\Delta) & = & \text{Zariski closure of the image of } f \\ & = & \operatorname{Proj} \mathbb{C}[C(\Delta) \cap \mathbb{Z}^{n+1}] \end{array}$$

where $C(\Delta)$ is the cone over Δ.

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  $H^0(\Sigma_\Delta, \mathcal{O}(1))$ has a basis corresponding to the lattice points of Δ.

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$(\mathbb{P}^n, \mathcal{O}(1))$ is given by the *n-dimensional simplex*

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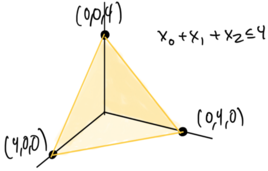
This is the d-th Veronese embedding of $\mathbb{P}^n$.

Ex



$(\mathbb{P}^3, \mathcal{O}(4))$

Example (Product polytopes)

If (X, L) and (Y, M) correspond to polytopes $P_X \subset \mathbb{R}^n$ and $P_Y \subset \mathbb{R}^m$, then the product

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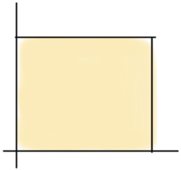
For instance $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b))$ is given by the rectangle

$$P_{a,b} = \{(x, y) \,|\, 0 \leq x \leq a, 0 \leq y \leq b\}$$



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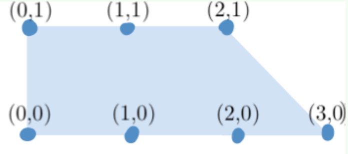

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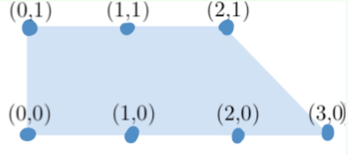

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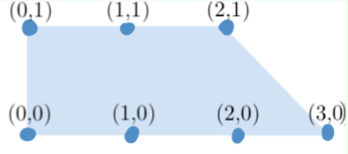

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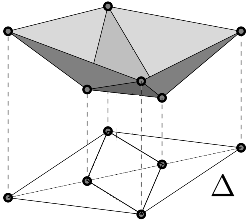

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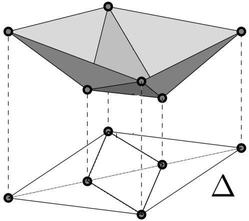

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$$\lambda : \Delta \to \mathbb{R}_{\geq 0}$$

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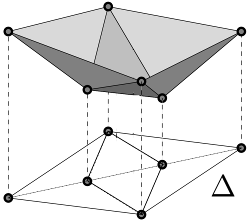

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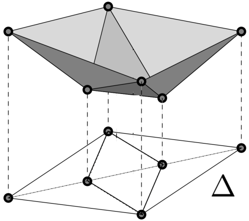

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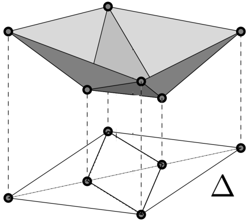

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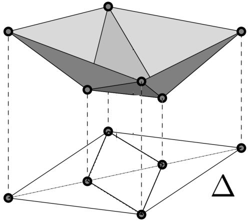

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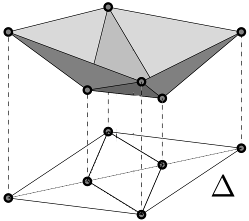
Toric degeneration



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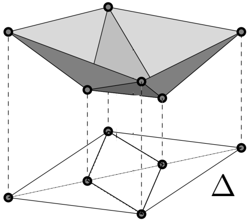

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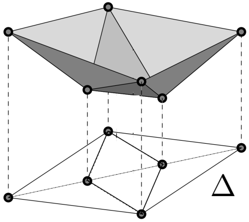

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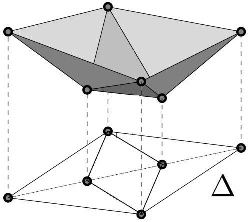

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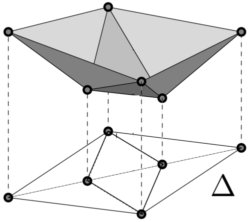

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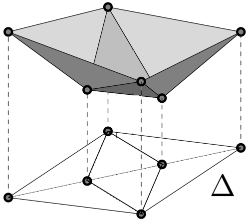
- The components intersect according to the combinatorics of the subdivision:

Toric degeneration



Given a regular subdivision $\mathscr{P}$, we can construct a (flat) degeneration

$$\mathcal{X} \to \mathbb{A}^1,$$

satisfying:

- $\mathcal{X} - \mathcal{X}_0 \simeq X(\Delta) \times \mathbb{C}^*$.
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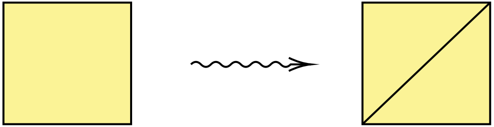

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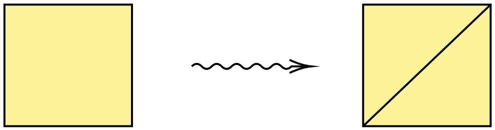

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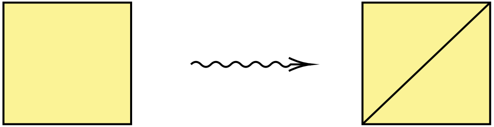

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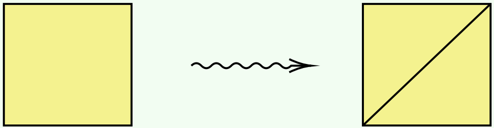

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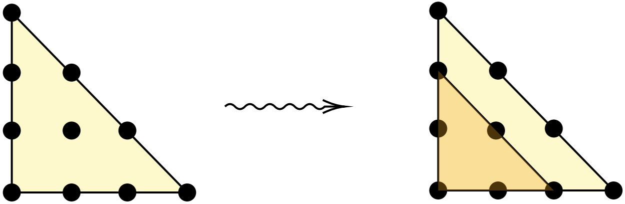

$\left(\mathbb{P}^2, 3H\right)$ $\quad$ $\left(\mathbb{P}^2, 2H\right) \cup \left(Bl_p\mathbb{P}^2, 2H - E\right)$

$$1 + x + x^2 + xy + y^2 + tx^3 + tx^2y + txy^2 + ty^3$$

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Proposition

Assuming that f is non-degenerate in the above sense, the corresponding degeneration is toroidal. Hence we can apply the motivic volume formula.

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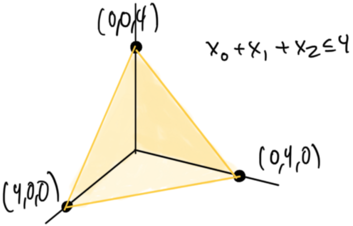

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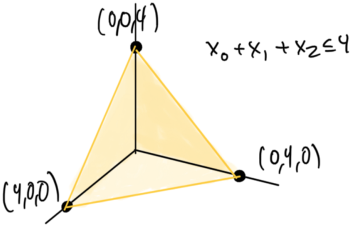

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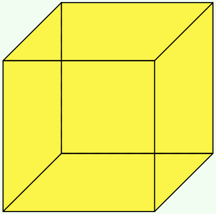
Example (Hassett–Pirutka–Tschinkel)

The product polytope $2\Delta_2 \times 2\Delta_3 \subset \mathbb{R}^5$ is stably irrational.



The following (2,2)-divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ is stably irrational.

$$xyU^2 + xzV^2 + yzW^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))T^2 = 0$$

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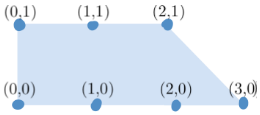

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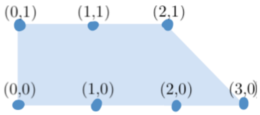

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The projection $(x, y) \to y$ induces a birational map $Z(F) \dashrightarrow \mathbb{A}^1$.

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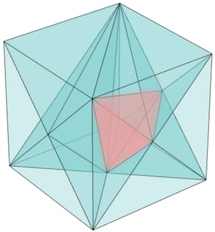

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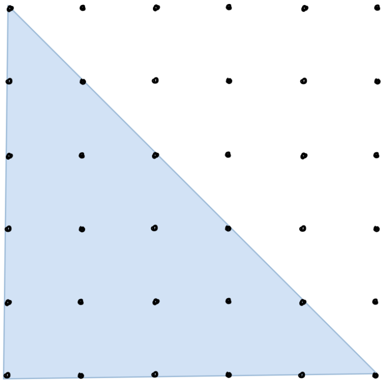

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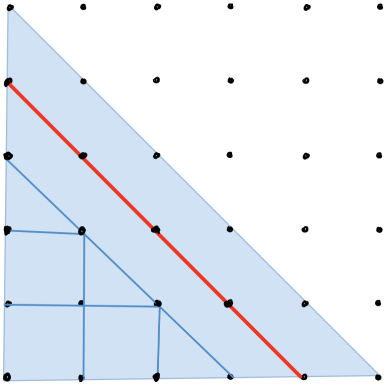

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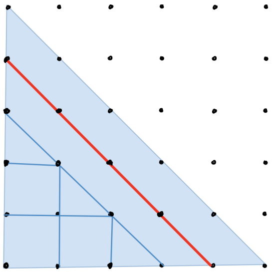

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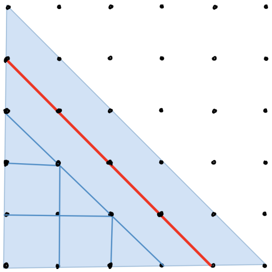

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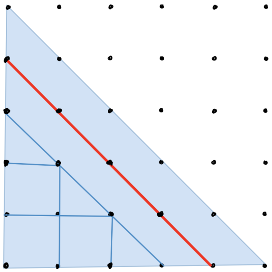

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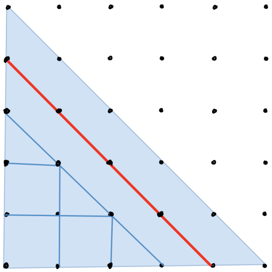

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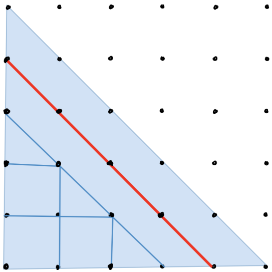

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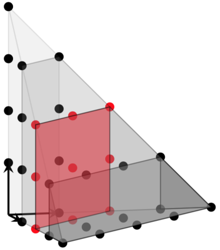

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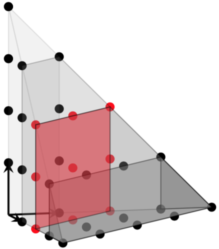

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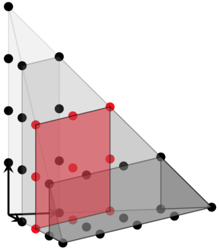

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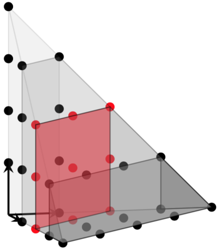

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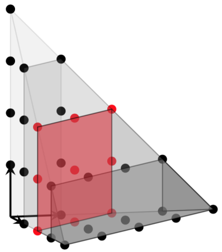

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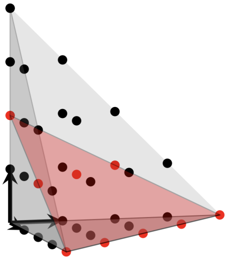

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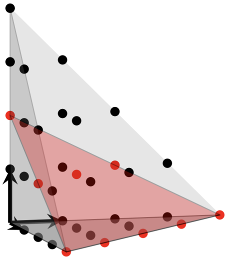

Red polytope = double quartic 4-fold.

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$\therefore$ we get all bidegrees corresponding to rational/irrational hypersurfaces.

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Hassett–Pirutka–Tschinkel/Schreieder:
Anything that specializes to Y does not admit a decomposition of Δ (hence is stably irrational).

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Concretely, the following bidegree $(2,3)$ polynomial

$$\begin{gathered} x_0^2 y_0^3 - 2x_0x_1y_0^3 + x_1^2y_0^3 - 2x_0^2y_0^2y_1 - 2x_0x_1y_0^2y_1 \\ + x_0^2y_0y_1^2 + x_0x_1y_1y_2^2 + x_0^2y_1y_3^2 + x_0x_1y_0y_4^2 \end{gathered}$$

dehomogenizes to the HPT quartic.

Let $\mathscr{P}$ denote the regular subdivision of the polytope $2\Delta_1 \times 3\Delta_4$ induced by the convex function

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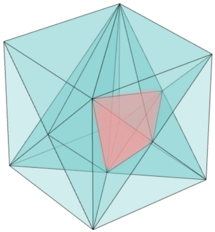

$\rightsquigarrow$ degeneration of $\mathbb{P}^1 \times \mathbb{P}^4$ into a union of 26 toric varieties.

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The case $\deg F = 1, \deg G = 2$ correspond to *Gushel–Mukai varieties*.

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$\rightsquigarrow$ a general hypersurface of degree 3 in $\mathbb{P}^9$ pulls back to a polynomial with Newton polytope $3P$.

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Define

$$\iota : \mathbb{R}^5 \to \mathbb{R}^6 \quad (t_1, t_2, t_3, t_4, t_5) \mapsto (t_5, t_4, t_1, t_5, t_2, t_3).$$

Then $\Delta_{\mathrm{HPT}} := \iota(\Delta_F)$ is contained in $3P$ and it is not contained in the boundary of $3P$.

Take the subdivision of $3P$ associated to the convex function

$$\lambda(z) = \max_{v \in \Delta_{\mathrm{HPT}}} \|z - v\|^2.$$

Using a computer, one checks that the resulting subdivision $\mathscr{P}$ contains 14 maximal polytopes, and all polytopes in except Δ_{HPT} have lattice width 1,

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