A singular path through toric geometry

Some aspects of
the work of Bernard
Teissier

CIRM
Marseille, 30/01/2025

Contents.

Introduction:

Toric geometry has developed transfer paths and a dictionary:

convex integral geometry $\longrightarrow$ Algebraic geometry and commutative algebra

convex integral geometry $\longleftarrow$ Algebraic geometry and commutative algebra

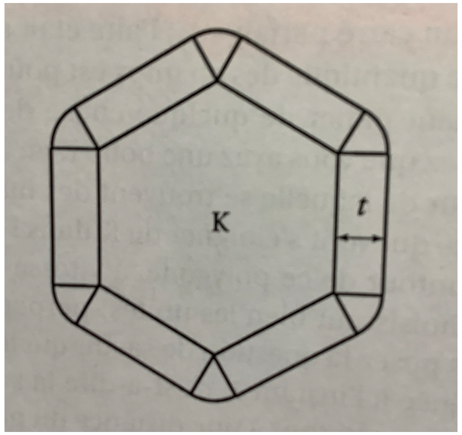
Polytopes, toric varieties and isoperimetric inequalities

Isoperimetric inequalities

Isoperimetric problem: Determine the relations between the volume of a convex compact set $K \subseteq \mathbb{R}^d$ and its boundary ∂K.

Take $B = B(0,1)$ and $K \subseteq \mathbb{R}^2$ with Surface $K = S$, Perimeter $K = L$

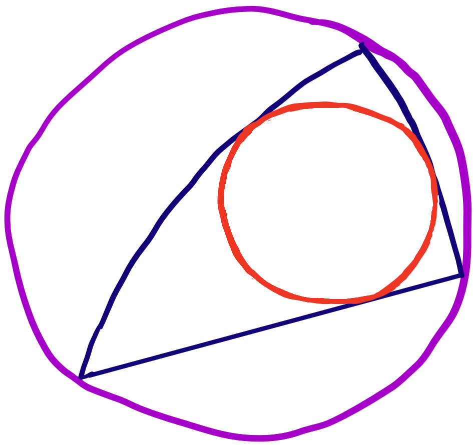
Take $t > 0$ a parameter:



$$\text{Surface}(K + tB) = S + Lt + \pi t^2$$

Iso. Inequality: $\boxed{L^2 - 4\pi S \geqslant 0}$

Bonnesen theorem (1921)



Take a convex compact set K in $\mathbb{R}^2$.

R = minimal ray of a cercle circunscribing K

r = maximal ray of a cercle inscribed in K

Then: $L^2 - 4\pi S \geqslant \pi^2 (R-r)^2$

Corollary: The extremal case $L^2 - 4\pi S = 0$ holds if and only if $R = r$, that is, K is a cercle.

Minkowski sum and mixed volumes.

Take compact convex subsets $K_1, K_2 \subseteq \mathbb{R}^d$

Th. (Minkowski - Steiner) If $s_1, s_2 \geqslant 0$ then

$\mathrm{Vol}(s_1 K_1 + s_2 K_2)$ is a homogeneous polynomial of degree d in s_1, s_2

$$\mathrm{vol}(s_1 K_1 + s_2 K_2) = \sum_{i=0}^{d} \binom{d}{i} v_i \, s_1^{i} \cdot s_2^{d-i}$$

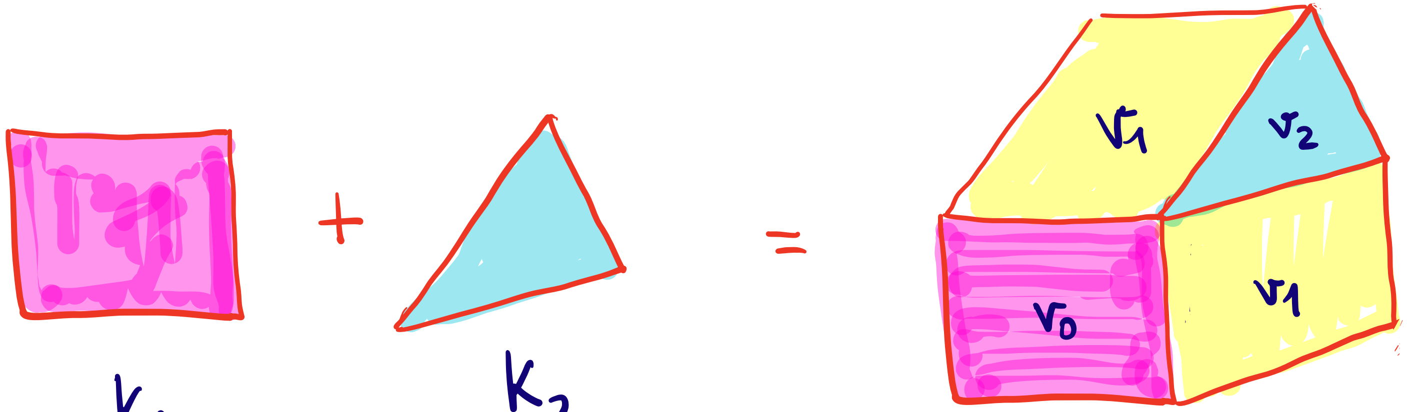
<u>Def</u>: $v_i = \mathrm{vol}(K_1^{[i]}, K_2^{[d-i]})$ = mixed volumes

<u>Rem</u>: $v_d = \mathrm{Vol}(K_1)$, $v_0 = \mathrm{Vol}(K_2)$

Generalized isoperimetric inequalities in $\mathbb{R}^2$.

Let $K_1, K_2 \subseteq \mathbb{R}^2$ be two compact convex subsets. Then:

$$\mathrm{vol}(s_1 K_1 + s_2 K_2) = v_0\, s_1^2 + 2 v_1\, s_1 \cdot s_2 + v_2 \cdot s_2^2$$



K_1 $\quad$ K_2 $\quad$ $K_1 + K_2$

$v_0 = \mathrm{vol}(K_1)$

$v_2 = \mathrm{vol}(K_2)$.

Theorem: [T] Take

$$r = \sup\{t \mid tK_2 \subseteq K_1 \text{ modulo translations}\}$$

$$R = \inf\{t \mid tK_2 \supseteq K_1 \text{ '' '' }\}$$

Then: $$\frac{1}{4}(R-r)^2\, v_2^2 \leq v_1^2 - v_0 v_2.$$

The toric variety determined by an integral polytope

Take $K \subseteq \mathbb{R}^d$ an integral polytope, i.e., with vertex set in the lattice $\mathbb{Z}^d$.

The polytope K is determined and determines the support function defined on the dual vector space $\check{\mathbb{R}}^d$.

$$H : \check{\mathbb{R}}^d \longrightarrow \mathbb{R}$$

$$H(u) = \min_{m \in K;} u(m)$$

Take a fan Σ subdividing $\check{\mathbb{R}}^d$ such H is linear in restriction to the cones $\sigma \in \Sigma$.

For each $\sigma \in \Sigma_i$ we have a finitely generated semigroup $\check{\sigma} \cap \check{\mathbb{Z}}^d$ which defines the affine normal toric variety $X_\sigma = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap \check{\mathbb{Z}}^d]$.

The toric variety $X_{\Sigma_i} = \bigcup_{\sigma \in \Sigma} X_\sigma / \sim$ is obtained by glueing the charts X_σ, where $X_\tau \underset{\text{open}}{\subset} X_\sigma$ if τ is a face of σ.

For each $\sigma \in \Sigma$ we have an integral vector $m_\sigma \in \mathbb{Z}^d$ such that $H(u) = u(m_\sigma)$ for all $u \in \sigma$.

This determines an invertible sheaf L on X_Σ

The space of global sections of L

$H^0(X_\Sigma, L)$ has a basis in bijection with $K \cap \mathbb{Z}^d$.

In particular, (X_Σ, L) determines K and

$$\dim_{\mathbb{C}} H^0(X, L) = \# K \cap \mathbb{Z}^d.$$

The case of two integral polytopes

- Take K_1 and $K_2 \subseteq \mathbb{R}^d$ two integral polytopes with support functions H_1 and H_2.
- Take Σ a fan such that for every $\sigma \in \Sigma$ the restriction of H_i to σ is linear, $i = 1, 2$.
- Take L_1 and L_2 the invertible sheafs defined as before from H_1 and H_2 on X_Σ.

The mixed degrees associated with L_1 and L_2

If $s_1, s_2 \in \mathbb{N}$ then the support function of $s_1 K_1 + s_2 K_2$ is also linear on each $\sigma \in \Sigma_n$.

The associated invertible sheaf is $L_1^{s_1} \otimes L_2^{s_2}$.

Then: $\chi(X_\Sigma, L_1^{s_1} \otimes L_2^{s_2}) = \dim_{\mathbb{C}} H^0(X_\Sigma, L_1^{s_1} \otimes L_2^{s_2})$

is a polynomial of degree d in s_1 and s_2

$$\chi(Z_\Sigma, L_1^{s_1} \otimes L_2^{s_2}) = \frac{1}{d!} \sum_i \binom{d}{i} r_i \, s_1^{\,i} \, s_2^{\,d-i} + (\text{l.d.t.})$$

r_i = mixed degrees of L_1 and L_2

Mixed degrees versus mixed volumes

On the other hand $\mathrm{vol}(s_1 K_1 + s_2 K_2)$ is also of the form:

$$\mathrm{vol}(s_1 K_1 + s_2 K_2) = \sum_i \binom{d}{i} v_i \; s_1^{i} \cdot s_2^{d-i}$$

v_i = mixed volumes of K_1 and K_2

Thanks to the equality: $\lim_{n \to \infty} \frac{1}{n^d} \# (nK \cap \mathbb{Z}^d) = \mathrm{Vol}(K)$

it can be proven that $\boxed{r_i = d! \, v_i, \quad 0 \le i \le d.}$

Application of the Hodge Index Theorem.

Th. [T]

In this situation the Hodge Index theorem implies that:

$$\boxed{r_{i-1}^2 \geq r_i \cdot r_{i-2}\ , \quad \text{for } 2 \leq i \leq d}$$

with equality if and only if there exists $a, b \in \mathbb{N}$ such that $L_1^a \cong L_2^b$.

Aleksandrov-Fenchel Inequalities

Corollary [T].

We get from the theorem the mixed volumes inequalities:

(AF) $\quad V_{i-1}^2 \geq V_i \cdot V_{i-2} \qquad$ for $\quad 2 \leq i \leq d.$

with equality if and only if there exists $a, b \in \mathbb{N}$ such that $\quad a K_1 = b K_2$ modulo translation.

Extension to compact convex subsets

The set of compact convex subsets of $\mathbb{R}^d$ is endowed with the Hausdorff metric δ:

$$\delta(K_1, K_2) = \min\{\varepsilon \mid K_1 \subseteq K_2 + \varepsilon B,\ K_2 \subseteq K_1 + \varepsilon B\}$$

- The mixed volumes are continuous functions with respect this metric.

- An approximation argument by integral polytopes with respect to sufficiently fine lattices implies that the A-F inequalities hold for pairs (K_1, K_2) of compact convex subsets of $\mathbb{R}^d$.

Back to isoperimetric inequalities

Take $K_1 \subseteq \mathbb{R}^d$ a compact convex subset and $K_2 = \mathbb{B} = B(0,1)$

In this case:

$$\boxed{V_{d-1} = \frac{1}{d} \operatorname{vol}(\partial K_1)}$$

<u>Rem:</u> We can reformulate (AF) in the form:

(AF) $$\frac{V_d}{V_{d-1}} \leq \frac{V_{d-1}}{V_{d-2}} \leq \cdots \leq \frac{V_1}{V_0}$$

Then $$\left(\frac{V_d}{V_{d-1}}\right)^{d-1} \leq \frac{V_{d-1}}{V_{d-2}} \cdot \frac{V_{d-2}}{V_{d-3}} \cdot \cdots \cdot \frac{V_1}{V_0} = \frac{V_{d-1}}{V_0}$$

Thus: $$\boxed{V_d^{d-1} \cdot V_0 \leq V_{d-1}^{d}}$$

The isoperimetric inequality in $\mathbb{R}^d$.

$v_{d-1} = \frac{1}{d} \operatorname{vol}(\partial K_1)$, $v_0 = \operatorname{vol}(B)$, $v_d = \operatorname{Vol}(K_1)$

Substituting in $v_d^{d-1} \cdot v_0 \leq v_{d-1}^d$ we get

$$\boxed{\operatorname{vol}(\partial K_1)^d \geq d^d \cdot \operatorname{vol}(K_1)^{d-1} \cdot \operatorname{vol}(B)}$$

If $d=2$ we recover the isoperimetric inequality:

$$L^2 \geq 4\pi S$$

Monomials, integral closure of ideals and multiplicities.

Integral closure of ideals

Def: Let A be a commutative ring and $I \subseteq A$ an ideal

$h \in A$ is integral over I if one has

$$h^k + a_1 h^{k-1} + \cdots + a_k = 0 \qquad \text{with } a_i \in I^i.$$

The set of elements of A which are integral over I is an ideal $\overline{I}$.

Theorem: If A is an analytic algebra then $h \in \overline{I}$ if and only if for all local morphism $\varphi : A \longrightarrow \mathbb{C}\{t\}$ one has

$$\boxed{\operatorname{ord}_t(\varphi(h)) \geq \operatorname{ord}_t(I) = \inf\{\operatorname{ord}_t(\varphi(g)) \mid g \in I\}.}$$

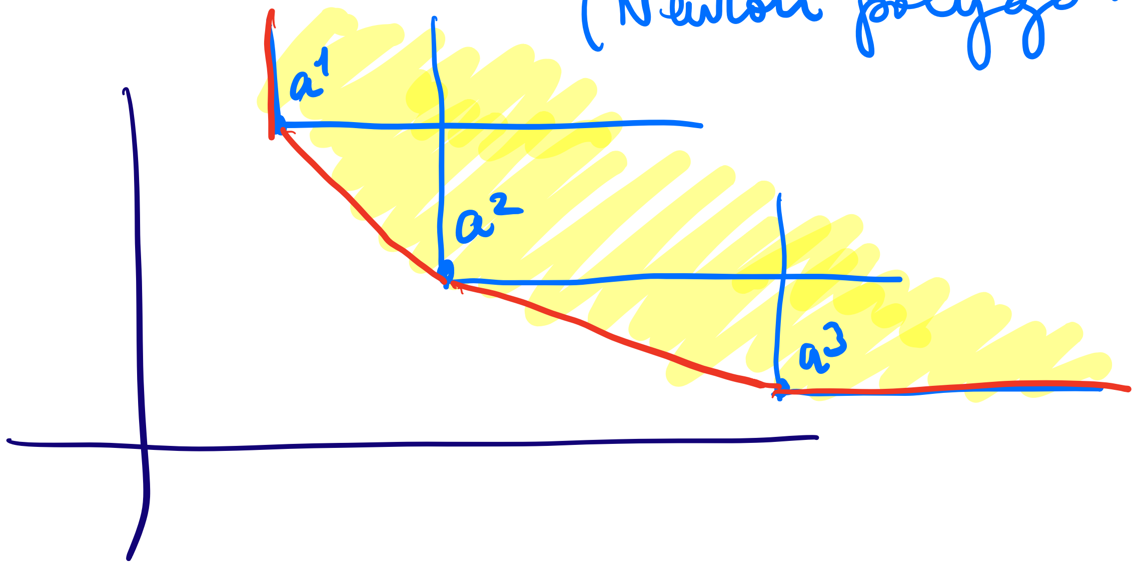
The case of a monomial ideal of $\mathbb{C}\{x_1, \dots, x_d\}$

$I = (\underline{x}^{a^1}, \dots, \underline{x}^{a_k}) \subseteq \mathbb{C}\{\underline{x}\}$ monomial ideal, $\underline{x} = (x_1, \dots, x_d)$.

Set $$E(I) := \bigcup_{i=1}^{k} a^i + \mathbb{R}^d_{\geq 0}$$

$N(I)$:= convex hull of $E(I)$

(Newton polygon of I)



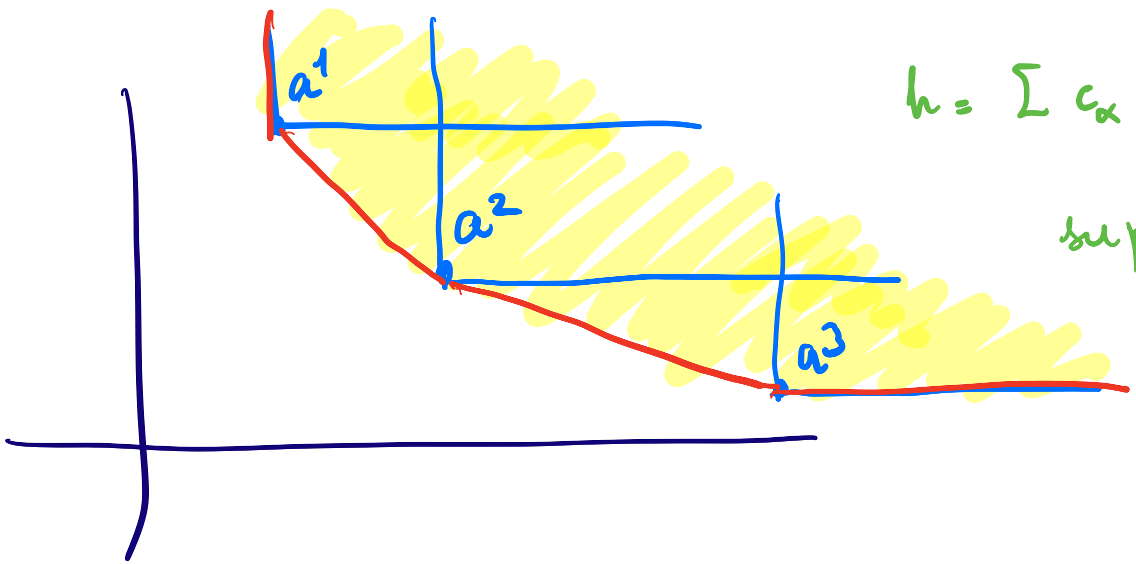
Then, $h \in \mathbb{C}\{\underline{x}\}$ is integral over I if for all

$$\varphi : \mathbb{C}\{\underline{x}\} \longrightarrow \mathbb{C}\{t\}, \quad x_i \mapsto x_i(t)$$

$$\operatorname{ord}_t \big(h(x_1(t), \dots, x_n(t)) \big) \geq \min \{ \langle a^1, \underline{v} \rangle, \dots, \langle a^k, \underline{v} \rangle \}$$

where $\underline{v} = (v_1, \dots, v_d)$, $v_i = \operatorname{ord}_t x_i(t)$.

Corollary: $h \in \overline{I} \iff \operatorname{supp}(h) \subseteq N(I)$



$h = \sum c_\alpha x^\alpha$,

$\operatorname{supp}(h) = \{ \alpha \in \mathbb{N}^d \mid c_\alpha \neq 0 \}$

Monomial Briançon-Skoda Theorem.

Caratheodory theorem
Let $E \subseteq \mathbb{R}^d$ be a connected set and $b \in \mathrm{Conv}(E)$ then $\exists\, e_1, \dots, e_d \in E$ such that $b \in \mathrm{Conv}(e_1, \dots, e_d)$.

We can deduce that:

1) $b \in N(I) \implies d \cdot b \in E(I)$.

2) $b_1, \dots, b_d \in N(I) \implies b_1 + \dots + b_d \in E(I)$

Corollary (Monomial Briançon-Skoda Th.) $\overline{I}^d \subseteq I$.

This is a particular case of a theorem of Briançon-Skoda for ideals of $\mathbb{C}\{x_1, \dots, x_d\}$.

Multiplicities

Def

Let A be a local noetherian ring and I a primary ideal.

There exists a polynomial $P_I \in \mathbb{Q}[t]$ with $\deg P_n = \dim A = d$ of the form $P_I(t) = \frac{e(I)}{d!} t^d + \ldots$ such that

$$P_I(s) = \lg_A A/I^s \quad \text{for } s >> 0$$

$e(I)$ = multiplicity of I.

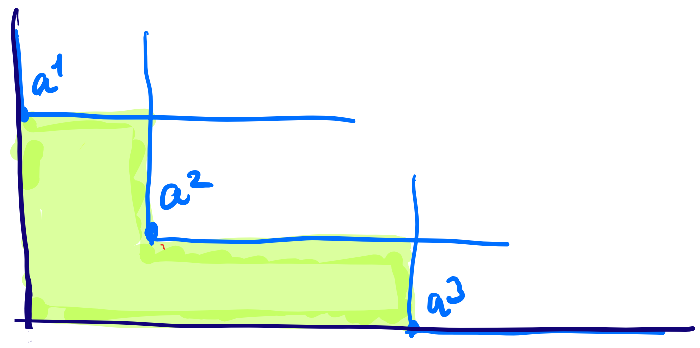
$P_I(t) ::=$ Hilbert-Samuel polynomial of I.

Multiplicities.

Let $I = (\underline{x}^{a^1}, \dots, \underline{x}^{a^k}) \subset \mathbb{C}\{x_1, \dots, x_d\}$ a monomial ideal

The following are equivalent:

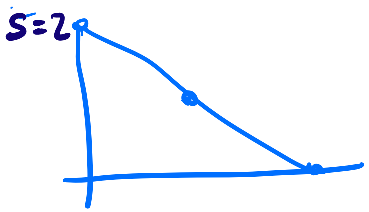
(1) I is primary



(2) $V(I) := \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \setminus E(I)\right) < \infty$

(3) $\dim_{\mathbb{C}} \mathbb{C}\{\underline{x}\}/I < \infty.$

Rem: $\dim \mathbb{C}\{x\}/I$ = n° of points of integral coordinates in $\mathbb{R}^d_{\geq 0} \setminus E(I)$



Rem: Take an integer $s \geq 1$.

n° of integral points of $\mathbb{R}^d_{\geq 0} - E(I^s) =$

n° of $\frac{1}{s}$-integral points of $\mathbb{R}^d_{\geq 0} - \frac{1}{s} \cdot E(I^s)$

and then we can show that

$$\lim_{s \to \infty} \frac{1}{s} E(I^s) = N(I)$$

thus

$$e(I) = d! \, V(N(I)) := \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \setminus N(I)\right)$$

(covolume of $N(I)$ in $\mathbb{R}^d_{\geq 0}$)

Corollary: If I_1, I_2 are primary monomial ideals.

① $\overline{I_1} = \overline{I_2} \Rightarrow e(I_1) = e(I_2)$

② $I_1 \subseteq I_2$ and $e(I_1) = e(I_2) \Rightarrow \overline{I_1} = \overline{I_2}$.

Since $N(I_1) \subseteq N(I_2)$ and the equality of multiplicities

$\Rightarrow V(N(I_1)) = V(N(I_2)) \Rightarrow N(I_1) = N(I_2) \Rightarrow \overline{I_1} = \overline{I_2}$.

Rem: This result is a particular case of Rees theorem, for primary ideals in a certain class of local rings.

Mixed covolumes

If I_1, I_2 are monomial ideals with finite covolumes:

$$V(N(I_1)), \quad V(N(I_2)) < \infty$$

Then, if $s_1, s_2 \in \mathbb{R}_{>0}$ we have that

$V(s_1 N(I_1) + s_2 N(I_2))$ is a homogeneous polynomial in s_1, s_2 of degree d and one can write:

$$\mathrm{vol}(s_1 N(I_1) + s_2 N(I_2)) = \sum_{i=0}^{d} \binom{d}{i} w_i \, s_1^{i} s_2^{d-i}.$$

The w_i are the mixed covolumes of $N(I_1)$ and $N(I_2)$.

Mixed multiplicities

If I_1, I_2 are two primary ideals of a noetherian local ring, then we have

$$e(I_1^{s_1} \cdot I_2^{s_2}) = \sum_{i=0}^{d} \binom{d}{i} e_i \, s_1^{i} s_2^{d-i}$$

The e_i are the mixed multiplicities of I_1, I_2

A Minkowsky type inequality for the multiplicities.

Theorem (B. Teissier, Rees)

Let A a formally equidimensional noetherian local ring of dimension d with infinite residue field and I_1, I_2 primary ideals. Then:

(1) $e_{i-1}^2 \leq e_i \cdot e_{i-2}$, $2 \leq i \leq d$

(2) $e(I_1 \cdot I_2)^{1/d} \leq e(I_1)^{1/d} + e(I_2)^{1/d}$

Equality in (1) holds if and only if

$$\frac{e_d}{e_{d-1}} = \frac{e_{d-1}}{e_{d-2}} = \cdots = \frac{e_1}{e_0} = \frac{a}{b}$$

and if A is normal then $\overline{I_1^a} = \overline{I_2^b}$.

Remark: The proof of the previous theorem involves a reduction to the case of surface by using Bertini's theorem, the existence of resolution of singularities of surfaces by Hironaka (char 0) and Abhyankar (char p), and that by Mumford's theorem the intersection matrix of the exceptional divisor of a resolution is negative definite.

This latter fact may be seen as a local expression of the Hodge index theorem.

A Minkowsky type inequality for the covolumes

Corollary: If I_1, I_2 are monomial primary ideals of $\mathbb{C}\{\underline{x}\}$ then

$$\boxed{e_i = d!\, w_i}$$

and then:

$$w_{i-1}^2 \leq w_i \cdot w_{i-2}, \quad 2 \leq i \leq d$$

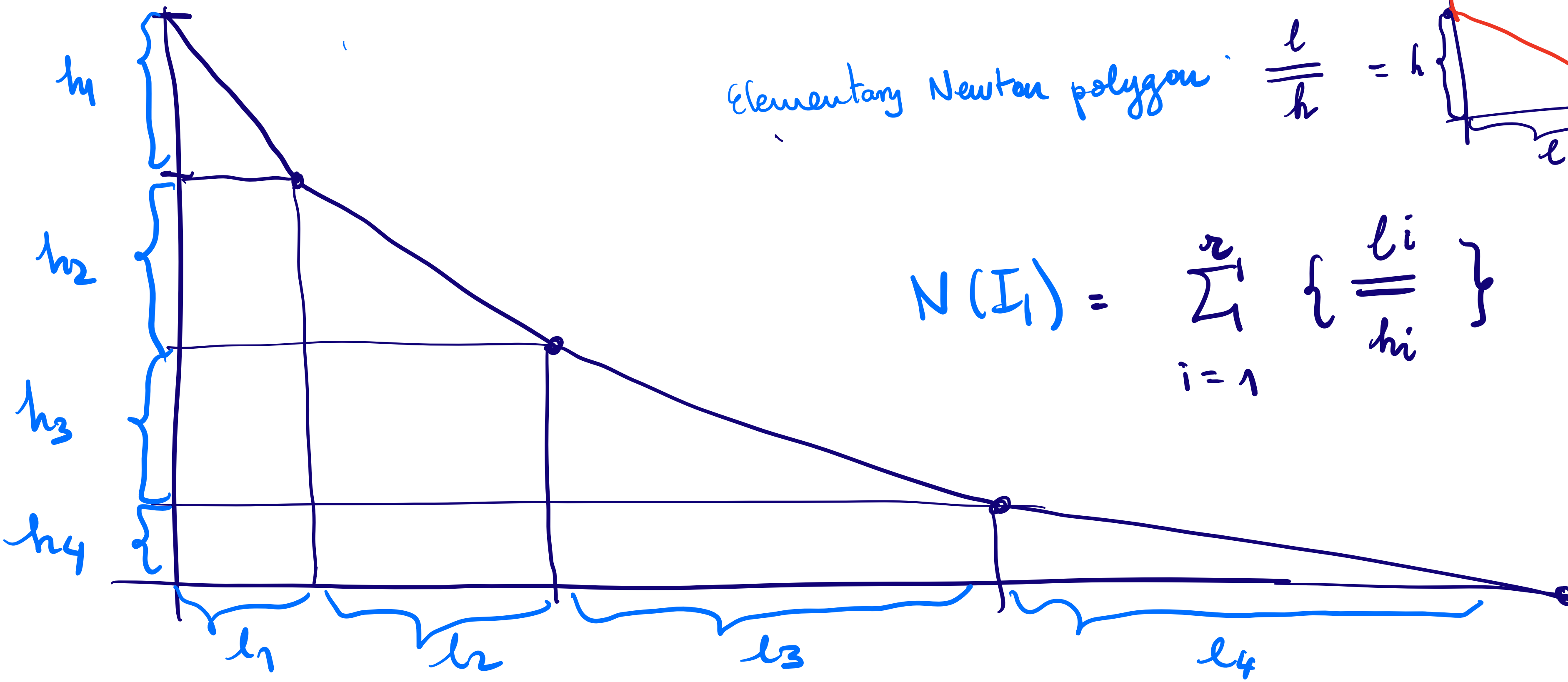
$$\operatorname{Vol}(N(I_1) + N(I_2))^{1/d} \leq \operatorname{Vol}(N(I_1))^{1/d} + \operatorname{Vol}(N(I_2))^{1/d}$$

with equality if and only if $\frac{a}{b} = \frac{w_d}{w_{d-1}} = \dots = \frac{w_1}{w_0}$

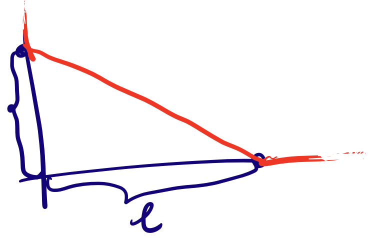
and $a\, N(I_1) = b\, N(I_2)$

A Minkowsky type inequality for the covolumes

Case: $I_1 \subseteq \mathbb{C}\{x_1, x_2\}$ and $I_2 = (x_1, x_2)$.



Elementary Newton polygon: $\dfrac{l}{h}$ =



$$N(I_1) = \sum_{i=1}^{r} \left\{ \frac{l_i}{h_i} \right\}$$

The inequality of the corollary becomes

$$\Big(\sum_i \min(l_i, h_i) \Big)^2 \leq 2 \cdot vol(N(I_1))$$

with equality if and only if $N(I_1) = m\, N(I_2)$ for some m.

Toric singularities and beyond.

Toric geometry and the Semple Nash modification

We proposed some material towards a general theory of toric varieties without the assumption of normality. This includes:

→ Combinatorial description: in term of a fan with attached semigroups subjet to glueing conditions.

→ Geometrical description: Good action of the torus, i.e., take Sumihiro's theorem as an axiom.

→ Blowing up of monomial ideals is a toric map.

The Semple Nash modification in the toric case

The Semple-Nash modification of a equidimensional variety is a canonical modification defined by taking the limits of tangents spaces at non singular points.

→ Def: Description of the Semple Nash modification in the toric case over an alg. closed field of characteristic zero, as the blowing up of a monomial ideal called the logarithmic jacobian ideal

→ Theorem: Local uniformization of any monomial valuation of maximal rank dominating a point of a toric variety by SN-modifications.

Overweighted deformations of affine toric varieties and local uniformization

Teissier has developed toric geometry methods in his program of local uniformization of valuations by overweighted deformations of affine toric varieties (see the references below).

References

B. Teissier, Two points of the boundary of toric geometry In "Singularities, Algebraic Geometry, Commutative Algebra and Related Topics. Festschrift for Antonio Campillo on the occasion of his 65 th Birthday", G.-M. Greuel, L. Narváez Macarro and S. Xambó-Descamp, Editors, Springer 2018, 107-117.

P. González Pérez and B. Teissier, Toric Geometry and the Semple-Nash modification, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Seria A, Matemáticas, 108, (1) (2014), 1--46.

B. Teissier, Overweight deformations of affine toric varieties and local uniformization, in "Valuation theory in interaction", Proceedings of the second international conference on valuation theory, Segovia--El Escorial, 2011. Edited by A. Campillo, F-V. Kuhlmann and B.Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474--565.

B. Teissier, Volumes des corps convexes; géométrie et algèbre. (French) Leçons de mathématiques d'aujourd'hui, Cassini, 2007.

B. Teissier, Monomial ideals, binomial ideals, polynomial ideals. Trends in commutative algebra, 211--246, Math. Sci. Res. Inst. Publ., 51, Cambridge Univ. Press, Cambridge, 2004.

B. Teissier, Valuations, deformations, and toric geometry. Valuation theory and its application, Vol. II (Saskatoon, SK, 1999), 361--459, Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 2003.

P. D. González Pérez and B. Teissier, Embedded resolutions of non necessarily normal affine toric varieties. C. R. Math. Acad. Sci. Paris, 334 (2002), no. 5,

R. Goldin and B. Teissier, Resolving singularities of plane analytic branches with one toric morphism. Resolution of singularities (Obergurgl, 1997), 315--340, Progr. Math., 181, Birkhauser, Basel, 2000.

B. Teissier, Monômes, volumes et multiplicités. (French) [Monomials, volumes and multiplicities] Introduction à la théorie des singularités, II, 127--141, Travaux en Cours, 37, Hermann, Paris, 1988.

B. Teissier, Bonnesen-type inequalities in algebraic geometry. I. Introduction to the problem. Seminar on Differential Geometry, pp. 85--105, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

B. Teissier, Variétés toriques et polytopes. (French) [Toric varieties and polytopes] Bourbaki Seminar, Vol. 1980/81, pp. 71--84, Lecture Notes in Math., 901, Springer, Berlin-New York, 1981.

B. Teissier, Du théorème de l'index de Hodge aux inégalités isopérimétriques. (French) C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 4, A287--A289.

B. Teissier, On a Minkowski-type inequality for multiplicities. II. C. P. Ramanujam---a tribute, pp. 347--361, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York, 1978.

D. Eisenbud, H.I. Levine, An algebraic formula for the degree of a $C\sp{\infty }$ map germ. With an appendix by Bernard Teissier, "Sur une inégalité à la Minkowski pour les multiplicités". Ann. Math. (2) 106 (1977), no. 1, 19--44.

Thanks Bernard for sharing all these beautiful mathematical paths, full of inspiration an perspectives, with great energy and enthusiasm!