Improved bounds for the Fourier uniformity conjecture

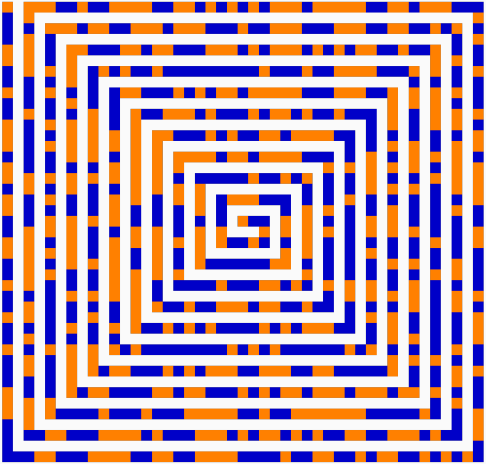
Prime numbers and arithmetic randomness – CIRM

Cédric Pilatte

23 June 2025 – Luminy

University of Oxford – Mathematical Institute

Introduction



Liouville pseudo-randomness

Guiding heuristic

Statistics of the completely multiplicative function $\lambda(n) := (-1)^{\Omega(n)}$

$$\approx$$

Statistics of random sequence of $+1$ and -1.

Liouville pseudo-randomness

Guiding heuristic

Statistics of the completely multiplicative function $\lambda(n) := (-1)^{\Omega(n)}$

$$\approx$$

Statistics of random sequence of $+1$ and -1.

Cancellation in long sums $\iff$ Prime Number Theorem

$$\sum_{n \leqslant X} \lambda(n) = o(X)$$

Liouville pseudo-randomness

Guiding heuristic

Statistics of the completely multiplicative function $\lambda(n) := (-1)^{\Omega(n)}$

$$\approx$$

Statistics of random sequence of $+1$ and -1.

Cancellation in long sums $\iff$ Prime Number Theorem

$$\sum_{n \leqslant X} \lambda(n) = o(X)$$

Square-root cancellation $\iff$ Riemann Hypothesis

$$\sum_{n \leqslant X} \lambda(n) = O(X^{1/2+\varepsilon})$$

Correlations of the Liouville function

Define $\mathbb{E}^*_{n\leqslant X} f(n) := \frac{1}{\log X}\sum_{n\leqslant X}\frac{1}{n}f(n)$.

Correlations of the Liouville function

Define $\mathbb{E}^*_{n\leqslant X} f(n) := \frac{1}{\log X}\sum_{n\leqslant X}\frac{1}{n}f(n)$.

Logarithmic Chowla conjecture

Fix distinct integers $h_1,\ldots,h_k$. Then

$$\mathop{\mathbb{E}^*}_{n\leqslant X} \lambda(n+h_1)\lambda(n+h_2)\cdots\lambda(n+h_k) = o(1)$$

as $X\to\infty$.

Correlations of the Liouville function

Define $\mathbb{E}^*_{n \leqslant X} f(n) := \frac{1}{\log X} \sum_{n \leqslant X} \frac{1}{n} f(n)$.

Logarithmic Chowla conjecture

Fix distinct integers $h_1, \ldots, h_k$. Then

$$\mathop{\mathbb{E}^*}_{n \leqslant X} \lambda(n+h_1)\lambda(n+h_2)\cdots\lambda(n+h_k) = o(1)$$

as $X \to \infty$.

Theorem (Tao 2016, Tao-Teräväinen, Helfgott-Radziwiłł, P. 2023)

The logarithmic Chowla conjecture is true for $k = 2$.

Correlations of the Liouville function

Define $\mathbb{E}^*_{n\leqslant X} f(n) := \frac{1}{\log X}\sum_{n\leqslant X} \frac{1}{n} f(n)$.

Logarithmic Chowla conjecture

Fix distinct integers $h_1, \ldots, h_k$. Then

$$\mathbb{E}^*_{n\leqslant X} \lambda(n+h_1)\lambda(n+h_2)\cdots\lambda(n+h_k) = o(1)$$

as $X \to \infty$.

Theorem (Tao 2016, Tao-Teräväinen, Helfgott-Radziwiłł, P. 2023)

The logarithmic Chowla conjecture is true for $k=2$. In fact,

$$\mathbb{E}^*_{n\leqslant X} \lambda(n+h_1)\lambda(n+h_2) \ll (\log X)^{-c}$$

for some absolute constant $c>0$.

Correlations of the Liouville function

Define $\mathbb{E}^*_{n\leqslant X} f(n) := \frac{1}{\log X}\sum_{n\leqslant X}\frac{1}{n}f(n)$.

Logarithmic Chowla conjecture

Fix distinct integers $h_1,\ldots,h_k$. Then

$$\mathop{\mathbb{E}^*}_{n\leqslant X} \lambda(n+h_1)\lambda(n+h_2)\cdots\lambda(n+h_k) = o(1)$$

as $X\to\infty$.

Theorem (Tao 2016, Tao-Teräväinen, Helfgott-Radziwiłł, P. 2023)

The logarithmic Chowla conjecture is true for $k=2$. In fact,

$$\mathop{\mathbb{E}^*}_{n\leqslant X} \lambda(n+h_1)\lambda(n+h_2) \ll (\log X)^{-c}$$

for some absolute constant $c>0$.

Theorem (Tao-Teräväinen 2017)

The logarithmic Chowla conjecture is true for $k=3,5,7,9,\ldots$

Cancellation in almost all short intervals

Cancellation in almost all short intervals

Theorem (Matomäki-Radziwiłł 2015)

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) \right| = o(HX)$$

as $X \to \infty$.

Cancellation in almost all short intervals

Theorem (Matomäki-Radziwiłł-Tao 2015)

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sup_{\alpha \in \mathbb{R}} \sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha) \right| = o(HX)$$

as $X \to \infty$.

Here $e(n\alpha) := e^{2\pi i n\alpha}$.

Fourier pseudo-randomness in almost all short intervals

Fourier uniformity conjecture

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sum_{X \leqslant x \leqslant 2X} \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha) \right| = o(HX)$$

as $X \to \infty$.

Consequences

Fourier uniformity conjecture

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sum_{X \leqslant x \leqslant 2X} \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha) \right| = o(HX) \qquad (\star)$$

as $X \to \infty$.

Consequences

Fourier uniformity conjecture

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sum_{X \leqslant x \leqslant 2X} \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha) \right| = o(HX) \qquad (\bigstar)$$

as $X \to \infty$.

To prove the *logarithmic Chowla* and *logarithmic Sarnak* conjectures, it suffices to establish either of the following (for nilsequences):

1. $(\bigstar)$ holds when $H := (\log X)^{\varepsilon}$, for all $\varepsilon > 0$;

Consequences

Fourier uniformity conjecture

Let $H = H(X) \leqslant X$ be a function tending to infinity with X. Then

$$\sum_{X \leqslant x \leqslant 2X} \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha) \right| = o(HX) \qquad (\bigstar)$$

as $X \to \infty$.

To prove the *logarithmic Chowla* and *logarithmic Sarnak* conjectures, it suffices to establish either of the following (for nilsequences):

1. $(\bigstar)$ holds when $H := (\log X)^{\varepsilon}$, for all $\varepsilon > 0$;
2. $\exists c > 0$ such that $(\bigstar)$ holds when $H := \exp((\log X)^{1/2-c})$, **and** the Helfgott-Radziwiłł approach can be extended to k-point correlations.

Known results

Theorem (Walsh 2023)

The Fourier uniformity conjecture holds for intervals of length

$$H \geqslant \exp((\log X)^{1/2+\varepsilon}).$$

Improves earlier work by Matomäki-Radziwiłł-Tao, M-R-T-Teräväinen-Ziegler.

Known results

Theorem (Walsh 2023)

The Fourier uniformity conjecture holds for intervals of length

$$H \geqslant \exp((\log X)^{1/2+\varepsilon}).$$

Improves earlier work by Matomäki-Radziwiłł-Tao, M-R-T-Teräväinen-Ziegler.

Theorem (Walsh 2023)

Assuming GRH, the Fourier uniformity conjecture holds for intervals of length

$$H \geqslant (\log X)^{\psi(X)}$$

for any given function $\psi(X)$ tending to infinity.

Theorem (P. 2025+)

The Fourier uniformity conjecture holds for intervals of length

$$H \geqslant \exp((\log X)^{2/5+\varepsilon}).$$

Proof ideas

General approach

Suppose that

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha_x) \right| \gg HX$$

for some unknown real numbers $(\alpha_x)_{x \in [X,2X]}$.

1. **Turán–Kubilius inequality.** Get local relations between frequencies.
2. **Combinatorial analysis.** Obtain globlal formula for the frequencies.
3. **Taylor expansion.** Reduction to the Matomäki-Radziwiłł theorem.

Application of Turán–Kubilius

1. Obtain local relations

Let $I \subset \mathbb{N}$ be a discrete interval of length H.

Let $f : I \to \mathbb{C}$ be an arbitrary 1-bounded function.

Turán–Kubilius inequality

We have

$$\mathop{\mathbb{E}}_{n \in I} f(n) = \mathop{\mathbb{E}}_{\substack{n \in I \\ p \mid n}} f(n) + O(\delta)$$

for "many" primes $H^{c(\delta)} \leqslant p \leqslant H^{1/2}$.

1. Obtain local relations

Let $I \subset \mathbb{N}$ be a discrete interval of length H.

Let $f : I \to \mathbb{C}$ be an arbitrary 1-bounded function.

Turán–Kubilius inequality

We have

$$\mathop{\mathbb{E}}_{n \in I} f(n) = \mathop{\mathbb{E}}_{\substack{n \in I \\ p \mid n}} f(n) + O(\delta)$$

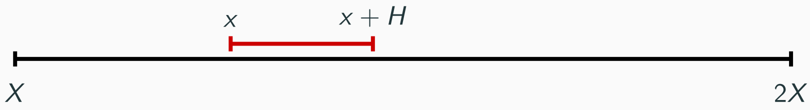
for "many" primes $H^{c(\delta)} \leqslant p \leqslant H^{1/2}$.

Parseval

Let $S \subset [0,1]$ be a $\frac{1}{H}$-separated set such that, for all $\alpha \in S$,

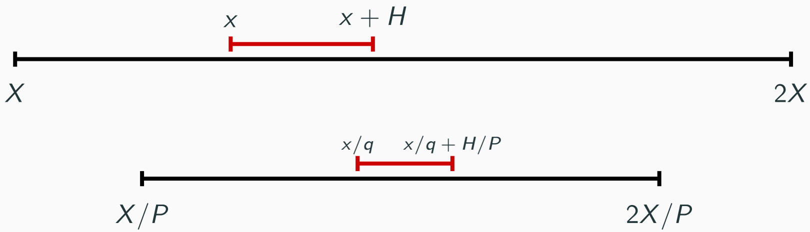
$$\Big| \mathop{\mathbb{E}}_{n \in I} f(n) e(n\alpha) \Big| \gg 1.$$

Then $|S| \ll 1$.



By Turán–Kubilius, for some scale $P = H^c$, there are many pairs (x, q) where q is a prime satisfying $P \leqslant q \leqslant (1+c)P$, such that

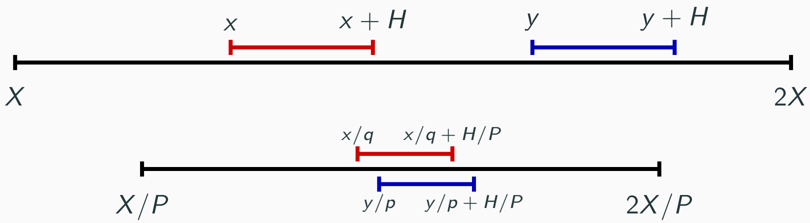
$$\mathop{\mathbb{E}}_{x \leqslant n \leqslant x+H} \lambda(n) e(\alpha_x n) \approx \mathop{\mathbb{E}}_{\substack{x \leqslant n \leqslant x+H \\ q \mid n}} \lambda(n) e(\alpha_x n)$$



By Turán–Kubilius, for some scale $P = H^c$, there are many pairs (x, q) where q is a prime satisfying $P \leqslant q \leqslant (1+c)P$, such that

$$\mathop{\mathbb{E}}_{x \leqslant n \leqslant x+H} \lambda(n) e(\alpha_x n) \approx \mathop{\mathbb{E}}_{\substack{x \leqslant n \leqslant x+H \\ q \mid n}} \lambda(n) e(\alpha_x n)$$

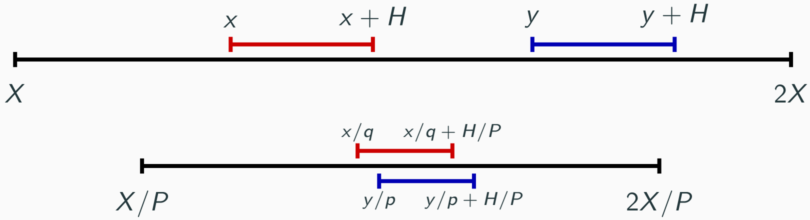
$$\approx - \mathop{\mathbb{E}}_{x/q \leqslant m \leqslant x/q + H/P} \lambda(m) e(\alpha_x q m).$$



If two such pairs (x, q) and (y, p) satisfy $\left|\frac{x}{q} - \frac{y}{p}\right| \leqslant c\frac{H}{P}$, then

$$[x/q,\, x/q + H/P] \quad \text{and} \quad [y/p,\, y/p + H/P]$$

are essentially the same interval I,

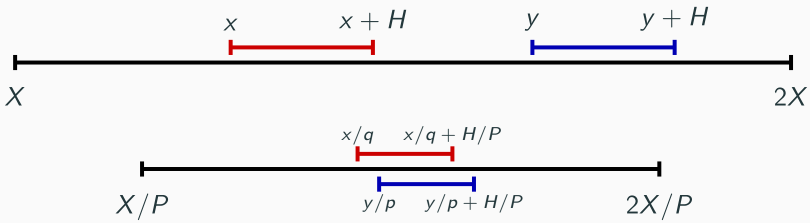


If two such pairs (x, q) and (y, p) satisfy $\left|\frac{x}{q} - \frac{y}{p}\right| \leqslant c\frac{H}{P}$, then

$$[x/q,\, x/q + H/P] \quad \text{and} \quad [y/p,\, y/p + H/P]$$

are essentially the same interval I, and

$$\begin{cases} \mathbb{E}_{x \leqslant n \leqslant x+H} \lambda(n) e(\alpha_x n) \approx - \mathbb{E}_{m \in I} \lambda(m) e(\alpha_x q m) \\ \mathbb{E}_{y \leqslant n \leqslant y+H} \lambda(n) e(\alpha_y n) \approx - \mathbb{E}_{m \in I} \lambda(m) e(\alpha_y p m). \end{cases}$$



If two such pairs (x, q) and (y, p) satisfy $\left|\frac{x}{q} - \frac{y}{p}\right| \leqslant c\frac{H}{P}$, then

$$[x/q,\, x/q + H/P] \quad \text{and} \quad [y/p,\, y/p + H/P]$$

are essentially the same interval I, and

$$\begin{cases} \mathop{\mathbb{E}}_{x \leqslant n \leqslant x+H} \lambda(n) e(\alpha_x n) \approx - \mathop{\mathbb{E}}_{m \in I} \lambda(m) e(\alpha_x q m) \\ \mathop{\mathbb{E}}_{y \leqslant n \leqslant y+H} \lambda(n) e(\alpha_y n) \approx - \mathop{\mathbb{E}}_{m \in I} \lambda(m) e(\alpha_y p m). \end{cases}$$

However, there are only $O(1)$ frequencies θ (up to a small error) such that

$$\left| \mathop{\mathbb{E}}_{m \in I} \lambda(m) e(\theta m) \right| \gg 1.$$

Suppose that

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e(n\alpha_x) \right| \gg HX.$$

Conclusion of Step 1

For some H-separated $A \subset [X, 2X]$ of size $|A| \gg X/H$, there are

$$\gg |A||\mathcal{P}|^2$$

quadruples $(x, y, p, q) \in A^2 \times \mathcal{P}^2$ satisfying

$$|px - qy| \leqslant \frac{P}{10} \quad \text{and} \quad \|q\alpha_x - p\alpha_y\| \leqslant \frac{P}{H}.$$

Here $\mathcal{P}$ is the set of primes in $[P, 2P]$, for some $P = H^c$.

2. Combinatorial analysis

Let $Y = X/H$ and $K = H/P$.

Definition

A configuration with concentration δ is a pair

$$\mathcal{A} = (A, (\alpha_x)_{x \in A})$$

where $A \subset [Y, 2Y]$ set of integers and $\alpha_x \in \mathbb{R}$ (the frequencies),

2. Combinatorial analysis

Let $Y = X/H$ and $K = H/P$.

Definition

A configuration with concentration δ is a pair

$$\mathcal{A} = (A, (\alpha_x)_{x \in A})$$

where $A \subset [Y, 2Y]$ set of integers and $\alpha_x \in \mathbb{R}$ (the frequencies), such that there are

$$\geqslant \delta |A| |\mathcal{P}|^2$$

quadruples $(x, y, p, q) \in A^2 \times \mathcal{P}^2$ satisfying

$$|px - qy| \leqslant \frac{P}{10} \quad \text{and} \quad \|q\alpha_x - p\alpha_y\| \leqslant \frac{1}{K}.$$

Example 1

Suppose

$$\alpha_x \approx \frac{T}{x} \pmod 1$$

for all $x \in A$, where T is constant.

Then, whenever $|px - qy| \leqslant \frac{P}{10}$, we have

$$\|q\alpha_x - p\alpha_y\| \approx \left\| \frac{T(px - qy)}{xy} \right\| \ll \frac{TP}{Y^2}.$$

This gives examples of configurations of concentration $\gg 1$.

Example 1

Suppose

$$\alpha_x \approx \frac{T}{x} \pmod 1$$

for all $x \in A$, where T is constant.

Then, whenever $|px - qy| \leqslant \frac{P}{10}$, we have

$$\|q\alpha_x - p\alpha_y\| \approx \left\| \frac{T(px - qy)}{xy} \right\| \ll \frac{TP}{Y^2}.$$

This gives examples of configurations of concentration $\gg 1$.

Goal: global formula

Show that the only configurations with size $|A| \gg Y$ and concentration $\gg 1$ are given by Example 1 (and slight variants).

3. Reduction to the Matomäki-Radziwiłł theorem

Suppose that

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e\left(n \frac{T}{x}\right) \right| \gg HX.$$

Suppose that

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e\left(n \frac{T}{x}\right) \right| \gg HX.$$

By a simple Taylor expansion, this implies

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H'} \lambda(n) n^{2\pi i T} \right| \gg H' X$$

for some H' slightly smaller than H.

Suppose that

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H} \lambda(n) e\left(n \frac{T}{x}\right) \right| \gg HX.$$

By a simple Taylor expansion, this implies

$$\sum_{X \leqslant x \leqslant 2X} \left| \sum_{x \leqslant n \leqslant x+H'} \lambda(n) n^{2\pi i T} \right| \gg H'X$$

for some H' slightly smaller than H.

But this is impossible, by the Matomäki-Radziwiłł theorem.

Heart of the proof: combinatorial analysis

Every α_x is, on average, related to $\asymp |\mathcal{P}|^2$ other frequencies α_y.

Every α_x is, on average, related to $\asymp |\mathcal{P}|^2$ other frequencies α_y.

In order to relate α_x to most frequencies, need an iterative argument with

$$\asymp \frac{\log Y}{\log P}$$

steps. We call these steps lifts (we will not define them).

Every α_x is, on average, related to $\asymp |\mathcal{P}|^2$ other frequencies α_y.

In order to relate α_x to most frequencies, need an iterative argument with

$$\asymp \frac{\log Y}{\log P}$$

steps. We call these steps lifts (we will not define them).

Difficulty

If every step loses a **constant factor**, then total loss is $\approx e^{\frac{\log Y}{\log P}}$.

We can only afford to lose a factor P^c, which forces

$$P \geqslant \exp((\log Y)^{1/2+o(1)}).$$

Walsh's iterations

Walsh proved the following dichotomy.

Key structure theorem (Walsh 2023)

Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \gg Y$ and concentration $\delta \gg 1$. Then:

Walsh's iterations

Walsh proved the following dichotomy.

Key structure theorem (Walsh 2023)

Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \gg Y$ and concentration $\delta \gg 1$. Then:

- either $\mathcal{A}$ has a lift with almost **no loss**,

Walsh's iterations

Walsh proved the following dichotomy.

Key structure theorem (Walsh 2023)

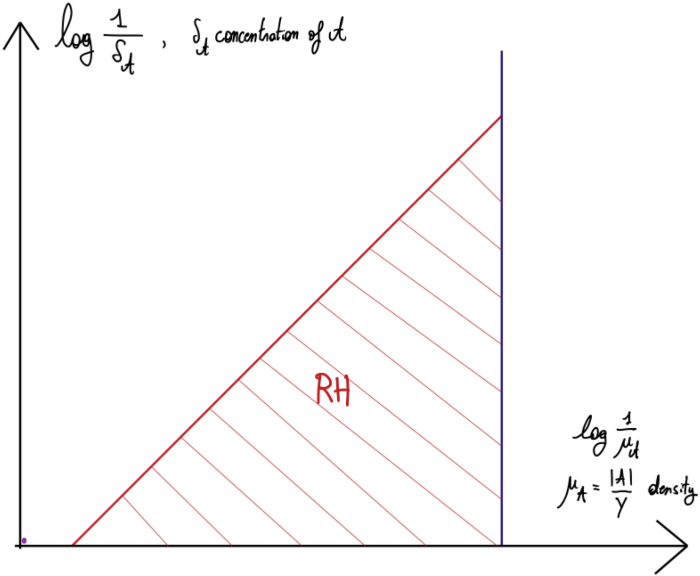
Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \gg Y$ and concentration $\delta \gg 1$. Then:

- either $\mathcal{A}$ has a lift with almost **no loss**,
- or there is a subset $A' \subset A$ of size $|A'| \geqslant |A| / \log Y$ such that the configuration $(A', (\alpha_x)_{x \in A'})$ has concentration

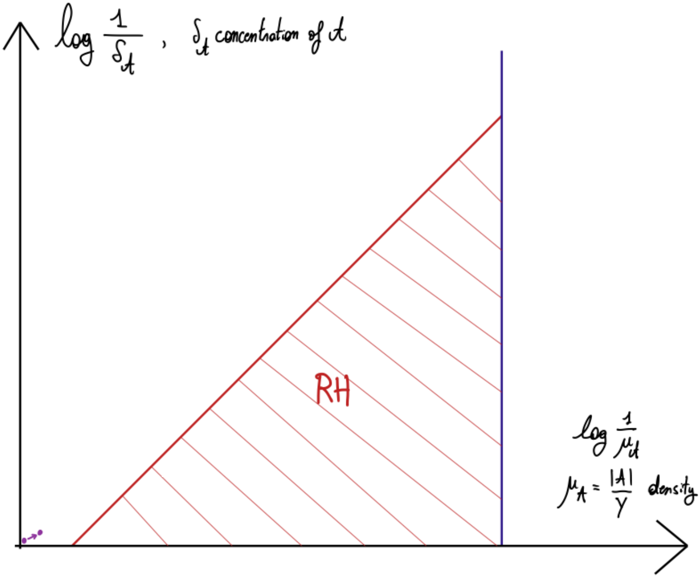
$$\geqslant \delta \left(\frac{|A'|}{|A|} \right)^{1/2}.$$

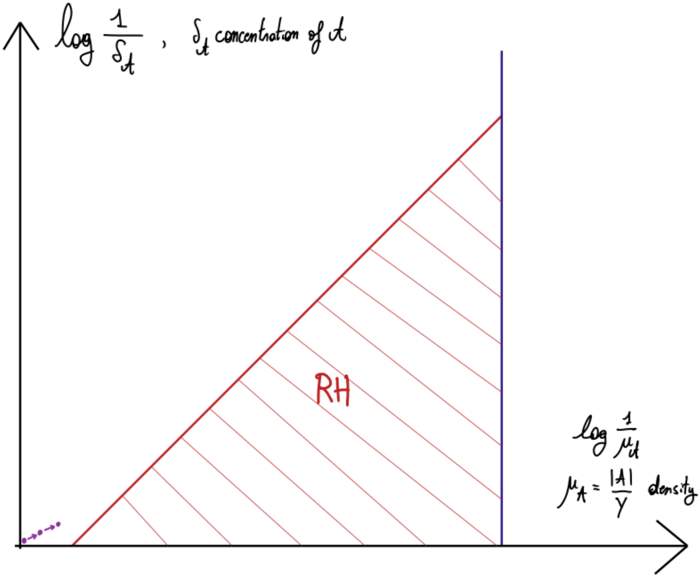
Assuming the Riemann Hypothesis

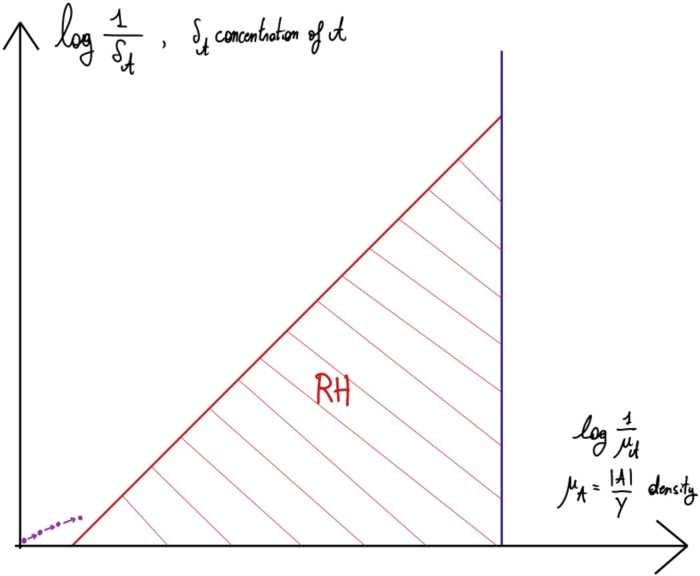
Let $P \geqslant (\log Y)^{10}$. Then, any configuration $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ with density $\mu_A := \frac{|A|}{Y} \geqslant P^{-c}$ has concentration $\delta \ll \mu_A$.



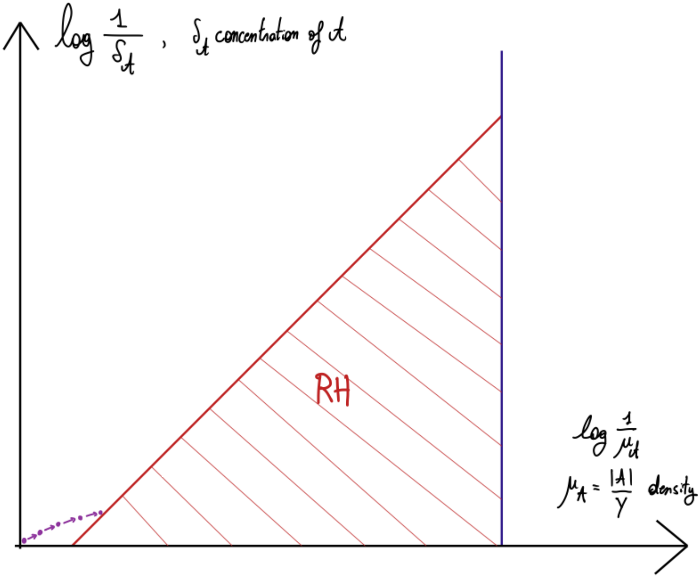
log 1/δ_A , δ_A concentration of A
RH
log 1/μ_A
μ_A = |A|/Y density


log 1/δ_A , δ_A concentration of A
RH
log 1/μ_A
μ_A = |A|/Y density


log 1/δ_A , δ_A concentration of A
RH
log 1/μ_A
μ_A = |A|/Y density



log 1/δ_A , δ_A concentration of A
RH
log 1/μ_A
μ_A = |A|/Y density



log 1/δ_A , δ_A concentration of A
RH
log 1/μ_A
μ_A = |A|/Y density

New work

Relative structure theorem (P. 2025+)

Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \geqslant P^{-c} Y$ and concentration $\delta \gg 1$. Then:

New work

Relative structure theorem (P. 2025+)

Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \geqslant P^{-c} Y$ and concentration $\delta \gg 1$. Then:

- either $\mathcal{A}$ has a lift with almost **no loss**,

New work

Relative structure theorem (P. 2025+)

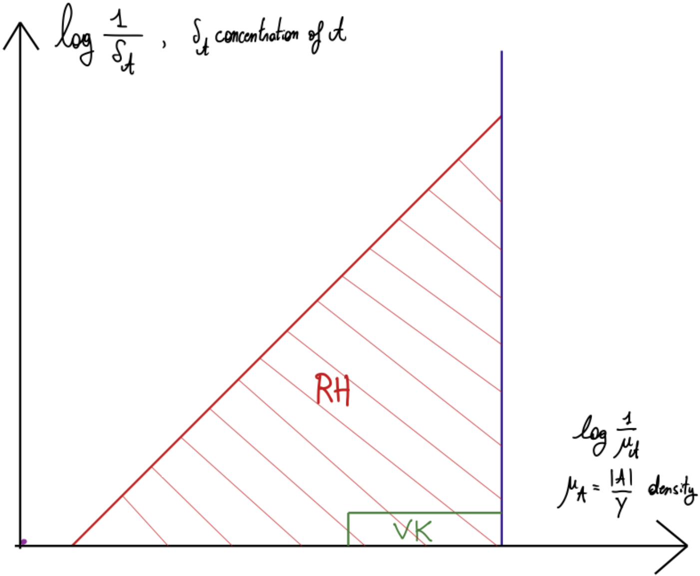
Let $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ be a configuration with $|A| \geqslant P^{-c} Y$ and concentration $\delta \gg 1$. Then:

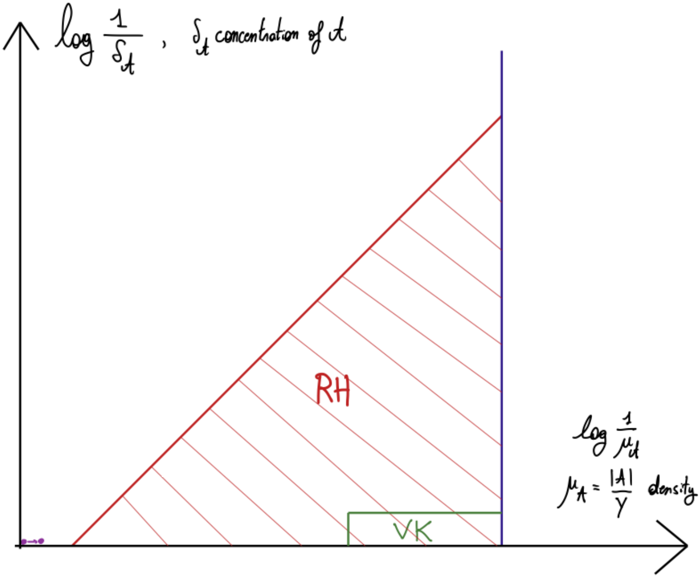
- either $\mathcal{A}$ has a lift with almost **no loss**,
- or there is a subset $A' \subset A$ with $|A|/\log Y \leqslant |A'| \leqslant |A|/2$ such that the configuration $(A', (\alpha_x)_{x \in A'})$ has concentration

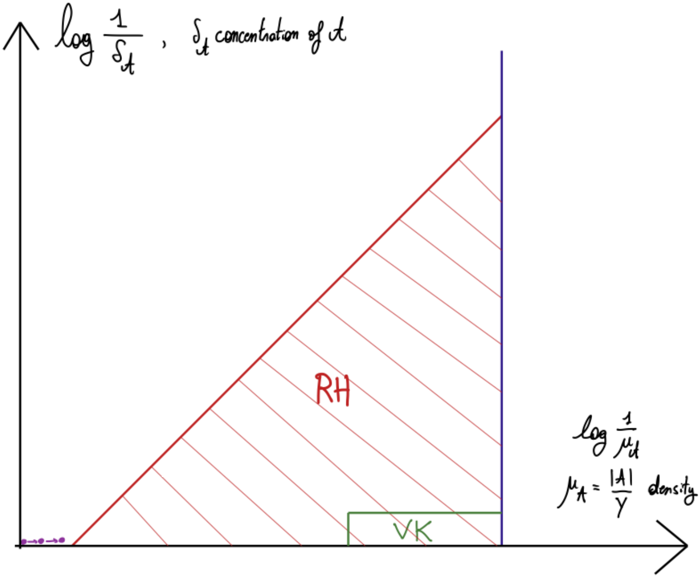
$$\geqslant \delta - \frac{1}{(\log P)^{1-o(1)}}.$$

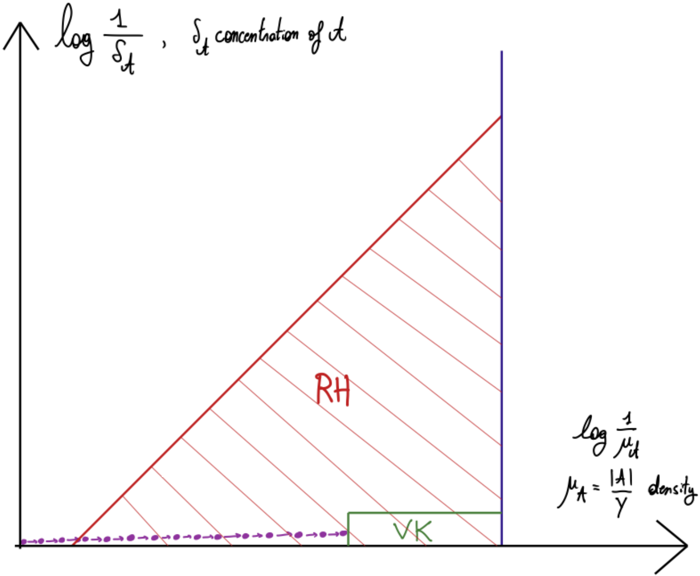
Unconditionally (Vinogradov-Korobov)

For $P = \exp((\log Y)^\theta)$, any configuration $\mathcal{A} = (A, (\alpha_x)_{x \in A})$ with concentration $\delta \gg 1$ has density $\mu_A := \frac{|A|}{Y} \geqslant \exp\left((\log Y)^{1-\frac{3\theta}{2}+o(1)}\right)$.


log 1/δ_A , δ_A concentration of A
RH
VK
log 1/μ_A
μ_A = |A|/Y density


log 1/δ_A , δ_A concentration of A
RH
VK
log 1/μ_A
μ_A = |A|/Y density


log 1/δ_A , δ_A concentration of A
RH
VK
log 1/μ_A
μ_A = |A|/Y density


log 1/δ_A , δ_A concentration of A
RH
VK
log 1/μ_A
μ_A = |A|/Y density

Open problems

Open problem 1

Let $\theta = \frac{2}{5} - \frac{1}{1000}$.

Let $P := \exp((\log Y)^\theta)$ and let $\mathcal{P}$ be the set of primes in $[P, 2P]$.

Let $A \subset [Y, 2Y] \cap \mathbb{N}$ be such that

$$N(A) := \left| \left\{ (x, y, p, q) \in A^2 \times \mathcal{P}^2 \, : \, |px - qy| \leqslant \tfrac{1}{10} P \right\} \right| \gg |A||\mathcal{P}|^2.$$

Prove that $|A| \gg P^{-0.0001} Y$.

Open problem 1 (implies Open problem 2)

Let $\theta = \frac{2}{5} - \frac{1}{1000}$.

Let $P := \exp((\log Y)^\theta)$ and let $\mathcal{P}$ be the set of primes in $[P, 2P]$.

Let $A \subset [Y, 2Y] \cap \mathbb{N}$ be such that

$$N(A) := \left| \left\{ (x, y, p, q) \in A^2 \times \mathcal{P}^2 \ : \ |px - qy| \leqslant \tfrac{1}{10} P \right\} \right| \gg |A| |\mathcal{P}|^2.$$

Prove that $|A| \gg P^{-0.0001} Y$.

Open problem 2

Let $\theta = \frac{2}{5} - \frac{1}{1000}$. Let $P := \exp((\log Y)^\theta)$ and $\mathcal{P} \subset [P, 2P]$ as above.

Let $A_1, \ldots, A_L$ be a partition of $[Y, 2Y] \cap \mathbb{N}$ with each $|A_i| \asymp Y/L$. Suppose that

$$\sum_{i=1}^{L} N(A_i) \gg Y |\mathcal{P}|^2.$$

Prove that $L \ll P^{0.0001}$.

Thank you!