Tropical and logarithmic techniques for the study of Milnor fibers

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Three hours course at the Research School

Logarithmic and non-archimedean methods in singularity theory

In honor of the 80th birthday of **Bernard Teissier**

CIRM, Marseille, 27–31 January 2025

I will present results understood during a long-term collaboration with:

- **Maria Angelica CUETO** (Ohio State University);
- **Dmitry STEPANOV** (Moscow Institute of Physics and Technology).

Thank you Angelica and Dmitry for your friendship and your collaboration!

(July 2017)

- A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***. In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, 2023, 629–709.

- A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.

In the article

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Complex surface singularities with integral homology sphere links

Walter Neumann and **Jonathan Wahl** formulated:

Conjecture 2 (Milnor Fiber Conjecture) *F is homeomorphic to the result $\overline{F}$ of pasting:*

$$\overline{F} := F_1^o \cup_{G_1 \times S^1} (G_1 \times G_2) \cup_{S^1 \times G_2} F_2^o,$$

where we identify $G_1 \times S^1$ with $G_1 \times \partial G_2$ and $S^1 \times G_2$ with $\partial G_1 \times G_2$.

In the statement of the conjecture we see the following **splicing operation** introduced by Neumann and Wahl:

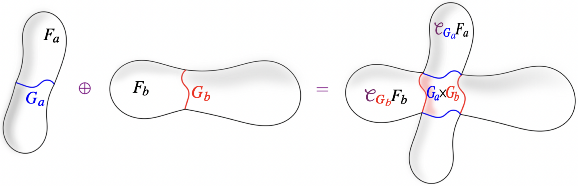


Figure 6. Splicing of two four-dimensional manifolds F_a and F_b with integral homology sphere boundaries along properly embedded surfaces called G_a and G_b, respectively (see Definition 2.45).

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***. In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, Berlin, 2023, 629–709.)

The conjecture states that the Milnor fiber of a **splice type singularity** is obtained by splicing the Milnor fibers of two **simpler such singularities**.

Plan of the course

In this course, I will explain successively:

- the prototypical decomposition occurring in singularity theory: the **plumbing decomposition of the link** of a complex normal surface singularity X, **once a normal crossings resolution of X is fixed**;
- an analogous **decomposition of the Milnor fibers** of a function f, **once a normal crossings resolution of f is fixed**;
- the properties of **log geometry in the sense of Fontaine and Illusie** which may be used to decompose **canonically** the Milnor fibers of f into pieces using **Kato and Nakayama's operation of rounding of a complex log structure**, **once a normal crossings resolution of X is fixed**;
- several viewpoints on **local tropicalization**, and how it leads to **toroidal resolutions** in the **Newton non-degenerate cases**;
- the definition of **splice type singularities**;
- the **principles of our proof of the Milnor fiber conjecture**;
- how to pass from the definition of **polar coordinates** to the general definitions of **log spaces** and of **rounding of complex log spaces**.

The **link** of a complex normal surface singularity

In 1961, Mumford described the **link** of an isolated complex analytic surface singularity as a **plumbed manifold**. Let us look at a few extracts from his paper.

THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE AND A CRITERION FOR SIMPLICITY

By DAVID MUMFORD

Let a variety V^n be embedded in complex projective space of dimension m. Let $P \in V$. About P, choose a ball U of small radius ε, in some affine metric $ds^2 = \Sigma dx_j^2 + \Sigma dy_j^2$, $z_j = x_j + iy_j$ affine coordinates. Let B be its boundary and $M = B \cap V$. Then M is a real complex of dimension $2n-1$, and a manifold if P is an isolated singularity. The topology of M together with its embedding in B ($=$ a $2m-1$-sphere) reflects the nature of the point P in V. The simplest case and the only one to be

Mumford's motivations

From the standpoint of the theory of algebraic surfaces, the really interesting case is that of a singular point on a *normal* algebraic surface, and *m* arbitrary. M is then by no means generally S^3 and consequently its own topology reflects the singularity P! In this paper, we shall consider this case, first giving a partial construction of $\pi_1(M)$ in terms of a resolution of the singular point P; secondly we shall sketch the connexion between $H_1(M)$ and the algebraic nature of P. Finally and principally, we shall demonstrate the following theorem, conjectured by Abhyankar:

Theorem. — $\pi_1(M)=(e)$ if and only if P is a simple point of F (a locally normal surface); and F topologically a manifold at P implies $\pi_1(M)=(e)$.

(Page 5 of D. Mumford, ***The topology of normal singularities of an algebraic surface and a criterion for simplicity.*** Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.)

I. — **ANALYSIS OF M AND PARTIAL CALCULATION OF** $\pi_1(M)$

A normal point P in F is given. A finite sequence of quadratic transformations plus normalizations leads to a non-singular surface F′ dominating F [15]. The inverse image of P on F′ is the union of a finite set of curves $E_1, E_2, \ldots, E_n$. By further quadratic transformations if necessary we may assume that all E_i are non-singular, and, if $i \neq j$, and $E_i \cap E_j \neq \emptyset$, then that E_i and E_j intersect normally in exactly one point, which does not lie on any other E_k. This will be a great technical convenience.

In the introduction, M is a level manifold of the positive C^∞ fcn.

$$p^2 = |Z_1|^2 + \ldots + |Z_n|^2,$$

(Z_i affine coordinates near $P \in F$). Now notice that M may also be defined as the level manifolds of p^2 on the non-singular F′ (p^2 being canonically identified to a fcn. on F′). It is as a "tubular neighborhood" of $\cup E_i \subset F'$ that we wish to discuss M. Now the general

(Page 6 of D. Mumford, ***The topology of normal singularities of an algebraic surface and a criterion for simplicity.*** Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.)

How to define **admissible** functions vanishing on the exceptional divisor?

barycentric subdivision of the given triangulation. I am informed that Thom [11] has considered it more from our point of view: for a suitably restricted class of positive C^∞ fcns. f such that $f(P)=0$ if and only if $P \in K$, define the tubular neighborhood of K to be the level manifolds $f=\varepsilon$, small ε. The catch is how to suitably restrict f; here the archtype for f^{-1} may be thought of as the potential distribution due to a uniform charge on K. In our case, as we have no wish to find the topological ultimate, we shall merely formulate a convenient, and convincingly broad class of such f, which includes the p^2 of the introduction.

Let us say that a positive C^∞ real fcn. f on F' such that $f(P)=0$ iff $P \in E_i$, is *admissible* if

What we must now show is that there is a unique manifold M such that, if f is any admissible fcn., M is homeomorphic to $\{P \mid f(P)=\varepsilon\}$ for all sufficiently small ε.

(Page 7 of D. Mumford, ***The topology of normal singularities of an algebraic surface and a criterion for simplicity.*** Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.)

The appearance of the term **plumbing**

possible form!). Note $\left(\frac{4^{n_j}\varepsilon}{\alpha_{ij}}\right)^{1/n_i} < 1/8$. Therefore, we see that $\psi_i^{-1}(E_i^*)$ and $\psi_j^{-1}(E_j^*)$ are patched by a standard "plumbing fixture":

$$\{(x, y, u, v) \mid (x^2+y^2) \leqslant 1/4,\ (u^2+v^2) \leqslant 1/4,\ (x^2+y^2)^n . (u^2+v^2)^m = \varepsilon < 1/8^{n+m}\}$$

where n and m are integers.

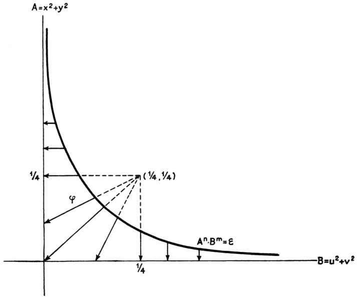


Fig. 1

(Pages 8 and 10 of D. Mumford, ***The topology of normal singularities of an algebraic surface and a criterion for simplicity.*** Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.)

The link M may be endowed with a projection φ onto the exceptional divisor $\cup E_i$:

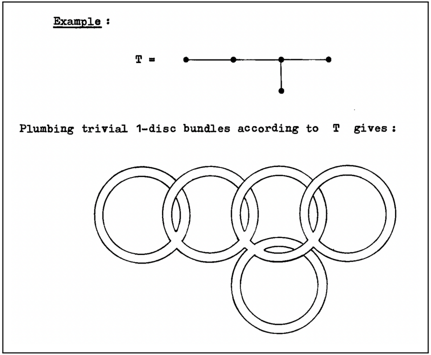
> This determines **M** uniquely. We have essentially found, moreover, not only **M** but also for any fixed f, maps
>
> $$\varphi : M \to \cup E_i$$
> $$\psi : \{P \mid 0 < f(P) \leqslant \varepsilon\} \to M$$

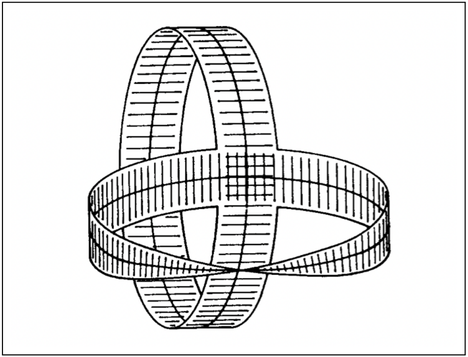
(Page 9 of D. Mumford, ***The topology of normal singularities of an algebraic surface and a criterion for simplicity.*** Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.)

The projection φ decomposes M into **elementary pieces**, the preimages of the irreducible components E_i of the exceptional divisor. Those pieces communicate through **2-dimensional tori**, which are the fibers of φ above the singular points of $\cup E_i$. The other fibers are **circles**.

One says that either the tubular neighborhood of $\cup E_i$ or its boundary M are **plumbed** or have **plumbing decompositions**.

One has analogous plumbing decompositions in other dimensions:





(Page 58 of F. Hirzebruch, W. D. Neumann, S. S. Koh, ***Differentiable manifolds and quadratic forms***. Appendix II by W. Scharlau. Lect. Notes in Pure and Applied Math. **4**. Marcel Dekker, Inc., 1971.)

(Page 533 of E. Brieskorn, H. Knörrer, ***Plane algebraic curves***. Birkhäuser, 1986.)

Let us pass to the discussion of **Milnor fibers**. The following are the two pictures of those objects in Milnor's book:

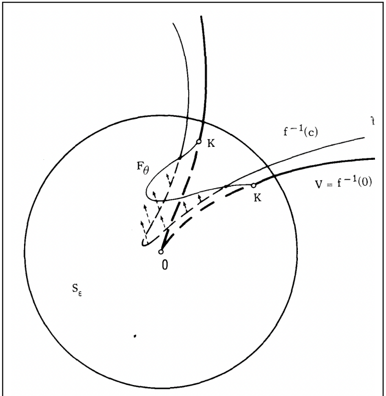


Figure 4.

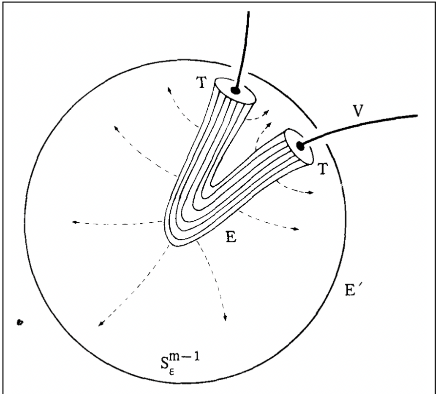


Figure 6.

(Pages 54 and 98 of J. Milnor, ***Singular points of complex hypersurfaces.*** Princeton Univ. Press, 1968.)

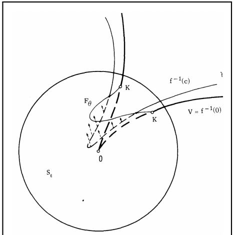


Figure 4.

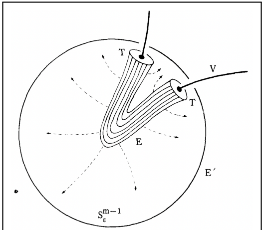


Figure 6.

One starts from a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ **with isolated critical point**. Two **isomorphic fibrations over a circle** are associated to it:

- **Milnor fibrations in the tube**: $\boxed{f : \partial T \to \mathbb{S}^1_\delta}$, $0 < \delta \ll 1$.
- **Milnor fibrations in the sphere**: $\boxed{\dfrac{f}{|f|} : \mathbb{S}^{2n-1}_\varepsilon \setminus Z(f) \to \mathbb{S}^1_1}$, $0 < \varepsilon \ll 1$.

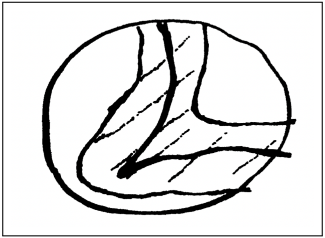
The closures of their fibers are smooth compact manifolds-with-boundary, called **Milnor fibers of f**.

Here is the beginning of a letter from John Milnor to John Nash (3 April 1966):

Dear John,

I enjoyed talking to you last week. The Brieskorn example is fascinating. After staring at it a while I think I know which manifolds of this type are spheres, but the statement is complicated and the proof doesn't exist yet. Let $\Sigma(p_1, \ldots, p_n)$ be the locus

$$z_1^{p_1} + \ldots + z_n^{p_n} = 0 \ , \ |z_1|^2 + \ldots + |z_n|^2 = 1$$



(Pages 47 and 48 of E. Brieskorn, *Singularities in the work of Friedrich Hirzebruch.* Surv. in Diff. Geom. **VII** (2000), 17–60.)

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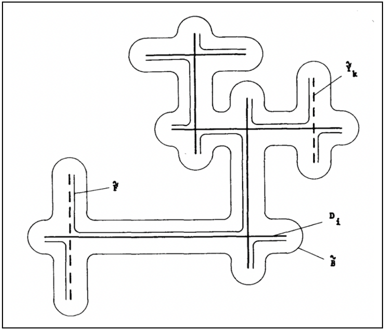
ON MILNOR'S FIBRATION THEOREM AND ITS OFFSPRING AFTER 50 YEARS

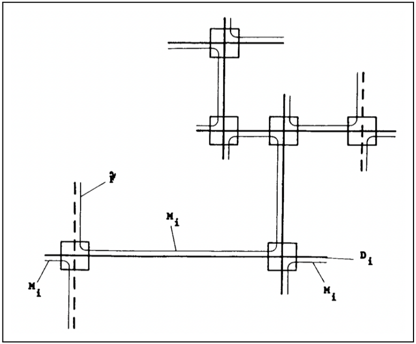
JOSÉ SEADE

To Jack, whose profoundness and clarity of vision seep into our appreciation of the beauty and depth of mathematics.

Studying a Milnor fiber using a **resolution with normal crossings**

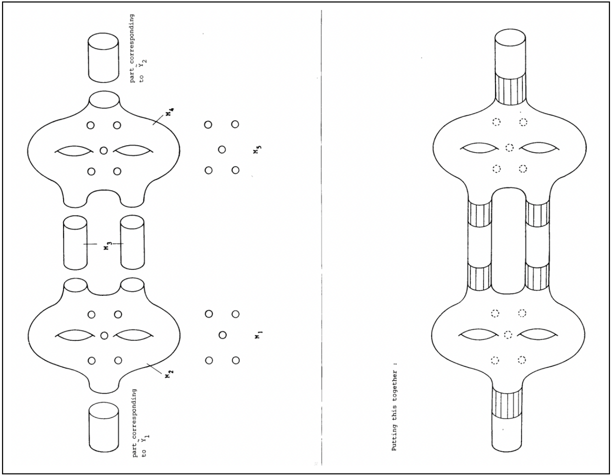
Similarly to what we saw for **links**, also **Milnor fibers** may be studied using **resolutions**. This time, one considers **resolutions** π **of functions** f, which are such that **the special fiber** $\boxed{Z(f \circ \pi)} := (f \circ \pi)^{-1}(0)$ **is a normal crossings divisor**:





(Pages 562 and 564 of E. Brieskorn, H. Knörrer, ***Plane algebraic curves***. Birkhäuser, 1986.)

The resulting decomposition of Milnor fibers



(Pages 566 and 567 of E. Brieskorn, H. Knörrer, ***Plane algebraic curves***. Birkhäuser, 1986.)

A complaint of Grothendieck around 1982

> *This naive vision immediately encounters various difficulties.*
> *The first is* **the somewhat vague nature of the very notion of tubular neighbourhood**, *which acquires a tolerably precise meaning only in the presence of structures which are much more rigid than the mere topological structure, such as "piecewise linear" or Riemannian (or more generally, space with a distance function) structure;*
> *the trouble here is that in the examples which naturally come to mind, one does not have such structures at one's disposal – at best an equivalence class of such structures, which makes it possible to rigidify the situation somewhat.*

(A. Grothendieck, ***Esquisse d'un programme***. Written around 1982. In ***Geometric Galois actions 1***, 5–48, Cambridge Univ. Press, 1997. English translation by L. Schneps and P. Lochak on pages 243–283 of the same volume: ***Sketch of a programme***.)

> *The study of the connectivity of a surface is based on its* **decomposition via transverse cuts**, *that is, lines which cut through the interior from one boundary point simply (no point occurring multiply) to another boundary point.*

(Section 6 of B. Riemann, ***Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse***. Inaugural dissertation, Göttingen, 1851.)

Using the notion of connected component of a topological space, it is not difficult to define **a canonical cutting operation of a smooth manifold along a smooth hypersurface**, by gluing local constructions.

Question: How to cut along higher-codimensional submanifolds or subvarieties, for instance *along a divisor with normal crossings in a complex manifold*?

Here $\cup_j C_j \hookrightarrow X$ denotes a **simple** normal crossings divisor in a complex manifold (each irreducible component C_j is *smooth*).

For $j = 1, \ldots, n$ let $\pi_j : Z_j \to X$ be the **real oriented blowup** *with center C_j. Therefore, above $x \in C_j$ lie the* **real oriented normal directions to C_j at x** *and π_j is a diffeomorphism outside C_j. Thus Z_j is a differentiable manifold-with-boundary and its boundary $\partial Z_j = \pi_j^{-1}(C_j)$ is diffeomorphic to the boundary of a tubular neighborhood of C_j in X. Let $\pi : Z \to X$ be the* **fibered product of the various π_j** *above X. Then Z is a differentiable manifold with corners and π is a diffeomorphism outside X_0. The boundary $\partial Z = \pi^{-1}(X_0) = N$ is a differentiable manifold with corners. The restriction of π to ∂Z is the map $\rho : N \to X_0$.* **The manifold N is homeomorphic to the boundary of every regular neighborhood of X_0 in X and ρ is a retraction.**

(Page 238 of N. A'Campo, ***La fonction zêta d'une monodromie***. Comment. Math. Helv. **50** (1975), 233–248.)

The real oriented blowup of the center of a disc

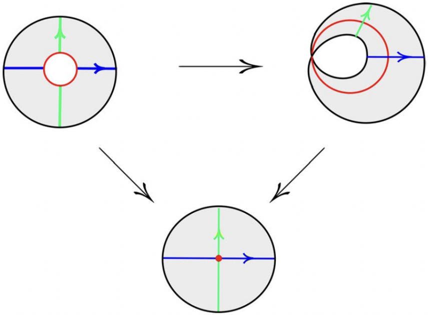


Figure 11. The left arrow is the real oriented blowup while the right one is the usual blowup of the center of a disc. The horizontal arrow is the natural factorisation of the left arrow through the second one.

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***.

In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, Berlin, 2023, 629–709.)

The boundary $\partial Z = \pi^{-1}(X_0) = N$ is a differentiable manifold with corners. The restriction of π to ∂Z is the map $\rho : N \to X_0$.
The manifold N is homeomorphic to the boundary of every regular neighborhood of X_0 in X and ρ is a retraction.

(N. A'Campo, ***La fonction zêta d'une monodromie***. Comment. Math. Helv. **50** (1975), 233–248.)

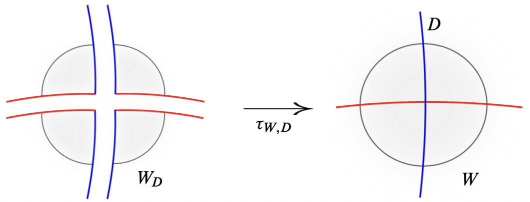


Figure 8. Real analog of A'Campo's real oriented blowup of W along the divisor D.

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***.

In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, Berlin, 2023, 629–709.)

Log structures give a different definition of real oriented blowups:

Proposition

Let D be a divisor with normal crossings in a complex manifold M. **It induces a canonical log structure on M**, *whose* **rounding** *is homeomorphic to the real oriented blowup of M along D,* **whenever D has simple normal crossings**.

Advantages of the logarithmic viewpoint on real oriented blowups:

- **rounding is functorial**;
- **rounding produces the desired topology for more general divisors than those with simple normal crossings**;
- **rounding may be avoided when one wants to prove homeomorphisms of boundaries of tubular neighborhoods**: it is enough to prove isomorphisms of log spaces.

A log structure on a topological space is a special type of morphism of sheaves. We will see the precise definition later on. For the moment, **we look at the theory more globally**, focusing on several **properties** of log structures and rounding.

1. A **log space** is a complex space endowed with a log structure. There exists a notion of **morphism** between log spaces.
2. If D is a reduced divisor in a complex variety W, then there is an induced structure of **divisorial log space** $\boxed{W^{\dagger}}$ on W.

Overview of **properties of log structures and rounding** II

3. If $E \hookrightarrow V$, $D \hookrightarrow W$ are reduced divisors in the complex varieties V, W and if

$$f : V \to W$$

is a holomorphic morphism such that $f^{-1}(D) \subseteq E$, then there is an associated morphism of log spaces

$$\boxed{f^{\dagger} : V^{\dagger} \to W^{\dagger}}$$

from V endowed with the divisorial log structure induced by E, to W endowed with the divisorial log structure induced by D.

4. Log structures may be **pulled back** by morphisms of complex spaces. The morphisms of log spaces obtained in this way are called **strict**.

5. There exists a **rounding functor** from the category of complex log spaces to that of topological spaces. We denote by $\boxed{\phi^{\odot} : V^{\odot} \to W^{\odot}}$ the rounding of the morphism $\phi : V \to W$ of log spaces.

6. If W is a log space with **underlying topological space** $\boxed{|W|}$, then there exists a canonical continuous map $\boxed{\tau_W : W^{\odot} \to |W|}$, called the **rounding morphism** of the log space W.

   Whenever W is a complex manifold endowed with the log structure induced by a simple normal crossings divisor, τ_W is homeomorphic to the real oriented blowup morphism of the manifold along the divisor.
   But, even if the normal crossings divisor D is not simple, $W^{\odot}$ is homeomorphic to the complement of the interior of a tubular neighborhood of D in W.

7. If $\phi : V \to W$ is a morphism of log spaces with associated continuous map $|\phi| : |V| \to |W|$, then the following diagram commutes:

$$\begin{array}{ccc} V^{\odot} & \xrightarrow{\phi^{\odot}} & W^{\odot} \\ {\scriptstyle \tau_V}\downarrow & & \downarrow{\scriptstyle \tau_W} \\ |V| & \xrightarrow[|\phi|]{} & |W|. \end{array}$$

8. If the morphism $\phi : V \to W$ is **strict**, the previous commutative diagram is moreover **cartesian = a pullback diagram**.

The special case of normal crossings divisors

In particular, if $\varphi : D \hookrightarrow W$ is a normal crossings divisor D in a complex manifold W, then the rounding $(D^\dagger)^\odot$ of the restriction $D^\dagger$ to D of the divisorial log structure $W^\dagger$ induced by D on W gets identified by $(\varphi^\dagger)^\odot : (D^\dagger)^\odot \hookrightarrow (W^\dagger)^\odot$ with the boundary of the real oriented blowup of W along D.

$$D \underset{\varphi}{\hookrightarrow} W \quad \text{(analytic category).}$$

$$D^\dagger \xrightarrow[\varphi^\dagger]{} W^\dagger \quad \text{is strict (log category).}$$

$$\begin{array}{ccc} (\mathbf{D}^\dagger)^\odot & \overset{(\varphi^\dagger)^\odot}{\hookrightarrow} & (W^\dagger)^\odot \\ \tau_{D^\dagger} \downarrow & & \downarrow \tau_{W^\dagger} \\ D & \underset{\varphi}{\hookrightarrow} & W \end{array} \quad \text{(topological category).}$$

Therefore, **the boundary of the tubular neighborhoods of D in W is encoded by $\mathbf{D}^\dagger$.**

Application to the study of **links** of isolated complex singularities

1. Start from a representative of the singularity:
$$(X, x).$$

2. Choose a normal crossings resolution of it:
$$(\tilde{X}, E) \xrightarrow{\pi} (X, x).$$

3. Consider the divisorial log structure on $\tilde{X}$ induced by E:
$$\tilde{\mathbf{X}}^{\dagger}.$$

4. Restrict that log structure to E:
$$\mathbf{E}^{\dagger} \text{ (the } \textbf{log exceptional divisor}).$$

5. **Round** that restriction:
$$(E^{\dagger})^{\odot} \xrightarrow{\tau} E.$$

This is a representative of the link, canonically decomposed into pieces!

1. Start from a Milnor tube representative of a smoothing:
$$(X, Z(f)) \xrightarrow{f} (\mathbb{D}, 0).$$

2. Choose a normal crossings resolution π of f:
$$(\tilde{X}, Z(f \circ \pi)) \xrightarrow{\pi} (X, Z(f)) \xrightarrow{f} (\mathbb{D}, 0).$$

3. Consider the divisorial log structures on $\tilde{X}$ and $\mathbb{D}$ induced by $Z(f \circ \pi)$ and 0:
$$\tilde{X}^{\dagger} \xrightarrow{(f \circ \pi)^{\dagger}} \mathbb{D}^{\dagger}.$$

4. Restrict those log structures to $Z(f \circ \pi)$ and 0:
$$(Z(f \circ \pi))^{\dagger} \xrightarrow{(f \circ \pi)^{\dagger}} 0^{\dagger} \quad \text{(the } \textbf{log special fiber}\text{)}.$$

5. **Round** this log morphism:
$$((Z(f \circ \pi))^{\dagger})^{\odot} \xrightarrow{((f \circ \pi)^{\dagger})^{\odot}} (0^{\dagger})^{\odot}.$$

This is a representative of the Milnor fibration!

The log representative

$$((\mathbf{Z}(\mathbf{f} \circ \pi))^{\dagger})^{\odot} \xrightarrow{((f\circ\pi)^{\dagger})^{\odot}} (\mathbf{0}^{\dagger})^{\odot}$$

of the Milnor fibration is **canonically decomposed into pieces**:

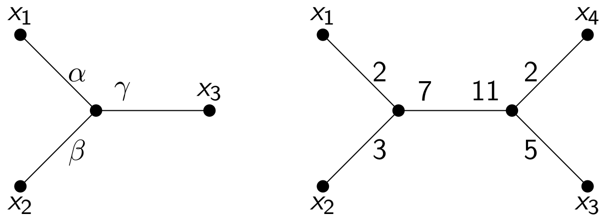
$$\begin{array}{ccc} ((\mathbf{Z}(\mathbf{f} \circ \pi))^{\dagger})^{\odot} & \xrightarrow{((f\circ\pi)^{\dagger})^{\odot}} & (\mathbf{0}^{\dagger})^{\odot} \\ \tau_{(Z(f\circ\pi))^{\dagger}} \downarrow & & \downarrow \tau_{0^{\dagger}} \\ \mathbf{Z}(\mathbf{f} \circ \pi) & \xrightarrow[f\circ\pi]{} & 0. \end{array}$$

It is now time to present a glimpse of **splice type singularities**.

The atomic splice type singularities are the **Pham-Brieskorn complete intersection surface singularities**. The other ones are obtained by **splicing their systems of equations**, as defined by Neumann and Wahl. ÂăFor instance:

$$x_1^{\alpha} + x_2^{\beta} + x_3^{\gamma} = 0, \qquad \begin{cases} x_1^2 - x_2^3 + x_3 x_4 = 0, \\ x_3^5 - x_4^2 + x_1 x_2^4 = 0. \end{cases}$$

The corresponding **splice diagrams** are:



Origin of the terminology of **splicing**

- Long splice – A splice used to join two rope ends forming one rope the length of the total of the two ropes. The long splice, unlike most splice types, results in a splice that is only very slightly thicker than the rope without the splice, but sacrifices some of the strength of the short splice. It does this by replacing two of the strands of each rope end with those from the other, and cutting off some of the extra strands that result. The long splice allows the spliced rope to still fit through the same pulleys, which is necessary in some applications.[8]
- Short splice – Also a splice used to join the ends of two ropes, but the short splice is more similar to the technique used in other splices and results in the spliced part being about twice as thick as the non spliced part, and has greater strength than the long splice. The short splice retains more of the rope strength than any knots that join rope ends.[9]

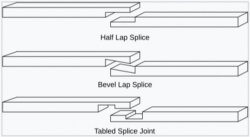
A short splice, with ends whipped

Types of splice joints [edit]

There are four main types of splice joints: half lap, bevel lap, tabled, and tapered finger.

Half lap splice joint [edit]

The half lap splice joint is the simplest form of the splice joint and is commonly used to join structural members where either great strength is not required or reinforcement, such as mechanical fasteners, are to be used.



Half lap splice, bevel lap splice and tabled splice joint

(Wikipedia, 19 January 2025.)

We will see the precise definition of **splice type singularities Y** later on.
What is important to note for the moment, is that **they are defined by concrete systems of equations in $\mathbb{C}^n$**.

In order to study their Milnor fibers, **we have to choose smoothings defined by suitable equations**. The total space **X** of the smoothing will be a singularity **defined by concrete systems of equations in** $\mathbb{C}^{n+1}$. The splice type singularity Y appears as the special fiber $Z(f)$ of the restriction

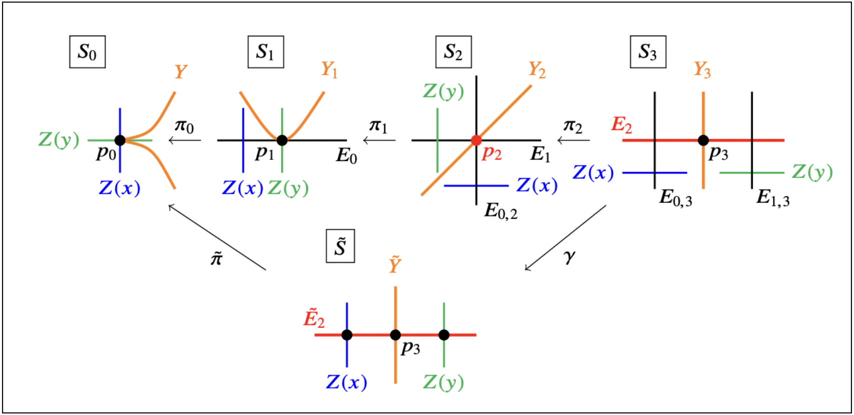
$$\mathbf{f} : \mathbf{X} \to \mathbb{C}$$

of the deformation parameter to the total space X of the deformation.
As Y is an isolated complete intersection, its Milnor fibers are diffeomorphic to the Milnor fibers of f, which may be studied using normal crossings resolutions of X and f, similarly to the case when $X = \mathbb{C}^n$.

Question: How to construct normal crossings resolutions of X and of f, starting from the defining equations?

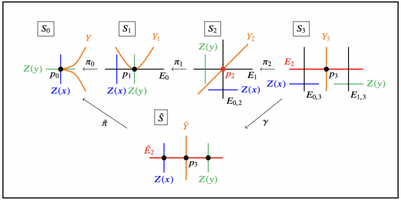
Resolution of a cuspidal plane cubic function

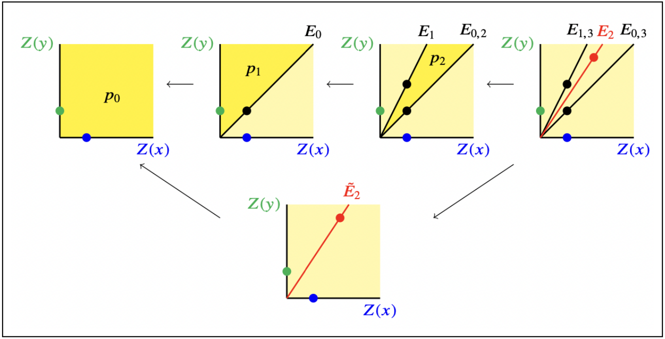
$$f(x,y) := y^2 - 2x^3 + x^2 y$$



(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

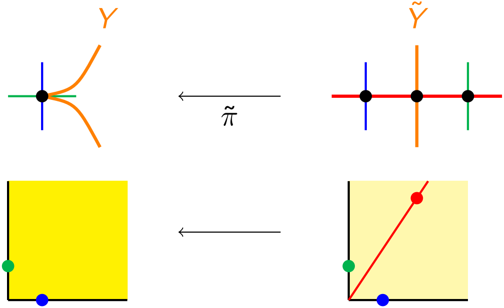
The previous morphisms are all **toric**





(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

The fiber $Z((x \cdot y \cdot f) \circ \tilde{\pi})$ endows the surface $\tilde{S}$ with a **toroidal structure**, locally analytically isomorphic to the toric boundary of a toric surface.



The **red ray** on the right lower corner encodes this morphism.
It is the **local tropicalization** of the germ of curve **Y** $:= (Z(f), 0)$.

Question: How to get it from the expression $f(x, y) := y^2 - 2x^3 + x^2 y$?

From the **Newton polygon** of Y to its **local tropicalization**

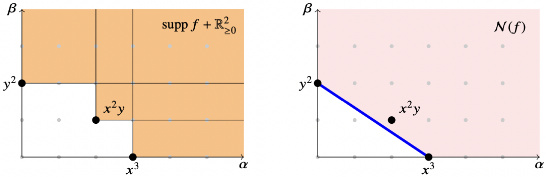


Fig. 1.3 The Minkowski sum supp $f + \mathbb{R}^2_{\geq 0}$ (on the left) and the Newton polygon $\mathcal{N}(f)$ of the series $f(x, y) = y^2 - 2x^3 + x^2y$ (on the right).

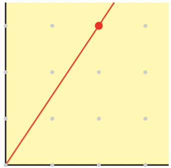


Fig. 1.4 The local tropicalization of the series $f(x, y) = y^2 - 2x^3 + x^2y$ is the ray spanned by the primitive vector $(2, 3)$, which is indicated with a filled dot.

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

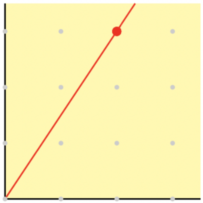


Fig. 1.4 The local tropicalization of the series $f(x, y) = y^2 - 2x^3 + x^2 y$ is the ray spanned by the primitive vector (2, 3), which is indicated with a filled dot.

All the **arcs** $t \mapsto (x(t), y(t))$ **such that** $f(x(t), y(t)) = 0$ satisfy:

$$(x(t), y(t)) = (a\ t^{2k} + \cdots, b\ t^{3k} + \cdots)$$

where $ab \neq 0$ and $k \in \mathbb{Z}_{>0}$.

This viewpoint leads to a **general definition of local tropicalization**.

A **general definition of local tropicalization** using **arcs**

But the shortest definition of the local tropicalization of the embedding $Y \hookrightarrow (\mathbb{C}^n, 0)$ is most likely as *the topological closure inside* $\mathbb{R}^n$ *of the union of rays* $\mathbb{R}_{\geq 0} w$ *generated by the weight vectors* $w = (w_1, \ldots, w_n)$ *which appear as initial exponent vectors of the formal arcs*

$$t \mapsto (c_1 t^{w_1} + \cdots , \ \ldots , \ c_n t^{w_n} + \cdots)$$

contained in Y *but not contained in* $\partial \mathbb{C}^n$ (that is, such that $c_j \in \mathbb{C}^*$ and $w_j \in \mathbb{Z}_{>0}$ for every $j \in \{1, \ldots, n\}$). For instance, if Y is a plane curve singularity defined by a series $f(x, y) \in \mathbb{C}[[x, y]]$, then its local tropicalization is the union of the rays orthogonal to the compact edges of the Newton polygon of f.

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

The local tropicalization of a Pham-Brieskorn surface singularity

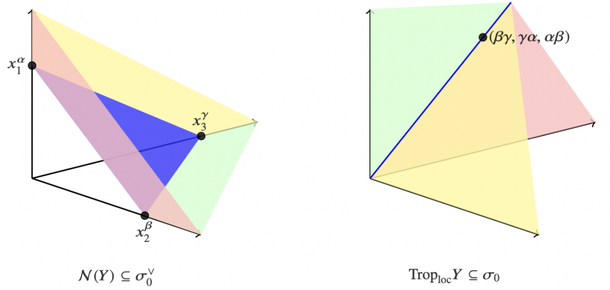


Fig. 1.5 The Newton polyhedron $\mathcal{N}(Y)$ of the Pham-Brieskorn singularity $Y \hookrightarrow (\mathbb{C}^3, 0)$ defined by equation $x_1^\alpha + x_2^\beta + x_3^\gamma = 0$ (on the left) and its local tropicalization (on the right)

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

The local tropicalization on the right is the support of a 2-dimensional fan $\mathcal{F}$ consisting of the colored cones and their faces. This fan defines **a toric birational morphism** $\pi_{\mathcal{F}} : \mathcal{X}_{\mathcal{F}} \to \mathbb{C}^3$.

Let us look at **toric birational morphisms**

For the geometrically-inclined reader, perhaps the most suggestive viewpoint on local tropicalization is the following *toric-geometrical* one. Seen as an affine toric variety, $\mathbb{C}^n$ has the set $\mathcal{F}_0$ of faces of the non-negative orthant $\sigma_0 := (\mathbb{R}_{\geq 0})^n$ of its vector space $\mathbb{R}^n$ of weight vectors as associated fan. Note that σ_0 also belongs to $\mathcal{F}_0$, as its nonproper face. Let $\mathcal{F}$ be a fan whose cones are contained in σ_0. There exists a canonical birational morphism

$$\pi_{\mathcal{F}} : X_{\mathcal{F}} \to \mathbb{C}^n$$

from the toric variety $X_{\mathcal{F}}$ associated to the fan $\mathcal{F}$ to the affine toric variety $\mathbb{C}^n = X_{\mathcal{F}_0}$. This birational morphism is a *modification*, that is, it is moreover *proper*, if and only if the *support* $|\mathcal{F}|$ of $\mathcal{F}$ (that is, if the union of its cones) is equal to σ_0. Denote by $Y_{\mathcal{F}}$ the strict transform of Y by $\pi_{\mathcal{F}}$ and let

$$\pi : Y_{\mathcal{F}} \to Y$$

be the restriction of $\pi_{\mathcal{F}}$ to $Y_{\mathcal{F}}$. *It may happen that π is a modification without $\pi_{\mathcal{F}}$ being so.*

Question: *Under which condition is the morphism $\pi : Y_{\mathcal{F}} \to Y$ proper?*

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

Question: *Under which condition is the morphism $\pi : Y_{\mathcal{F}} \to Y$ proper?*

Answer: *Precisely when the support $|\mathcal{F}|$ of $\mathcal{F}$ contains the local tropicalization of $Y \hookrightarrow (\mathbb{C}^n, 0)$!* That is, this local tropicalization may be defined as *the intersection of the supports of the fans $\mathcal{F}$ such that $\pi : Y_{\mathcal{F}} \to Y$ is proper.*

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

One is led to look at special fans $\mathcal{F}$: **those whose support is equal to the local tropicalization of $Y \hookrightarrow (\mathbb{C}^n, 0)$**. In the best cases, the modification $\pi : Y_{\mathcal{F}} \to Y$ is a **toroidal resolution**. This is the **Newton non-degenerate** situation.

Foundational papers relating **Newton non-degeneracy** and **toric resolutions**

Inventiones math. 32, 1 – 31 (1976)

Inventiones mathematicae

Polyèdres de Newton et nombres de Milnor

A. G. Kouchnirenko (Moscou)

Inventiones math. 37, 253 – 262 (1976)

Inventiones mathematicae

Zeta-Function of Monodromy and Newton's Diagram

A. N. Varchenko

Khovanskii's extension to **complete intersection singularities**

4. The local behavior of the system of functions $f_1, \ldots, f_k$ around the point 0 in C^n is determined on the whole by the parts of the Newton polyhedra $\Delta_1, \ldots, \Delta_k$ that are turned to the point zero. These parts are called Newton diagrams (see [9, 10] for a similar definition). The definition of nonsingularity can be carried over directly to systems of functions with given Newton diagrams. Here it is necessary to require that the condition (ξ) holds for all covectors ξ with positive coordinates. The theorem on resolution of singularities also carries over to the local case with the help of a suitable toroidal variety. Here it is necessary to take a sufficiently fine decomposition of the positive octant in R^{n*}. We note that the subject of Newton polyhedra began precisely from local problems: from the rich empirical material and conjectures of V. I. Arnol'd and the first results of A. G. Kushnirenko.

(Page 292 of A. G. Khovanskii, ***Newton polyhedra and toroidal varieties***. Funct. Analysis Appl. **11** (1977), 289–296.)

ANALYSE MATHÉMATIQUE. — *Sur la résolution des singularités des surfaces*. Note de M. GUSTAVE DUMAS, présentée par M. P. Appell.

Soit

$$(1) \qquad F(x_i) = \sum A_{\alpha\beta\gamma}\, x_1^{\alpha} x_2^{\beta} x_3^{\gamma} = 0$$

l'équation d'un élément de surface analytique dans le voisinage du point

$$x_i = 0,$$

supposé singulier.

L'objet de cette Note est de montrer comment on peut, dans ce cas, représenter cet élément.

Une *surface polyédrale convexe* Π, jouant le rôle des polygones de Newton dans la résolution des singularités des courbes analytiques planes, est à la base de la théorie.

(G. Dumas, ***Sur la résolution des singularités de surfaces***. C. R. Acad. Sci. Paris **152** (1911), 682–684.)

Precursors: the 1911 and 1912 articles of Gustave Dumas II

ANALYSE MATHÉMATIQUE. — *Sur les singularités des surfaces.* Note de M. **Gustave Dumas**, présentée par M. Appell.

Dans une précédente Note (1), à laquelle se rapporte ce qui suit, j'ai donné un moyen de résolution des singularités des surfaces.

Aujourd'hui, je voudrais, en résolvant *complètement* la singularité que la surface

$$z^{10} - 4y^{12} + 4x^3y^8 + x^6y^4 - x^9 + 25x^4y^5z^2 = 0 \quad (1)$$

présente au point

$$x = y = z = 0, \quad (2)$$

montrer, avec quelques détails, en quoi consiste la méthode. Le polyèdre ne possède ici qu'une face finie, triangulaire, et les arêtes qui la limitent seront les arêtes I, II, III.

A chacune d'elles, on fait correspondre une substitution désignée par le même chiffre romain :

I.	II.	III.
$x = \xi_1^{20}\eta_1^{21}u_1^{12}$,	$x = \xi_2^{20}\eta_2^{91}u_2^{50}$,	$x = \xi_3^{20}\eta_3^{26}u_3^{7}$,
$y = \xi_1^{15}\eta_1^{16}u_1^{9}$,	$y = \xi_2^{15}\eta_2^{69}u_2^{38}$,	$y = \xi_3^{15}\eta_3^{19}u_3^{5}$,
$z = \xi_1^{18}\eta_1^{19}u_1^{11}$,	$z = \xi_2^{18}\eta_2^{82}u_2^{45}$,	$z = \xi_3^{18}\eta_3^{23}u_3^{6}$.

(G. Dumas, ***Sur les singularités des surfaces***. C. R. Acad. Sci. Paris **154** (1912), 1495–1497.)

Definition

Consider $w \in \sigma^\circ$ and $f = \sum_{m \in \text{ supp } f} f_m \chi^m \in \mathbb{C}[[x_1, \dots, x_n]]$.

The **basis** $\boxed{\delta_w(f)} \subseteq \text{supp } f$ **of** $\text{supp } f$ **relative to** w is the locus where the restriction of the linear form $w : M_{\mathbb{R}} \to \mathbb{R}$ to the support $\text{supp } f$ of f achieves its *minimum*. The w-weighted homogeneous polynomial

$$\boxed{\text{in}_w f} := \sum_{m \in \delta_w(f)} f_m \chi^m$$

is called the ***w*-initial form** of the series f.

If $I \hookrightarrow \mathbb{C}[[x_1, \dots, x_n]]$ is an ideal, then its ***w*-initial ideal** $\boxed{\text{in}_w I}$ is the ideal of $\mathbb{C}[[x_1, \dots, x_n]]$ generated by the w-initial forms $\text{in}_w f$, for all $f \in I$.

The definition of **Newton non-degenerate complete intersections**

Definition 4.3 Fix a positive integer s and a regular sequence $(f_1, \ldots, f_s)$ in $\hat{\mathcal{O}}$. Consider the germ $(Y, 0) \hookrightarrow \mathbb{C}^n$ defined by the ideal $\langle f_1, \ldots, f_s \rangle \hat{\mathcal{O}}$. The sequence $(f_1, \ldots, f_s)$ is *a Newton non-degenerate complete intersection system* for $(Y, 0)$ if for any positive weight vector $w \in (\mathbb{R}_{>0})^n$, the hypersurfaces of $(\mathbb{C}^*)^n$ defined by each $\mathrm{in}_w(f_i)$ form a normal crossings divisor in a neighborhood of their intersection. Equivalently, the differentials of the initial forms $\mathrm{in}_w(f_1), \ldots, \mathrm{in}_w(f_s)$ must be linearly independent at each point of this intersection.

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.)

A general question of Teissier

Finally, it seems that the following coordinate-free definition of nondegeneracy is appropriate:

DEFINITION. An algebraic or formal subscheme X of an affine space $\mathbf{A}^d(k)$ is nondegenerate at a point $x \in X$ if there exist local coordinates $u_1, \dots, u_d$ centered at x and an open (étale or formal) neighborhood U of x in $\mathbf{A}^d(k)$ such that there is a proper birational toric map $\pi : Z \to U$ in the coordinates $u_1, \dots, u_d$ with Z nonsingular and such that the strict transform X' of $X \cap U$ by π is nonsingular and transversal to the exceptional divisor at every point of $\pi^{-1}(x) \cap X'$.

If X admits a system of equations which in some coordinates is nondegenerate with respect to its Newton polyhedra, it is also nondegenerate in this sense as we saw. The converse will not be discussed here.

QUESTION [Teissier 2003]. Given a reduced and equidimensional algebraic or formal space X over an algebraically closed field k, is it true that for every point $x \in X$ there is a local formal embedding of X into an affine space $\mathbf{A}^N(k)$ such that X is nondegenerate in $\mathbf{A}^N(k)$ at the point x?

A subsequent problem is to give a geometric interpretation of the systems of coordinates in which an embedded toric resolution for X exists.

(Page 228 of B. Teissier, ***Monomial ideals, binomial ideals, polynomial ideals***. In ***Trends in commutative algebra.***

ÂăMSRI Publications **51** (2004), 211–246.)

The importance of **standard tropicalizing fans**

Remark 3.8. Notice that Trop Y has no canonical fan structure. Particularly useful to us are those fan structures where the initial ideals of I are constant along the relative interiors of all its cones. A fan $\mathcal{F}$ with this property and support equal to Trop Y is called a *standard tropicalizing fan* (see Definition 5.4).

The use of standard tropicalizing fans is convenient when dealing with *Newton non-degenerate germs* (see Definition 5.8).

Proposition 3.9. *Assume that $\mathcal{F}$ is a standard tropicalizing fan of a Newton non-degenerate germ $(Y, 0) \hookrightarrow (\mathbb{C}^{n+1}, 0)$, and let $\tilde{Y}$ be the strict transform of Y under the toric morphism $\pi_{\mathcal{F}} : \mathcal{X}_{\mathcal{F}} \to \mathbb{C}^{n+1}$. Then, $\tilde{Y}$ is transversal to the toric boundary $\partial \mathcal{X}_{\mathcal{F}}$ of $\mathcal{X}_{\mathcal{F}}$ in the sense of Definition 4.1.*

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***.

In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, 2023, 629–709.)

Boundary-transversality

A **toroidal variety** is a pair $(W, \partial W)$ locally analytically isomorphic to toric pairs $(\mathcal{X}_{\mathcal{F}}, \partial \mathcal{X}_{\mathcal{F}})$. Toroidal varieties are unavoidable in a toric context, as **toric structures induce toroidal structures on boundary-transversal subvarieties**.

Definition 4.1. Let $(W, \partial W)$ be a toroidal variety. A reduced closed equidimensional subvariety V of W is called *boundary-transversal*, or *∂-transversal* for short, if the following conditions are satisfied for each stratum S of the toroidal stratification of W:

(1) the analytic space $V \cap S$ is a (possibly empty) equidimensional complex manifold;

(2) if $V \cap S \neq \emptyset$, then $\mathrm{codim}_V(V \cap S) = \mathrm{codim}_W(S)$.

As Theorem 5.12 below shows, our main example of ∂-transversal subvarieties are strict transforms of Newton non-degenerate germs $(X, 0) \hookrightarrow \mathbb{C}^n$ by toric birational morphisms defined by standard tropicalizing fans of $(X, 0) \hookrightarrow \mathbb{C}^n$.

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***The Milnor fiber conjecture of Neumann and Wahl, and an overview of its proof***.

In ***Essays in Geometry, dedicated to Norbert A'Campo***, A. Papadopoulos ed., EMS Publishing House, 2023, 629–709.)

Question: Why not to work only with normal crossings divisors?

There are several reasons:

- The **most economical standard tropicalizing fans** of a germ $Y \hookrightarrow (\mathbb{C}^n, 0)$, those directly related to the defining system of equations of Y, **are not necessarily regular**. That is, their associated toric varieties are not necessarily *smooth*.

- Any fan has regular subdivisions, but **non-canonical** (excepted in dimension 2). The regular standard tropicalizing fans are hard to relate to the defining system of equations.

- It is not necessary to look for regular subdivisions, as **rounding has the same topological properties in the toroidal context as in the normal crossings context**. Caveat: a subdivisor of a toroidal boundary is not necessarily a toroidal boundary.

Two other viewpoints on **local tropicalization**

A reduced subtoric germ $Y \hookrightarrow (\mathbb{C}^n, 0)$ is **interior** if it has no irreducible component included in the toric boundary $\partial\mathbb{C}^n$. We saw two viewpoints on the **local tropicalization of Y**, using **toric morphisms** or **weight vectors of arcs**. One may describe it in two other ways, using the **weight vectors whose initial ideal does not contain monomials**, and using the **weight vectors defined by the real-valued semivaluations of the local ring $\mathcal{O}_{Y,0}$, whose support is an interior subtoric germ**.

Theorem 1.6.2 *Let $Y \hookrightarrow (X_\sigma, 0)$ be an interior subtoric germ. Then, the following subsets of the weight cone $\sigma \subset N_{\mathbb{R}}$ coincide:*

1. *the compactifying cone of Y;*
2. *the cone-closure of the set of arcwise weight vectors of Y;*
3. *the cone-closure of the set of initial weight vectors of Y;*
4. *the cone-closure of the set of valuative weight vectors of Y.*

(P. Popescu-Pampu, D. Stepanov, ***An introduction to local tropicalization.*** Under review. Soon on ArXiv.)

Our initial **valuative viewpoint** on **local tropicalization**

In our 2013 paper ***Local tropicalization***, we wanted to build a notion which applies **both locally and globally**. We also wanted to respect the basic philosophy of commutative algebra: in order to **apply to both algebraic geometry and number theory**, do not assume that rings contain subfields.

Local tropicalization

Patrick Popescu-Pampu and Dmitry Stepanov

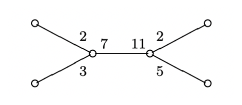
Abstract. In this paper we propose a general functorial definition of the operation of *local tropicalization* in commutative algebra. Let R be a commutative ring, Γ a finitely generated subsemigroup of a lattice, $\gamma : \Gamma \to R/R^*$ a morphism of semigroups, and $\mathcal{V}(R)$ the topological space of valuations on R taking values in $\mathbb{R} \cup \infty$. Then we may *tropicalize* with respect to γ any subset $\mathcal{W}$ of the space of valuations $\mathcal{V}(R)$. By definition, we get a subset of a rational polyhedral cone canonically associated to Γ, enriched with strata at infinity. In particular, when R is a local ring, γ is a *local* morphism of semigroups, and $\mathcal{W}$ is the space of valuations which are either positive or non-negative on R, we call these processes *local tropicalizations*. They depend only on the ambient toroidal structure, which in turn allows to define tropicalizations of subvarieties of toroidal embeddings. We prove that with suitable hypothesis, these local tropicalizations are the supports of finite rational polyhedral fans enriched with strata at infinity and we compare the global and local tropicalizations of a subvariety of a toric variety.

It is now time to define **splice diagrams**. They are finite trees whose vertices are either **leaves** or **nodes** and whose edges are decorated by positive integer **weights near the nodes**. Those weights have to satisfy:

- the weights around a node are positive and pairwise coprime;
- the weight on an edge ending in a leaf is > 1;
- all edge determinants are positive.

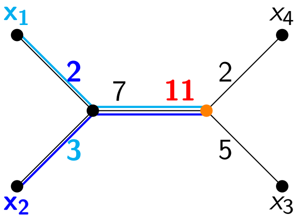
More general splice diagrams appear for other situations (see, eg, [7, 29, 30]), but we will only consider splice diagrams satisfying the above conditions here.

Theorem 1.1 [7] *The homology spheres that are singularity links are in one-one correspondence with splice diagrams satisfying the above conditions.*



(Pages 763 and 764 of W. Neumann, J. Wahl, *Complex surface singularities with integral homology sphere links*. Geom. Topol. **9** (2005), 757–811.)

The splice diagrams which lead to **splice type systems** have to satisfy a supplementary constraint: the **semigroup condition**.



$$\mathbf{11} = \mathbf{1} \cdot \mathbf{3} + \mathbf{4} \cdot \mathbf{2} \quad (\implies \mathbf{11} \in \text{Semigroup } \langle \mathbf{3}, \mathbf{2} \rangle.)$$

Here is an associated **splice type system**:

$$\begin{cases} x_1^2 - x_2^3 + x_3^1 x_4^1 = 0, \\ \mathbf{x_3^5 - x_4^2 + x_1{}^1 x_2{}^4 = 0}. \end{cases}$$

What happens at nodes of valency ≥ 4?

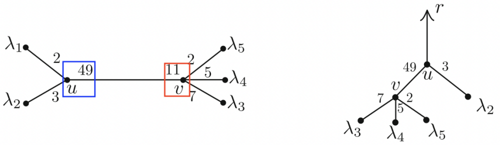


Fig. 1 From left to right: a splice diagram and its associated rooted diagram obtained by fixing one of the leaves as its root r, and removing one weight from the star of each node

The semigroup condition is also satisfied, since

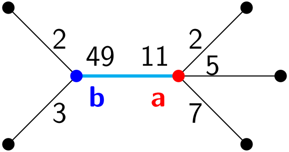
$$\boxed{49 = 0 \cdot (2 \cdot 5) + 1 \cdot (2 \cdot 7) + 1 \cdot (5 \cdot 7)} \quad \text{and} \quad \boxed{11 = 1 \cdot (3) + 4 \cdot (2)} = 3 \cdot (3) + 1 \cdot (2).$$

Thus, we may take as exponents $m_{u,[u,v]} = (0, 0, 0, 1, 1)$ and $m_{v,[u,v]} = (1, 4, 0, 0, 0)$ or $(3, 1, 0, 0, 0)$ in $\mathbb{Z}^5$. A possible strict splice type system for Γ is:

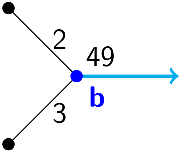
$$\begin{cases} f_{u,1} := z_1^2 - 2\, z_2^3 + z_4\, z_5, \\ f_{v,1} := z_1 z_2^4 + z_3^7 + z_4^5 - 2155\, z_5^2, \\ f_{v,2} := 33\, z_1 z_2^4 + z_3^7 + 2\, z_4^5 - 2123\, z_5^2. \end{cases} \tag{2.11}$$

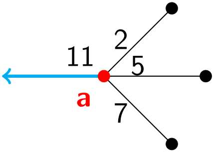
(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.)

The conjecture states for instance that the Milnor fiber of any splice type singularity corresponding to the splice diagram



is obtained by splicing the Milnor fibers of the splice type singularities corresponding to the *a*-**side** and *b*-**side** diagrams





along the surfaces cut by the variables corresponding to the **arrows**.

Splice type singularities are **Newton non-degenerate complete intersections**

Theorem 1.1 *Splice type systems are Newton non-degenerate complete intersection systems of equations. The associated splice type singularities are isolated, irreducible and not contained in any coordinate subspace of the corresponding ambient space* $\mathbb{C}^n$.

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.)

In order to prove the Milnor fiber conjecture, we construct a special kind of smoothing of a splice type singularity, **whose total space is again Newton non-degenerate**.

The **local tropicalizations of splice type singularities**

Our proof of Newton non-degeneracy passes through the description of the **local tropicalizations of splice type singularities**: **they are cones over suitable embeddings of the corresponding splice diagram**.

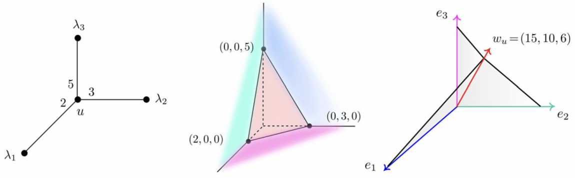


Fig. 2 Splice diagram, Newton polyhedron and local tropicalization of the E_8 surface singularity

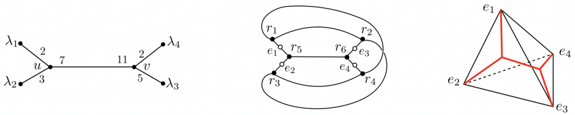


Fig. 3 From left to right: Splice diagram, and representations of the global and local tropicalizations of Example 3.9

(A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.)

For details, one may consult our 2023 paper, in which our proof is decomposed into 28 steps.

- We construct **a special smoothing** $f : Y \to \mathbb{D}$ of the given splice type singularity $X \hookrightarrow \mathbb{C}^n$, by adding suitable powers of a new variable.
- We determine **a standard tropicalizing fan** $\mathcal{F}$ of the total space $Y \hookrightarrow \mathbb{C}^{n+1}$ of the smoothing f.
- We prove that $Y \hookrightarrow \mathbb{C}^{n+1}$ is a **Newton non-degenerate complete intersection**.
- Consider the strict transform $\tilde{Y}$ of Y by the toric birational morphism $\mathcal{X}_{\mathcal{F}} \to \mathbb{C}^{n+1}$. Then $\pi : \tilde{Y} \to Y$ is a modification and $f \circ \pi : \tilde{Y} \to \mathbb{D}$ satisfies the hypotheses of **Nakayama and Ogus' local triviality theorem** (explained later on).
- We perform **similar analyses for the a-side and b-side**.
- We relate the **log special fibers** of $f \circ \pi$, $f_a \circ \pi_a$, $f_b \circ \pi_b$.

The principles of our proof of the Milnor fiber conjecture II

Let us look at the **rounding of the log special fiber** of $f \circ \pi$:

$$\begin{array}{ccc} ((Z(f\circ\pi))^{\dagger})^{\odot} & \xrightarrow{((f\circ\pi)^{\dagger})^{\odot}} & (0^{\dagger})^{\odot} \\ \Big\downarrow \tau_{(Z(f\circ\pi))^{\dagger}} & & \Big\downarrow \tau_{0^{\dagger}} \\ Z(f\circ\pi) & \xrightarrow[f\circ\pi]{} & 0. \end{array}$$

The upper horizontal arrow is a representative of the Milnor fibration of f.

We construct the deformed splice type system such that:

$$Z(f \circ \pi) = \tilde{X} + \partial_{\mathbf{a}} \tilde{\mathbf{Y}} + \partial_{\mathbf{ab}} \tilde{\mathbf{Y}} + \partial_{\mathbf{b}} \tilde{\mathbf{Y}},$$

where **the dual complex of the divisor $\partial_{\mathbf{a}} \tilde{\mathbf{Y}} + \partial_{\mathbf{ab}} \tilde{\mathbf{Y}} + \partial_{\mathbf{b}} \tilde{\mathbf{Y}}$ is isomorphic to the given splice diagram, rooted at an interior point of $[a, b]$.**

We prove that the preimage of $\partial_{\mathbf{ab}} \tilde{\mathbf{Y}}$ by the left vertical arrow intersects the Milnor fibers along pieces isomorphic to $G_a \times G_b$, as predicted by the Milnor fiber conjecture.

We study analogous diagrams for the ***a*-side** and ***b*-side** systems. We relate them to the previous diagram, using adequate toric morphisms. In particular, we determine a subdivisor $\partial_0^- \tilde{\mathbf{Y}}_\mathbf{a} \hookrightarrow Z(f_a \circ \pi_a)$ which is sent isomorphically onto $\partial_\mathbf{a} \tilde{\mathbf{Y}}$.

The *a*-side pieces of the Milnor fibers of f are related to the Milnor fibers of f_a using the following diagram in the log category:

> By restricting the commutative triangle (7.15) to those compact subspaces of the source and target of the embedding Φ_a, we get the following commutative triangle in the logarithmic category:
>
> $$\begin{array}{ccc} (\partial_0^- \tilde{Y}_a, \mathcal{O}^\star_{\mathcal{X}_{\mathcal{F}^-_{a,r}} | \partial_0^- \tilde{Y}_a}(-Z(\tilde{x}_0))) & \longrightarrow & (\partial_a \tilde{Y}, \mathcal{O}^\star_{\mathcal{X}_{\mathcal{F}} | \partial_a \tilde{Y}}(-Z(\tilde{z}_0))) \\ & \searrow \quad \swarrow & \\ & 0^\dagger . & \end{array} \tag{7.16}$$
>
> As $\Phi_a^\dagger$ is strict by Step (23), the horizontal arrow from (7.16) is an isomorphism.

(Page 702 of A. Cueto, P. Popescu-Pampu, D. Stepanov, ***Local tropicalizations of splice type surface singularities***. With an appendix written by J. Wahl. Math. Annalen **390** (2024), 811–887.)

It is now time to turn to the definition of **log structures** and of **rounding**.

Look again at the drawing:

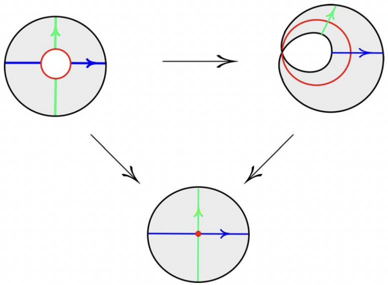


Figure 11. The left arrow is the real oriented blowup while the right one is the usual blowup of the center of a disc. The horizontal arrow is the natural factorisation of the left arrow through the second one.

The left diagonal arrow may be realized as **the inverse of the passage to polar coordinates**.

Let us look at the passage to **polar coordinates**

The **change of variables from cartesian to polar coordinates** is usually written

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

or

$$z = r \cdot e^{i\theta}$$

if $\boxed{z} := x + iy$. In what follows we will not use any of the transcendental functions $\cos, \sin, e^{\bullet}$, but we will rather write this change of variables as:

$$z = |z| \cdot \mathrm{sign}(z),$$

The **sign function** $\mathrm{sign} : \mathbb{C}^* \to \mathbb{S}^1$ is the morphism of multiplicative abelian groups defined by:

$$\boxed{\mathbf{sign}(z)} := z/|z|.$$

We have a pair of inverse maps:

$$\boxed{\psi_{\mathbb{C}}} : \begin{array}{ccc} \mathbb{C} & \dashrightarrow & \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \\ z & \mapsto & (|z|, \mathrm{sign}(z)) \end{array}, \qquad \boxed{\tau_{\mathbb{C}}} : \begin{array}{ccc} \mathbb{R}_{\geq 0} \times \mathbb{S}^1 & \to & \mathbb{C} \\ (r, u) & \mapsto & r \cdot u \end{array}.$$

The dashed arrow $\dashrightarrow$ indicates that $\psi_{\mathbb{C}}$ is not defined at the origin of $\mathbb{C}$ and that it cannot be extended by continuity to the whole complex plane $\mathbb{C}$.

But its inverse $\tau_{\mathbb{C}}$ is everywhere defined and surjective! It is this inverse which allows **to interpret the real oriented blowup of the center of a disc using polar coordinates**.

Making the passage to polar coordinates intrinsic I

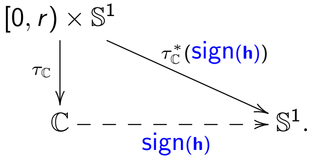
Consider a germ $h \in \mathcal{O}_{\mathbb{C},0} \setminus \{0\}$. We may write it in a unique way:

$$h = z^m \cdot v$$

for some $m \in \mathbb{N}$ and $v \in \mathcal{O}^{\star}_{\mathbb{C},0}$. Thus, one has on some $\mathbb{D}_r \setminus \{0\}$:

$$\mathrm{sign}(\mathbf{h}) = \frac{h}{|h|} = \left(\frac{z}{|z|}\right)^m \cdot \frac{v}{|v|} = \mathrm{sign}(z)^m \cdot \mathrm{sign}(v).$$

The lift $\tau_{\mathbb{C}}^{*}(\mathrm{sign}(\mathbf{h})) : (0, r) \times \mathbb{S}^1 \to \mathbb{S}^1$ extends by continuity to $[0, r) \times \mathbb{S}^1$:



If $h_1, h_2 \in \mathcal{O}_{\mathbb{C},0} \setminus \{0\}$, then we have on some $\mathbb{D}_r \setminus \{0\}$:

$$\text{sign}(h_1) \cdot \text{sign}(h_2) = \text{sign}(h_1 \cdot h_2).$$

As a consequence, the relation

$$\tau_{\mathbb{C}}^*(\text{sign}(h_1)) \cdot \tau_{\mathbb{C}}^*(\text{sign}(h_2)) = \tau_{\mathbb{C}}^*(\text{sign}(h_1 \cdot h_2))$$

is true over a neighborhood of the boundary $\partial(\mathbb{R}_{\geq 0} \times \mathbb{S}^1)$ of $\mathbb{R}_{\geq 0} \times \mathbb{S}^1$.

Proposition

Consider a point $\mathbf{P} \in \partial(\mathbb{R}_{\geq 0} \times \mathbb{S}^1)$. *Then, the map*

$$\begin{array}{ccc} (\mathcal{O}_{\mathbb{C},0} \setminus \{0\}, \cdot) & \to & (\mathbb{S}^1, \cdot) \\ h & \mapsto & \tau_{\mathbb{C}}^*(\text{sign}(h))(\mathbf{P}) \end{array}$$

is a **morphism of multiplicative monoids extending the standard morphism of groups** $(\mathcal{O}_{\mathbb{C},0}^{\star}, \cdot) \to (\mathbb{S}^1, \cdot)$ **defined by** $h \mapsto \text{sign}(h(0))$. *Moreover, this morphism induces* **a homeomorphism between** $\partial(\mathbb{R}_{\geq 0} \times \mathbb{S}^1)$ **and the group of such morphisms of monoids**.

We get the following **intrinsic description** of the real oriented blowup of a point of a Riemann surface:

Definition

Let S be a Riemann surface and s be a point on it. Denote by $\boxed{\mathcal{O}_S^\star(-s)}$ the subsheaf of monoids of $(\mathcal{O}_S, \cdot)$ consisting of the holomorphic functions which do not vanish outside s. Define the **real oriented blowup of S at s** by:

$$\boxed{S_s^\odot} := \{(x,u), x \in S, u \in \mathbf{Hom}(\mathcal{O}^\star_{\mathbf{S},\mathbf{x}}(-\mathbf{s}), \mathbb{S}^{\mathbf{1}}), u(f) = \mathrm{sign}(f(x)) \ \forall f \in \mathcal{O}^\star_{S,x}\}$$

where **the morphisms are taken in the category of monoids**. Define the associated **real oriented blowup morphism** by:

$$\boxed{\tau} : \begin{array}{ccc} S_s^\odot & \to & S \\ (x,u) & \mapsto & x \end{array}.$$

The previous definition of **real oriented blow up** proceeds in two steps:

1. consider the sheaf of monoids $(\mathcal{O}^{\star}_{\mathbf{S}}(-\mathbf{s}), \cdot)$;
2. construct the real oriented blowup $\mathbf{S}^{\odot}_{\mathbf{s}}$ of S at s from it.

One may perform these steps starting from any reduced divisor D in a complex variety W:

1. consider the sheaf of monoids $\boxed{(\mathcal{O}^{\star}_{W}(-D), \cdot)}$ consisting of the **holomorphic functions which do not vanish outside** D;
2. construct the real oriented blowup $\boxed{W^{\odot}_{D}}$ of W along D from it, by the same formula as above.

But one may be **still more general**, defining the **rounding of an arbitrary complex log structure**.

The inclusion morphism $(\mathcal{O}^{\star}_{\mathbf{W}}(-\mathbf{D}), \cdot) \hookrightarrow (\mathcal{O}_{\mathbf{W}}, \cdot)$ is an example of **log structure in the sense of Fontaine and Illusie**:

Definition

A **logarithmic space** $\boxed{W}$ or a **log space** for short is a ringed space $(|W|, \mathcal{O}_W)$, endowed with a sheaf of monoids $\boxed{\mathcal{M}_W}$ and a morphism

$$\boxed{\alpha_W} : \mathcal{M}_{\mathbf{W}} \to (\mathcal{O}_{\mathbf{W}}, \cdot)$$

of sheaves of monoids, **which restricts to an isomorphism between their subsheaves of invertible elements $(\mathcal{M}^{\star}_W, \cdot)$ and $(\mathcal{O}^{\star}_W, \cdot)$**. The pair $(\mathcal{M}_W, \alpha_W)$ is called a **logarithmic structure** on the ringed space W, or a **log structure** for short. The log space W and its log structure are called **complex** if the structure sheaf $\mathcal{O}_W$ is a sheaf of complex algebras.

Question: Why is the morphism $\alpha_W : \mathcal{M}_W \to (\mathcal{O}_W, \cdot)$ not required to be injective, as is the case for the divisorial log structure $\mathcal{O}^\star_{\mathbf{W}}(-\mathbf{D}) \hookrightarrow \mathcal{O}_{\mathbf{W}}$?

Because one wants to be able to **pull back** log structures. And even if one starts from $(\mathcal{O}^\star_W(-D), \cdot) \hookrightarrow (\mathcal{O}_W, \cdot)$, its pullback $(\mathcal{O}^\star_{W|D}(-D), \cdot) \to (\mathcal{O}_D, \cdot)$ is no longer injective!

In fact, if one simply takes the pullback to D of the sheaf $(\mathcal{O}^\star_W(-D), \cdot)$,
one does not get a log structure: the condition of isomorphicity of the subgroups of invertible elements is not satisfied. One obtains only a **prelog structure**,
Âăand one has to take the **associated log structure**.

This illustrates the care which has to be taken when building the foundations of the theory.

Kato's foundational 1988 paper

LOGARITHMIC STRUCTURES OF FONTAINE-ILLUSIE

By Kazuya Kato

1. Logarithmic structures.
2. Fine logarithmic structures.
3. Smooth morphisms.
4. Several types of morphisms.
5. Crystalline sites.
6. Crystals and crystalline cohomology.

Complements.

Introduction. In this note, we present a general formulation of "logarithmic structure" on a scheme found by J. M. Fontaine and L. Illusie. Following their plan, we develop the theory of crystals with logarithmic poles using this logarithmic structure.

Kato and Nakayama's version of **real oriented blowup**

K. KATO AND C. NAKAYAMA
KODAI MATH. J.
22 (1999), 161–186

LOG BETTI COHOMOLOGY, LOG ÉTALE COHOMOLOGY, AND LOG DE RHAM COHOMOLOGY OF LOG SCHEMES OVER C

KAZUYA KATO AND CHIKARA NAKAYAMA

(1.2). We define the topological space $X^{\log}$ as follows. In §3, we will endow $X^{\log}$ with a structure of a ringed space.

As a set, we define $X^{\log}$ by

$$X^{\log} = \left\{ (x,h) \;\middle|\; x \in X, h \in \mathrm{Hom}(M^{\mathrm{gp}}_{X,x}, \mathbf{S}^1), h(f) = \frac{f(x)}{|f(x)|} \text{ for any } f \in \mathcal{O}^*_{X,x} \right\}.$$

We have an evident map $X^{\log} \to X; (x,h) \mapsto x$ which we will denote by τ.

The **rounding** $X^{\odot}$ is also called the **Kato-Nakayama space** or the **Betti realization** of X and is usually denoted $X^{\log}$ or $X_{\log}$.

Alternative description of the **rounding** of X

The set $X^{\log}$ is defined also as follows: Let T be the analytic space $\operatorname{Spec}(\boldsymbol{C})$ endowed with the log structure M_T given by:

$$\Gamma(T, M_T) = \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1$$

where

$$\boldsymbol{R}_{\geq 0} = \{x \in \boldsymbol{R}; x \geq 0\}$$

$$\boldsymbol{S}^1 = \{x \in \boldsymbol{C}; |x| = 1\}$$

with the multiplicative semi-group laws, and where $M_T \to \mathcal{O}_T$ is given by

$$\alpha_T : \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1 \to \boldsymbol{C}; (x, y) \to xy.$$

Note that this log structure on T is not fs.

As a set, $X^{\log}$ is the set of all morphisms $T \to X$ of log analytic spaces over $\boldsymbol{C}$: We associate to $(x, h) \in X^{\log}$ the morphism $T \to X$ defined by the homomorphism $M_{X,x} \to \boldsymbol{R}_{\geq 0} \times \boldsymbol{S}^1; a \mapsto (|(\alpha(a))(x)|, h(a))$.

(Page 165 of K. Kato, C. Nakayama, ***Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over C***. Kodai Math. J. **22** (1999), 161–186.)

The rounding of X has a **natural topology**

We define the topology of $X^{\log}$ as follows.

(1.2.1). Assume there exists a chart $\beta : P \to M_X$ with P fs. When fixing such β, we can identify $X^{\log}$ with a closed subset of $X \times \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1)$ via the map

$$X^{\log} \hookrightarrow X \times \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1); \quad (x, h) \mapsto (x, h_P)$$

where h_P is the composite $P^{\mathrm{gp}} \to M^{\mathrm{gp}}_{X,x} \xrightarrow{h} \mathbf{S}^1$. The image is closed because $(x, \sigma) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1)$ is contained in the image if and only if for any $p \in P$, $(\beta(p))(x) = \sigma(p)|(\beta(p))(x)|$.

We endow $X^{\log}$ with the induced topology from $X \times \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1)$. Here the topology of $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1)$ is the evident one. This topology does not depend on the choice of a chart $P \to M_X$ because for another $\beta' = \beta \circ u$ with $u : P' \to P$, the map $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{S}^1) \to \mathrm{Hom}(P'^{\mathrm{gp}}, \mathbf{S}^1)$ is closed and continuous.

(Page 165 of K. Kato, C. Nakayama, ***Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over C***. Kodai Math. J. **22** (1999), 161–186.)

Reminder: application of rounding to the study of Milnor fibers

1. Start from a Milnor tube representative of a smoothing:

$$(X, Z(f)) \xrightarrow{f} (\mathbb{D}, 0).$$

2. Choose a **normal crossings resolution π of f**:

$$(\tilde{X}, Z(f \circ \pi)) \xrightarrow{\pi} (X, Z(f)) \xrightarrow{f} (\mathbb{D}, 0).$$

3. Consider the divisorial log structures on $\tilde{X}$ and $\mathbb{D}$ induced by $Z(f \circ \pi)$ and 0:

$$\tilde{X}^{\dagger} \xrightarrow{(f \circ \pi)^{\dagger}} \mathbb{D}^{\dagger}.$$

4. Restrict those log structures to $Z(f \circ \pi)$ and 0:

$$(Z(f \circ \pi))^{\dagger} \xrightarrow{(f \circ \pi)^{\dagger}} 0^{\dagger} \quad \text{(the log special fiber)}.$$

5. **Round** those restrictions:

$$((Z(f \circ \pi))^{\dagger})^{\odot} \xrightarrow{((f \circ \pi)^{\dagger})^{\odot}} (0^{\dagger})^{\odot},$$

getting a representative of the Milnor fibration.

As explained before, we use modifications defined by standard tropicalizing fans. They are **not normal crossings resolutions** of the smoothing f under scrutiny, but **only toroidal resolutions**. Luckily, **Chikara Nakayama** and **Arthur Ogus** proved a **local triviality theorem** which covers this level of generality.

Geometry & Topology 14 (2010) 2189–2241 **2189**

Relative rounding in toric and logarithmic geometry

CHIKARA NAKAYAMA
ARTHUR OGUS

We show that the introduction of polar coordinates in toric geometry smoothes a wide class of equivariant mappings, rendering them locally trivial in the topological category. As a consequence, we show that the Betti realization of a smooth proper and exact mapping of log analytic spaces is a topological fibration, whose fibers are orientable manifolds (possibly with boundary). This turns out to be true even for certain noncoherent log structures, including some families familiar from mirror symmetry. The moment mapping plays a key role in our proof.

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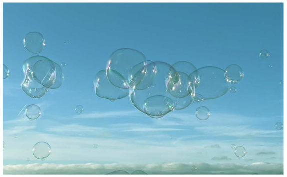

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Healthy, happy and creative ninth decade, Bernard!
Thank you for opening my horizons during many unforgettable mathematical rambles!

(June 2001)