Recent progress on some irrationality questions of Erdős

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CIRM: Prime numbers and arithmetic randomness

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Erdős and irrationality

- Paul Erdős was interested in problems in many different branches of mathematics
- At sundry times and seasons, he thought about various irrationality questions
- He was not, in the main, interested in the central questions in irrationality
- Erdős: *We will discuss here only special series which do not connect up with the general theory at all but which seem attractive to us and where often clever special methods are needed which usually are not available in general.*

Joseph Fourier's 1815 proof that e is irrational

- Assume $e = \frac{a}{b} = \sum_{n\geq 0} \frac{1}{n!}$
- For any $N \in \mathbb{N}$ with $N \geq b$, have $A := N!\left(e - \sum_{n=0}^{N} \frac{1}{n!}\right)$ is an integer
- Note $A = N! \sum_{n=N+1}^{\infty} \frac{1}{n!} = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots \leq \frac{1}{N}$
- If $N \geq 2$, say, then $A \in \mathbb{Z}$ satisfies $0 < A < 1$, contradiction!

General principles from the proof

1. $A \in \mathbb{R}$ given by infinite series, expect $A \notin \mathbb{Q}$
2. If $A \in \mathbb{Q}$, can obtain an integrality condition (rigidity)
3. If series for A converges quickly, don't have to consider too many terms
4. Show terms of A are too "flexible" to always satisfy rigid integrality condition

Conjecture on irrationality of sum-of-divisor-powers series

Conjecture (Erdős and Kac, 1953)

For positive integers k, n define $\sigma_k(n) = \sum_{d|n} d^k$. Then

$$\alpha_k := \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational for every $k \in \mathbb{N}$.

- Exercise: Show $\alpha_1, \alpha_2 \notin \mathbb{Q}$
- α_3 independently shown, using sieve methods, to be irrational by Schlage-Puchta and Friedlander–Luca–Stoiciu ($\sim$ 2006)
- Also showed α_k is irrational for every k assuming Schinzel's Hypothesis H, or appropriate version of prime k-tuples conjecture

Irrationality of α_4

Theorem (P, 2023)

The number

$$\alpha_4 = \sum_{n=1}^{\infty} \frac{\sigma_4(n)}{n!} = 42.30104\ldots$$

is irrational.

- Proof relies on sieve theory, as in previous works, but more complicated arguments
- Substantial new ideas seem needed for $k \geq 5$

Ideas in the proof, I

- Assume $\alpha_4 = a/b$. For large prime $p \sim x$, $A = (p-1)! \sum_{n=p}^{\infty} \frac{\sigma_4(n)}{n!} = (p-1)! \left(\alpha_4 - \sum_{n=1}^{p-1} \frac{\sigma_4(n)}{n!} \right)$ is an integer.
- $\sigma_4(n) \ll n^4$
- Hence

$$\frac{\sigma_4(p)}{p} + \frac{\sigma_4(p+1)}{p(p+1)} + \frac{\sigma_4(p+2)}{p(p+1)(p+2)} + \frac{\sigma_4(p+3)}{p(p+1)(p+2)(p+3)}$$

  is within distance $\ll x^{-1}$ of an integer.
- This is the rigid "integrality" condition.

Ideas in the proof, II

- For $\theta \in \mathbb{R}$, write $\|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n|$
- $\|\frac{\sigma_4(p)}{p} + \frac{\sigma_4(p+1)}{p(p+1)} + \frac{\sigma_4(p+2)}{p(p+1)(p+2)} + \frac{\sigma_4(p+3)}{p(p+1)(p+2)(p+3)}\| \ll x^{-1}$
- $\sigma_4(p)/p = p^3 + p^{-1} = p^3 + O(x^{-1})$
- Impose condition: $\frac{p+3}{2}$ has no prime factors $\leq (\log x)^{100}$
- More severe condition: $p + 2$ has no prime factors $\leq x^{1/4+\epsilon}$
- Hence, for all such prime $p \sim x$, have $\|\frac{\sigma_4(p+1)}{p(p+1)} + \frac{1}{16}\| \leq (\log x)^{-100}$

Ideas in the proof, III

- $\|\frac{\sigma_4(p+1)}{p(p+1)} + \frac{1}{16}\| \leq (\log x)^{-100}$
- Most $p \sim x$ have factorization $p + 1 = \ell q$, where $q \approx x^{\epsilon}$ is a prime
- $\frac{\sigma_4(p+1)}{p(p+1)} \approx \frac{\sigma_4(p+1)}{(p+1)^2} = \frac{\sigma_4(\ell)\sigma_4(q)}{\ell^2 q^2} = \frac{\sigma_4(\ell)}{\ell^2}(q^2 + q^{-2})$
- By positivity arguments and Fourier analysis, can reduce to bounding exponential sums $\sum_{n \sim Q} e(A(n^2 + n^{-2}))$ for some A much larger than Q; here n runs over integers
- Conclusion: one can find primes $p \sim x$ for which $\|\frac{\sigma_4(p+1)}{p(p+1)} + \frac{1}{16}\| \leq (\log x)^{-100}$ fails, contradiction!

E-functions

- Conjecture of Erdős–Kac about α_k reminiscent of Siegel E-functions
- E-function: $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$, where a_n are algebraic integers, say, that do not grow too quickly
- If f satisfies suitable differential equation, results of Siegel, Shidlovskii, many others gives relatively satisfactory transcendence theory
- No such differential structure for $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!} z^n$
- Still, expect $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!} \gamma^n$ is transcendental for every $k \in \mathbb{N}$ and nonzero algebraic γ
- Can "arithmetic" structure overcome lack of differential structure?

Erdős irrationality result for $\tau(n)$

Theorem (Erdős, 1948)

Let $\tau(n)$ denote the divisor function. Then $s := \sum_{n=1}^{\infty} \frac{\tau(n)}{2^n}$ is irrational.

- Assume $s = a/b$. For large $N \sim x$,
  $A = b\sum_{k\geq 1} \frac{\tau(N+k)}{2^k} = b2^N \left(s - \sum_{n=1}^{N} \frac{\tau(n)}{2^n}\right)$ is an integer.
- For $K \approx (\log x)^{1/10}, L \approx 100 \log x$, split sum into three parts: $k \leq K, K < k \leq L, k > L$
- Choose N such that $\tau(N+k) \equiv 0 \pmod{2^k}$ for $k \leq K$, and $\tau(N+k) \leq (\log x)^{100}$ for $K < k \leq L$
- Sum over $k \leq K$ is an integer, and the sum over $k > K$ is too small

Erdős conjectures for other arithmetic functions

- Let $\omega(n), \varphi(n), \sigma(n) = \sigma_1(n)$ be number of distinct prime divisors of n, Euler totient of n, and sum of divisors of n

Conjecture (Erdős, 1948)

The following numbers are all irrational:

$$\sum_{n=1}^{\infty} \frac{\omega(n)}{2^n}, \quad \sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n}, \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{2^n}.$$

- Try to follow outline for $\tau(n)$: since $\varphi(n), \sigma(n) \approx n$, need strong control on many initial terms in the sum
- $\omega(n) \approx \log\log n$ is small on average, but need "global" information about prime factors of n, rather than only "local" information on prime factors as with $\tau(N+k)$

Nesterenko's theorem

- Problem with $\sum_{n=1}^{\infty} \frac{\sigma(n)}{2^n}$ is solved

Theorem (Nesterenko, 1996)

Let γ be an algebraic number with $0 < |\gamma| < 1$. Then $\sum_{n=1}^{\infty} \sigma(n)\gamma^n$ is transcendental.

- Weight two Eisenstein series $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) z^n$ is a quasimodular form
- Ring $\mathbb{Q}[E_2(z), E_4(z), E_6(z)]$ is closed under action of differential operator $z\frac{d}{dz}$
- Construct auxiliary polynomial $P(z, E_2(z), E_4(z), E_6(z))$ with large order of vanishing at $z = 0$
- Key input is deep "zero estimate" on how many zeros such a polynomial can have

Conditional irrationality for $\omega(n)$

Theorem (P, 2025)

Assume a suitable version of the prime k-tuples conjecture. Then the number

$$\beta_t := \sum_{n=1}^{\infty} \frac{\omega(n)}{t^n}$$

is irrational for every integer $t \geq 2$.

- For $1 \leq i \leq k$, let $L_i(n) = a_i n + b_i$ be linear form with $a_i, b_i \in \mathbb{N}$
- Need that if $\{L_1, \ldots, L_k\}$ is "admissible," then we have expected asymptotic formula for number of $n \leq x$ with $L_i(n)$ all prime, $1 \leq i \leq k$
- Need some uniformity, so that k, a_i, b_i can grow with x

Ideas in the proof, I

- $\beta = \beta_t = \sum_{n=1}^{\infty} \frac{\omega(n)}{t^n}$
- Assume $\beta = a/b$. For any $N \in \mathbb{N}$,

$$A = b \sum_{k \geq 1} \frac{\omega(N+k)}{t^k} = bt^N \left(\beta - \sum_{n \leq N} \frac{\omega(n)}{t^n} \right)$$

  is an integer
- We choose $N \sim x$. Set $K \approx 5 \log\log\log x$, $L \approx 2 \log\log x$. Choose N with $k \mid N$ for all $k \leq K$
- Split sum over k into three parts: $k \leq K, K < k \leq L, k > L$
- Contribution from $k > L$ is trivially $O(\frac{\log x}{t^L})$

Ideas in the proof, II

- Write sum over $k \leq K$ as

$$= b \sum_{k \leq K} \frac{\omega(N+k)}{t^k} = b \sum_{k \leq K} \frac{\omega(k(\frac{N}{k}+1))}{t^k}$$

$$= b \sum_{k \leq K} \frac{\omega(k)}{t^k} + b \sum_{k \leq K} \frac{\omega(\frac{N}{k}+1)}{t^k}$$

$$= a + O\left(\frac{\log K}{t^K}\right) + b \sum_{k \leq K} \frac{\omega(\frac{N}{k}+1)}{t^k}$$

- Use prime k-tuples conjecture to find $N \sim x$ with $\frac{N}{k} + 1$ prime for all $k \leq K$
- Sum over $k \leq K$ is then $a + \frac{b}{t-1} + O\left(\frac{\log K}{t^K}\right)$

Ideas in the proof, III

- $b\sum_{k\le L}\frac{\omega(N+k)}{t^k}$ is close to integer
- Sum over $k \le K$ is $a + \frac{b}{t-1} + o(1)$
- For sum over $K < k \le L$: find $N \sim x$ with $\omega(N+k) \le (\log\log x)^2$, say, for $K < k \le L$
- Also want $\omega(N+K+1) > \frac{1}{10}\log\log x$, say
- $b\sum_{K<k\le L}\frac{\omega(N+k)}{t^k} = o(1)$, but also not too small (by lower bound on $\omega(N+K+1)$)
- Some careful analysis gives existence of integer strictly between 0 and 1, contradiction!

Ideas in the proof, IV

- Need a choice of $N \sim x$ such that $\frac{N}{k} + 1$ is prime for all $k \leq K$ and $\omega(N + K + 1) > \frac{1}{10} \log \log x$
- By prime k-tuples conjecture, number of $N \sim x$ with $\frac{N}{k} + 1$ prime for all $k \leq K$ is $\approx x(\log x)^{-K}$
- Want to show

$$\sum_{\substack{N \sim x \\ N/k+1 \text{ is prime}, \forall k \leq K \\ \omega(N+K+1) \leq \frac{1}{10} \log \log x}} 1 \lessapprox \frac{x}{(\log x)^K} \cdot \frac{1}{(\log x)^c},$$

for some constant $c > 0$; can do this unconditionally using sieve theory

G-functions

- Erdős problems with $\sum_{n\geq 1} a(n)t^{-n}$ are reminiscent of Siegel G-functions
- G-function: $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where a_n are algebraic integers, say, with $|a_n| \leq c^n$
- If $f(z)$ satisfies suitable differential equation, then work of Bombieri, Chudnovskys, others gives some irrationality results
- Currently, no transcendence results for G-functions via Siegel's theory
- No reason to expect this differential structure for most arithmetic functions $a(n)$
- Can "arithmetic" structure overcome lack of differential structure?

Other Erdős questions: Are the following irrational?

$$\sum_{n \geq 1} \frac{\varphi(n)}{2^n},$$

$$\sum_{n \geq 2} \frac{1}{n! - 1},$$

$$\sum_{n \geq 1} \frac{p_n}{2^n}, \qquad (p_n \text{ is n-th prime}),$$

$$\sum_{n \geq 1} \mu^2(n) \frac{n}{2^n},$$

$\vdots$

Problem of a different flavor

- Let $t \geq 2$ an integer, $q \in \mathbb{Q}$ with $q \neq -t^n$ for any $n \in \mathbb{N}$.
- Borwein (1991) showed $\sum_{n=1}^{\infty} \frac{1}{t^n+q}$ is irrational, resolving conjecture of Erdős and Graham (1980)

Conjecture (Stolarsky, Erdős, $\leq$ 1980)

Let (a_n) be an infinite sequence of positive integers such that $\sum_n \frac{1}{a_n}$ converges. There is some $t \in \mathbb{N}$ such that $\sum_n \frac{1}{a_n+t}$ is irrational.

Theorem (Kovač–Tao, 2024)

There is a strictly increasing sequence of positive integers a_n such that $\sum_n \frac{1}{a_n+t}$ converges to a rational number for all $t \in \mathbb{Q} \backslash \{-a_1, -a_2, \ldots\}$.

Thank you for your attention!