Hamiltonian Dysthe equation for hydroelastic waves in a compressed ice sheet

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J. Fluid Mech (2025)

Hydroelastic waves

- Hydroelastic waves or gravity-flexural waves refer to water waves propagating at the surface of an ideal fluid covered by a thin ice sheet.[1] (Waves in polar regions).
- Hydroelasticity problems dealing with the interaction between moving fluids and deformable bodies.
- The ice sheet is assumed to coincide with the fluid surface and bend in unison with it.
- Waves can deform the sea ice, move it vertically and horizontally, and possibly break it up.
- An important problem is the modeling the ice deformations subject to water wave motion.

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[1] Greenhill, A. (1886) *Wave motion in hydrodynamics.*

Modeling: Ice seen a continuous, thin elastic plate of infinite extent[2].

- Ice rigidity: measures its resistance to bending, stretching, or any permanent change in shape under stress.
- Ice compression: measures the ability of ice to withstand compressive forces without deforming permanently.
- Inertia: inertial property of the ice plate (usually small).
- Large literature on linear models, valid for small-amplitude waves and small ice deflections.
- Linear theory is not sufficient to describe intense wave-in-ice events (Marko, 2003, passage of an intense cyclone in the sea of Okhotsk, Iceland, with waves of large amplitude]

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[2] Squire, Hosking, Kerr, Langhorne, Moving loads on ice plates (1996)

Nonlinear models

- Based on Kirchhoff–Love plate theory

Forbes, L. *Surface waves of large amplitude beneath an elastic sheet. High-order series solution*. JFM, 1986

Liu, Mollo-Christensen *Wave propagation in a solid ice pack*, J. Phys. Oceanogr., 1988

Părău, Dias, *Nonlinear effects in the response of a floating ice plate to a moving load.* JFM 2002

- Plotnikov and Toland formulation[3]
  - based on the Cosserat theory of hyperelastic shells.
  - More nonlinear dependence of the bending force exerted by the ice cover on the water surface.
  - Conservative form. Can be framed within the classical Hamiltonian formulation of the water wave problem.

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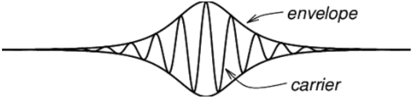
[3] Plotnikov and Toland, *Modelling nonlinear hydroelastic waves*, Phil. Trans. R. Soc. A (2011), 309, 2942–2956

Approximated solutions. Modulational regime

Solutions approximated by slowly modulated monochromatic waves.



$$\eta \sim \varepsilon\, A(X,\tau) e^{i(k_0 x - \omega(k_0) t)} + c.c$$

Here, $\omega(k)$ is the dispersion relation. $X = \varepsilon(x - \omega'(k_0)t), \tau = \varepsilon^2 t$.

Amplitude envelope A is *modulated* according to the 1d NLS equation

- Field observations and direct numerical simulations support the relevance of the modulational regime for wave-ice interactions in the ocean.
- Measurements from the Arctic Ocean have found groupiness is a common trait of the wave field under open-water and ice-covered conditions (Gemmrich, Mudge, Thomson 2021)
- Group structure enhanced when waves interact with sea ice.
  Wave groups may be subject to wave focusing due to modulational instability.
  May produce unusually large amplitudes (Collins et al. 2015).

  Implications for ice breaking and ice decline.

Derivation of NLS from water wave: Zakharov'68, Hasimoto-Ono'72
Dysthe (1979) proposed an NLS equation that includes *higher order* dispersion and nonlinear terms for 2d deep water waves.

$$
\begin{aligned}
i\,\partial_t A &= -\frac{\omega_0''}{2}\partial_x^2 A + |A|^2 A \\
&\quad + \varepsilon\Big(\frac{i\omega_0'''}{6}\partial_x^3 A - \frac{3i}{2}|A|^2\partial_x A - \frac{i}{4}A^2\partial_x \overline{A} + A\,\partial_x\Phi\Big)\,,
\end{aligned}
$$

where the wave-induced mean flow Φ satisfies $\partial_x\Phi = -\frac{1}{2}|D|(|A|^2)$.

- The Dysthe's equation has been extended many other settings: finite depth (Brinch-Nielsen Jonsson 1986), gravity-capillary waves (Hogan 1985), 3d water waves → 2D Dysthe (Gramstad-Truslen (2011).
- It is widely used in water wave community due to its efficiency to describe realistic waves, in particular with relatively large steepness.
- Observations from experiments by Keller (1982) and Su (1982): The slow evolution of deep-water waves exhibits certain asymmetric features: An initially symmetric wave packet of sufficiently large wave slope tends to lean forward. Beyond applicability of cubic NLS but captured by Dysthe. [4]
  [Lo-Mei, JFM 1982]

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[4] Lo, Mei, *A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation,* JFM (1985)

- Unlike NLS, the classical Dysthe equation not known to be Hamiltonian.
- From a modeling and numerical point of view, it is important that structural properties such as energy conservation are inherited by the asymptotic model.
- Goal: Derive a Dysthe equation that preserves the Hamiltonian character of the original water wave problem. (Craig-Guyenne-S. '21, Guyenne-Kairzhan-S. '22, '23, '25).
- The starting point is the Hamiltonian formulation of Euler equations; the methodology is based on a sequence of canonical transformations. Introduction of the modulational Ansatz. By construction, the resulting Dysthe equation is Hamiltonian.

For example, Hamiltonian Dysthe derived from surface gravity waves in deep water:

$$\begin{aligned} i\,\partial_\tau u &= -\frac{1}{2}\omega_0''\partial_X^2 u + k_0^3|u|^2 u \\ &\quad + \varepsilon\left(\frac{i}{6}\omega_0'''\partial_X^3 u - 3ik_0^2|u|^2\partial_X u - k_0^2 u|D_X||u|^2\right). \end{aligned}$$

- Wellposedness results

  - LWP in $H^3(\mathbb{R}^2)$ (Chihara, 2004), in $H^{3/2}(\mathbb{R}^2)$ (Koch-Saut 2007-large class of equations)

  - LWP in $H^1(\mathbb{R}^2)$, ill-posedness in $H^s(\mathbb{R}^2)$, $s < 0$ in the sense that the initial data-to-solution map, from $H^s(\mathbb{R}^2)$ to $C([0,T], H^s(\mathbb{R}^2))$ is not C^3. (Grande-Kurianski-Staffilani 2021)

  - GWP and scattering for small data in $L^2(\mathbb{R}^2)$. Sharp result in view of previous result. (Mosincat-Pilot-Saut 2021)

Derivation of Dysthe equation for hydroelastic water waves

- Hamiltonian formulation for nonlinear potential flow.
  - Nonlinear representation of the ice cover based on Cosserat theory (Plotnikov-Toland).
- Derivation of a Hamiltonian Dysthe equation for the wave envelope.
  - Method of Birkhoff normal form transformations.
  - Resonant cubic terms due to the more complicated dispersion relation.

- Linear analysis of the modulational stability of Stokes waves
  - Validation against numerical solutions of the Dysthe equation.
  Competition of ice rigidity/compression affects the focusing-defocusing character of modulational regime.
- Validation of the approximation
  - Comparison with NLS predictions and with direct numerical simulations of the full Euler system.
  - Surface reconstruction: non-perturbative calculation, by inverting the normal form transformation.

The water wave system

Fluid domain

$$S(\eta) := \{(x,y) : \ x \in \mathbb{R}, \ -\infty < y < \eta(x,t)\}.$$

$u(x,y,t)$: velocity of a particle of fluid located at (x,y), at time t.
Classical formulation : Potential flow $u = \nabla\varphi$

- Incompressible: $div\ u = 0$; Irrotational : $Curl\ u = 0$

$$\Delta\varphi = 0 \quad \text{in} \quad S(t)$$

Two *nonlinear* boundary conditions at the free surface $y = \eta(x,t)$

- Kinematic condition:

$$\eta = \partial_y\varphi - (\partial_x\eta)(\partial_x\varphi).$$

- Dynamical condition: *Bernouilli balance of forces*

$$\partial_t\varphi = -g\eta - \frac{1}{2}\left(\partial_x^2\varphi + \partial_y^2\varphi\right) - \mathcal{D}\left(\partial_s^2\kappa + \frac{1}{2}\kappa^3\right) - \mathcal{P}\kappa,$$

$$\kappa = \text{Curvature fluid-ice interface} = \frac{\partial_x^2\eta}{\left(1+(\partial_x\eta)^2\right)^{3/2}}$$

g: acceleration due to gravity;

$\mathcal{D}$: flexural rigidity coefficient; s = arclength along interface.

$\mathcal{P}$: ice compression coefficient.

$$\partial_s^2\kappa + \frac{1}{2}\kappa^3 = \frac{1}{\sqrt{1+(\partial_x\eta)^2}}\partial_x\left(\frac{1}{\sqrt{1+(\partial_x\eta)^2}}\partial_x\left(\frac{\partial_x^2\eta}{\left(1+(\partial_x\eta)^2\right)^{3/2}}\right)\right)$$
$$+\frac{1}{2}\left(\frac{\partial_x^2\eta}{\left(1+(\partial_x\eta)^2\right)^{3/2}}\right)^3.$$

Ice parameters

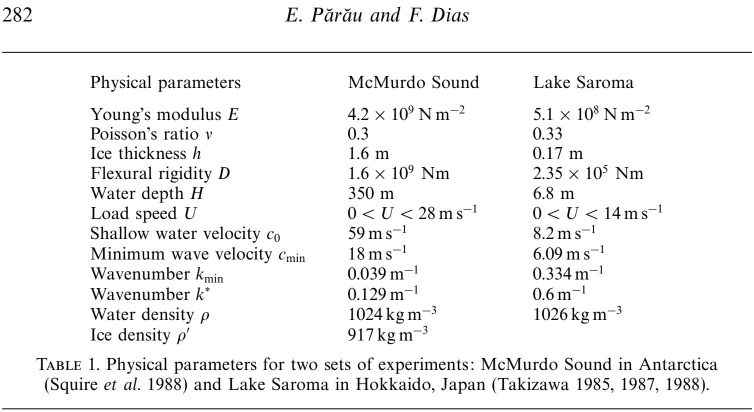
$$\mathcal{D} = \frac{\sigma}{\rho}, \quad \sigma = \frac{Eh^3}{12(1-\nu^2)}, \quad \mathcal{P} = \frac{Ph}{\rho},$$

ρ = density of the fluid; h = thickness of the ice sheet
ν = Poisson's ratio for ice (transverse/longitudinal deformation)
E = Young's modulus (measure of stiffness) , P = compressive stress.

$$\mathcal{P} \rightarrow \frac{\mathcal{P}}{\sqrt{\sigma\rho g}} = \frac{\mathcal{P}}{\sqrt{g\mathcal{D}}}, \quad \mathcal{D} \rightarrow 1, \quad g \rightarrow 1,$$

The new $\mathcal{P}$ measures the relative importance of compression to gravity and rigidity.

In typical physical situations, $\mathcal{P} \sim O(1)$.
Numerical values of these physical parameters are listed in Părău-Dias (2002) for two sets of experimental data.



Physical parameters	McMurdo Sound	Lake Saroma
Young's modulus E	$4.2 \times 10^9\,\mathrm{N\,m^{-2}}$	$5.1 \times 10^8\,\mathrm{N\,m^{-2}}$
Poisson's ratio v	0.3	0.33
Ice thickness h	1.6 m	0.17 m
Flexural rigidity D	1.6×10^9 Nm	2.35×10^5 Nm
Water depth H	350 m	6.8 m
Load speed U	$0 < U < 28\,\mathrm{m\,s^{-1}}$	$0 < U < 14\,\mathrm{m\,s^{-1}}$
Shallow water velocity c_0	$59\,\mathrm{m\,s^{-1}}$	$8.2\,\mathrm{m\,s^{-1}}$
Minimum wave velocity c_{min}	$18\,\mathrm{m\,s^{-1}}$	$6.09\,\mathrm{m\,s^{-1}}$
Wavenumber k_{min}	$0.039\,\mathrm{m^{-1}}$	$0.334\,\mathrm{m^{-1}}$
Wavenumber k^*	$0.129\,\mathrm{m^{-1}}$	$0.6\,\mathrm{m^{-1}}$
Water density ρ	$1024\,\mathrm{kg\,m^{-3}}$	$1026\,\mathrm{kg\,m^{-3}}$
Ice density ρ'	$917\,\mathrm{kg\,m^{-3}}$	

TABLE 1. Physical parameters for two sets of experiments: McMurdo Sound in Antarctica (Squire *et al*. 1988) and Lake Saroma in Hokkaido, Japan (Takizawa 1985, 1987, 1988).

Hamiltonian formulation

The water wave problem can be written as a Hamiltonian system in terms of the surface variables (η, ξ), $\xi(x,t) = \varphi(x, \eta(x,t), t)$ only:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix} = J\nabla H$$

$$\begin{aligned} H(\eta,\xi) = {} & \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)\xi \, dx \\ & + \frac{1}{2}\int_{\mathbb{R}} \left(g\eta^2 + \mathcal{D}\frac{(\partial_x^2\eta)^2}{\left(1+(\partial_x\eta)^2\right)^{5/2}} - 2\mathcal{P}\left(\sqrt{1+(\partial_x\eta)^2} - 1\right)\right) dx. \end{aligned}$$

$G(\eta) : \xi \longmapsto \sqrt{1+(\partial_x\eta)^2}\ (\vec{n}\cdot\nabla\varphi)\big|_{y=\eta}$ is the Dirichlet-Neumann operator, $\vec{n}$ is the outward normal to the fluid surface.

In Fourier variables, the system preserves its canonical form

$$\partial_t \begin{pmatrix} \hat{\eta}_{-k} \\ \hat{\xi}_{-k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{\hat{\eta}_k} H \\ \partial_{\hat{\xi}_k} H \end{pmatrix},$$

Taylor expansion of the Hamiltonian near equilibrium

From analyticity of $G(\eta)$ in η and its convergent Taylor series expansion near $\eta = 0$:

$$G(\eta) = \sum_{m=0}^{\infty} G^{(m)}(\eta),$$

one gets the expansion of Hamiltonian H,

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \dots,$$

where each term $H^{(m)}$ is homogeneous of degree m in (η, ξ).

$$H^{(2)} = \frac{1}{2}\int \Big(|k||\xi_k|^2 + (g - \mathcal{P}k^2 + \mathcal{D}k^4)|\eta_k|^2\Big)dk,$$

$$H^{(3)} = -\frac{1}{2\sqrt{2\pi}}\int (k_1k_3 + |k_1||k_3|)\xi_1\eta_2\xi_3\delta_{123}dk_{123},$$

and

$$H^{(4)} = -\frac{1}{8\pi}\int |k_1||k_4|(|k_1| + |k_4| - 2|k_3 + k_4|)\xi_1\eta_2\eta_3\xi_4\,\delta_{1234}\,dk_{1234}$$

$$+\frac{1}{4\pi}\int \Big(\frac{5\mathcal{D}}{2}k_1^2k_2^2k_3k_4 + \frac{\mathcal{P}}{4}k_1k_2k_3k_4\Big)\,\eta_1\eta_2\eta_3\eta_4\delta_{1234}\,dk_{1234},$$

Compact notations: $(\eta_j, \xi_j) = (\eta_{k_j}, \xi_{k_j})$, $dk_{1\dots n} = dk_1 \dots dk_n$,
$\delta_{1\dots n} = \delta(k_1 + \dots + k_n)$, where $\delta(\cdot)$ is the Dirac distribution.

Dispersion relation

Linearized system around $(\eta, \xi) = (0, 0)$:

$$\partial_t \eta = |D|\xi,$$
$$\partial_t \xi = -g\eta - \mathcal{P}\partial_x^2 \eta - \mathcal{D}\partial_x^4 \eta.$$

For $\omega_k^2 = |k| \left(g - \mathcal{P}k^2 + \mathcal{D}k^4\right)$: (dispersion relation)

Periodic plane-wave solutions

$$\eta(x,t) \propto e^{i(kx-\omega_k t)} \quad \text{and} \quad \xi(x,t) \propto e^{i(kx-\omega_k t)}.$$

Depending on k and choice of parameters, $\omega_k^2 > 0$ or $\omega_k^2 < 0$.

When $\omega_k^2 > 0$, the solutions are travelling waves.

When $\omega_k^2 < 0$, the solutions are either evanescent or growing waves.

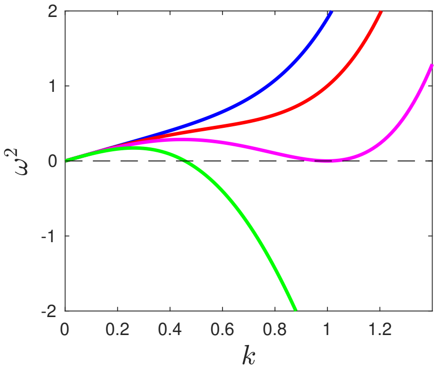


Figure: Linear dispersion relation $\omega^2(k)$ as a function of k for $\mathcal{P} = 0.1$, $\mathcal{P} = 1$, $\mathcal{P} = 2$, $\mathcal{P} = 5$.

(In dimensionless variables, $\mathcal{P} \to \frac{\mathcal{P}}{\sqrt{g\mathcal{D}}}, \quad \mathcal{D} \to 1, \quad g \to 1.$)

Complex symplectic coordinates:

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_k & i\,a_k^{-1} \\ a_k & -i\,a_k^{-1} \end{pmatrix} \begin{pmatrix} \eta_k \\ \xi_k \end{pmatrix},$$

where $a_k^2 := \omega_k / |k|$. The quadratic term $H^{(2)}$ becomes

$$H^{(2)} = \int \omega_k |z_k|^2 dk.$$

Then,

$$H = H(z_k, \bar{z}_{-k}) = H^{(2)} + H^{(3)} + H^{(4)} + \ldots,$$

where each term $H^{(m)}$ is homogeneous of degree m in the $(z, \bar{z})$

Resonant triads

Existence of nonzero *resonant triads* (k_1, k_2, k_3) satisfying

$$\begin{cases} k_1 + k_2 + k_3 = 0, \\ \omega_1 \pm \omega_2 \pm \omega_3 = 0, \end{cases}$$

Equivalently,

$$\begin{cases} k_1 + k_2 + k_3 = 0, \quad k_j \neq 0, \\ d_{123}(k_1, k_2, k_3) = 0, \end{cases}$$

$$d_{123} = (\omega_1 + \omega_2 + \omega_3)(\omega_1 + \omega_2 - \omega_3)(\omega_1 - \omega_2 + \omega_3)(\omega_1 - \omega_2 - \omega_3).$$

the function d_{123} appears together with the constraint

$$k_1 k_3 + |k_1||k_3| \neq 0.$$

In the case of pure gravity waves ($\mathcal{P} = \mathcal{D} = 0$, $\omega_k = \sqrt{g|k|}$), $d_{123} = -4g^2 k_1 k_3$.

Lemma

Assuming $\operatorname{sgn} k_1 = \operatorname{sgn} k_3$ *and* $k_1 + k_2 + k_3 = 0$*, one has*

$$d_{123} = k_1 k_3 \widetilde{d}(k_1, k_3),$$

$$\begin{aligned}\widetilde{d}(k_1, k_3) := {}& k_1 k_3 (k_1 + k_3)^2 \left(3\mathcal{P} - 5\mathcal{D}(k_1^2 + k_1 k_3 + k_3^2)\right)^2 \\ & - 4(g - \mathcal{P}k_1^2 + \mathcal{D}k_1^4)(g - \mathcal{P}k_3^2 + \mathcal{D}k_3^4).\end{aligned}$$

Symmetry properties: $\widetilde{d}(k_1, k_3) = \widetilde{d}(k_3, k_1)$, $\widetilde{d}(k_1, k_3) = \widetilde{d}(-k_1, -k_3)$.
$k_3 \approx (4g/(25\mathcal{D}k_1))^{1/3}$ when $k_1 \ll 1$.

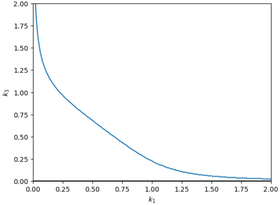


Figure: Resonant curve, $\mathcal{P} = 1$; similar curve for $\mathcal{P} < 2$

Transformation theory and Birkhoff normal forms

- Extend the point of view of transformation theory of finite dimensional Hamiltonian Mechanics to problems with infinitely many degrees of freedom, in particular the use of *Birkhoff normal forms* to eliminate 'non-essential' nonlinearities.
- A Birkhoff normal form is a canonical change of variables up to a given order m, so that the Taylor expansion of the transformed Hamiltonian up to order m contains only resonant terms.

$$H = H^{(2)} + \left(Z^{(3)} + \cdots + Z^{(m)}\right) + R^{(m+1)}$$

- Dyachenko-Zakharov '94, '95; Craig-Worfolk '94 carried this reduction up to $m = 4$ for 2d water waves, infinite depth. The Hamiltonian has no three-wave resonances ($Z^{(3)} = 0$), and $Z^{(4)}$ has a special integrable form.

  Cubic terms and non-resonant quartic terms can be removed by canonical transformations. The remaining resonant quartic terms are expressed in terms of action variables alone.

- The integrability of the water wave system truncated at fourth order, has been at the centre of recent analytic works.

- Long-time existence result for the 2d water wave problem with periodic boundary conditions[Berti, Feola and Pusateri (2020)]. Almost global in time gravity-capillary water waves with constant vorticity [Berti, Maspero, Murgante (2023)].

- Our approach is to use this formalism to derive a high-order NLS equation (Dysthe equation) that inherits the Hamiltonian character of the water wave problem.

We look for a transformation $\tau : w = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \mapsto w'$

defined *in a neighborhood of the origin* such that :

1. The transformation τ is canonical, so the new equations are

$$\partial_t w' = J \, \nabla H'(w') \,, \qquad H'(w') = H(\tau^{-1}(w'))$$

2. The new Hamiltonian is

$$H'(w') = H^{(2)}(w') + \left(Z^{(3)} + \cdots + Z^{(M)}\right) + R^{(M+1)}$$

where each $Z^{(m)}$ retains only resonant terms.

3. The transformation τ is constructed as a Hamiltonian flow ψ from "time" $s = -1$ to "time" $s = 0$, associated to auxiliary Hamiltonian K.

$$\partial_s \psi = J \, \nabla K(\psi) \,, \quad \psi(w')|_{s=0} = w' \,, \quad \psi(w')|_{s=-1} = w \,,$$

This transformation is canonical and preserves the Hamiltonian structure of the system.

- Taylor series expansion of new Hamiltonian H' near $s = 0$

$$H'(w') = H(\psi_s(w'))_{|s=0} - \frac{d}{ds} H(\psi_s(w'))_{|s=0} + \frac{1}{2}\frac{d^2}{ds^2} H(\psi_s(w'))_{|s=0} - \dots$$

$$H'(w') = H(w') - \{K, H\}(w') + \frac{1}{2}\{K, \{K, H\}\}(w') + \dots.$$

where the Poisson bracket is $\{K, H\} = \int (\partial_\eta H \partial_\xi K - \partial_\xi H \partial_\eta K)$

- Return to the expansion of $H = H^{(2)} + H^{(3)} + \dots.$

$$\begin{aligned} H'(w) = H^{(2)}(w) &+ H^{(3)}(w) + \dots \\ &-\{K, H^{(2)}\}(w) - \{K, H^{(3)}\}(w) - \dots. \\ &+ \frac{1}{2}\{K, \{K, H^{(2)}\}\}(w) + \frac{1}{2}\{K, \{K, \mathcal{H}^{(3)}\}\}(w) + \dots \end{aligned}$$

If K is homogeneous of degree m, and $H^{(n)}$ homogeneous of degree n, $\{K, H^{(n)}\}$ is of degree $m + n - 2$.

If we can find $K = K^{(3)}$ homogeneous of degree 3 satisfying

$$H^{(3)} - \{K^{(3)}, H^{(2)}\} = 0,$$

we will have eliminated the (non resonant) cubic terms in the transformed Hamiltonian H'.

Lemma. The coordinates z_j and $\bar{z}_{-j}$ diagonalize the coadjoint operator $\mathrm{coad}_{H^{(2)}} := \{\cdot, \mathcal{H}^{(2)}\}$, where $H^{(2)} = \int \omega_k |z_k|^2 dk$. When applying the operator to $\mathcal{I} := \int z_1 z_2 \bar{z}_{-3} \delta_{123} dk_{123}$, we have

$$\{\mathcal{I}, H^{(2)}\} = i \int (\omega_1 + \omega_2 - \omega_3) z_1 z_2 \bar{z}_{-3} \delta_{123} dk_{123}.$$

- Write $H^{(3)}$ is the form of linear combination of 3rd order monomials in z_{k_i} and $\bar{z}_{-k_j}$.
- Look for $K^{(3)}$ in the form of a linear combination of all possible monomials of degree 3 and identify coefficients.

Third order Birkhoff normal form: Case of deep-water gravity waves

Proposition. *In the 2D, infinite depth case, zero-vorticity, $K^{(3)}$ simplifies to* (Craig-S. 2016)

$$K^{(3)} = -\frac{1}{2\sqrt{2\pi}} \int_{k_1+k_2+k_3=0} \mathrm{sgn}(\mathrm{k}_1)\mathrm{sgn}(\mathrm{k}_2)\eta_{\mathrm{k}_1}\eta_{\mathrm{k}_2}|\mathrm{k}_3|\xi_{\mathrm{k}_3}$$

$$= \int_{-\infty}^{\infty} (-i\mathrm{sgn}(\mathrm{D})\eta)^2 \; |\mathrm{D}|\xi \, \mathrm{dx} \; .$$

The new coordinates are obtained as the solutions at $s = 0$ of the system

$$\frac{d}{ds}\begin{pmatrix}\eta \\ \xi\end{pmatrix} = \begin{pmatrix}0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\eta K^{(3)} \\ \partial_\xi K^{(3)}\end{pmatrix}$$

with the “initial” condition at $s = -1$ being the original variables $(\eta(t), \xi(t))$.

Introduce: $\widetilde{\eta} = -i\mathrm{sgn}(\mathrm{D})\eta, \ \ \widetilde{\xi} = -\mathrm{isgn}(\mathrm{D})\xi$

$$K^{(3)} = \int_{-\infty}^{\infty} (\widetilde{\eta})^2 \, \partial_x \widetilde{\xi} \, dx \, .$$

$$\widetilde{\eta}_s = \partial_{\widetilde{\xi}} K^{(3)} = -\tfrac{1}{2} \partial_x (\widetilde{\eta})^2 \, , \quad \partial_s \widetilde{\xi} = -\partial_{\widetilde{\eta}} K_3 = -\widetilde{\eta} \partial_x \widetilde{\xi}$$

Proposition. *The Hamiltonian system that defines the 3rd order normal form transformation has the form of two coupled PDEs.*

$$\begin{aligned} \widetilde{\eta}_s + \widetilde{\eta} \, \partial_x \widetilde{\eta} = 0 \\ \widetilde{\xi}_s + \widetilde{\eta} \, \partial_x \widetilde{\xi} = 0 \end{aligned}$$

The equation for $\widetilde{\eta}$ *is the Burgers equation.*
The equation for $\widetilde{\xi}$ *is its linearization along Burger's flow.*

Third-order Birkhoff normal form: Case of hydroelastic waves

The presence of resonant triads does not allow us to eliminate $H^{(3)}$ completely.

$$\{K^{(3)}, \mathcal{H}^{(2)}\} = H^{(3)}_{\text{NoRes}},$$

where $H^{(3)}_{\text{NoRes}}$ is the non-resonant part of the third-order Hamiltonian. $H^{(3)}_{\text{NoRes}} := H^{(3)} - H^{(3)}_{\text{Res}}$.

$$H^{(3)}_{\text{Res}} = \frac{1}{8\sqrt{\pi}} \int \chi_\mu(k_1, k_2, k_3) S_{123}(z_1\bar{z}_{-2}z_3 + \bar{z}_{-1}z_2\bar{z}_{-3})\delta_{123}dk_{123},$$

$$S_{123} = S(k_1, k_2, k_3) := (k_1k_3 + |k_1||k_3|)\frac{a_1a_3}{a_2},$$

and χ_μ is a characteristic function :

$$\chi_\mu(k_1,k_2,k_3) := \begin{cases} 1, & \text{if } (k_1,k_2,k_3) \in \text{a nbhd } C_\mu \\ 0, & \text{otherwise,} \end{cases}$$

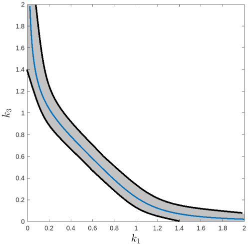


Figure: Blue curve: the (k_1,k_3) values of resonant triads.

Proposition

There exists a unique solution $K^{(3)}$ which, in complex symplectic coordinates, is

$$K^{(3)} = \frac{1}{8i\sqrt{\pi}} \int S_{123} \Big[\frac{z_1 z_2 z_3 - \bar{z}_{-1}\bar{z}_{-2}\bar{z}_{-3}}{\omega_1 + \omega_2 + \omega_3} - 2 \frac{z_1 z_2 \bar{z}_{-3} - \bar{z}_{-1}\bar{z}_{-2} z_3}{\omega_1 + \omega_2 - \omega_3} \\ + (1 - \chi_\mu(k_1, k_2, k_3)) \frac{z_1 \bar{z}_{-2} z_3 - \bar{z}_{-1} z_2 \bar{z}_{-3}}{\omega_1 - \omega_2 + \omega_3} \Big] \delta_{123} dk_{123}.$$

The new Hamiltonian is

$$H(w) = H^{(2)}(w) + H^{(3)}_{\mathrm{Res}}(w) + H^{(4)}_{+}(w) + R^{(5)},$$

where $R^{(5)}$ denotes all terms of order 5 and higher,
$H^{(4)}_{+}$ is the new fourth-order term

$$H^{(4)}_{+} = H^{(4)} - \frac{1}{2}\{K^{(3)}, H^{(3)}_{\mathrm{NoRes}}\} - \{K^{(3)}, H^{(3)}_{\mathrm{Res}}\}.$$

Modulational Ansatz

- Look for solutions in the form of near-monochromatic waves with carrier wavenumber k_0, $k_0 > 0$.
  In the physical space
  $$z(x,t) = \varepsilon\, u(X,t)e^{ik_0 x}, \quad X = \epsilon x$$
- In Fourier space, this corresponds to a narrow band approximation, with z_k and $\bar{z}_k$ localized near k_0.
- Need to choose the thickness of the little tube around the resonant curve. We choose $\mu = \varepsilon$.
- Substitute into truncated Hamiltonian
  $$H = H^{(2)} + H_{+}^{(4)}$$
  and expand in powers of ε.
- Homogenization over short scales naturally selects among all the possible quartic interactions 4-wave resonances for which fast oscillations exactly cancel out.

Dysthe equation

The Hamiltonian system is

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \partial_u H \\ \partial_{\overline{u}} H \end{pmatrix} \implies i\partial_t u = \varepsilon^{-1} \partial_u H,$$

$$H = \varepsilon \int \overline{u}\, \omega(k_0 + \varepsilon D_X)\, u\, dX + \varepsilon^3 \alpha \int |u|^4 dX$$
$$+ \varepsilon^4 \beta \int |u|^2 \mathrm{Im}(\overline{u} \partial_X u) dX - \varepsilon^4 \gamma k_0^2 \int |u|^2 |D_X| |u|^2\, dX + \mathcal{O}(\varepsilon^5).$$

Coefficients α, β, γ involve integrals containing the cut-off function.

For example, expressions of the form ($k_j = k_0 + \varepsilon\lambda_j$),

$$[1 - \chi_\mu(k_1, -k_1 - k_2, k_2)\chi_\mu(k_3, -k_3 - k_4, k_4)]$$

$$[1 - \chi_\mu(k_1, -k_1 - k_2, k_2)\chi_\mu(k_3, -k_3 - k_4, k_4)]$$

and

$$[1 - \chi_\mu(k_3 - k_1, k_1, -k_3)\chi_\mu(k_4 - k_2, -k_4, k_2)]\,(1 + \mathrm{sgn}(\lambda_1 - \lambda_3)).$$

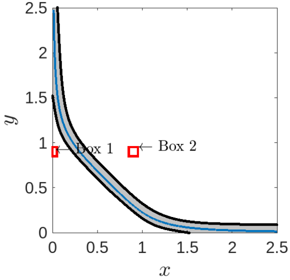
Case $k_0 = 0.9$

We have the following identities:

$$\chi_\mu(k_1, -k_1 - k_2, k_2) = 0,$$

and

$$\chi_\mu(k_4 - k_2, -k_4, k_2) = 0 \quad \text{if} \quad \operatorname{sgn}(k_4 - k_2) = +1 \quad \text{and} \quad k_4 - k_2 \sim \varepsilon.$$



$$\alpha = F_\alpha(k_0) + G_\alpha(k_0), \quad \beta = F_\beta(k_0) + G_\beta(k_0), \quad \gamma = 2.$$

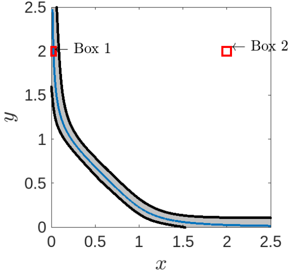
Case $k_0 = 2$.

We have the following identities:

$$\chi_\mu(k_1, -k_1 - k_2, k_2) = 0,$$

and

$$\chi_\mu(k_4 - k_2, -k_4, k_2) = 1 \quad \text{if} \quad \operatorname{sgn}(k_4 - k_2) = +1 \quad \text{and} \quad k_4 - k_2 \sim \varepsilon.$$



$$\alpha = F_\alpha(k_0) + G_\alpha(k_0), \quad \beta = F_\beta(k_0) + G_\beta(k_0), \quad \gamma = 1.$$

Hamiltonian Dysthe equation for two-dimensional hydroelastic waves on deep water ($\tau = \varepsilon^2 t$)

$$
\begin{aligned}
i\,\partial_\tau u &= -\frac{1}{2}\omega_0''\partial_X^2 u + \alpha|u|^2 u \\
&\quad + \frac{i}{6}\varepsilon\omega_0'''\partial_X^3 u - i\varepsilon\beta|u|^2\partial_X u - \varepsilon\frac{\gamma k_0^2}{2}u|D_X||u|^2.
\end{aligned}
$$

- Agrees with the NLS equation obtained by Trichtchenko, Milewski and Părău '19 with $\mathcal{P} = 0$.
- γ depends on the choice of k_0. For example, if $k_0 = 0.9, \gamma = 2$; if $k = 2, \gamma = 1$.
- Natural to expect that, in the limit $\mathcal{D} \to 0$ and $\mathcal{P} \to 0$, the coefficients in the above Dysthe eq. tend to those of the Hamiltonian Dysthe eq. for 2d deep-water pure gravity waves. In this case, no resonant triads and $\gamma = 2$ for all k_0.

$$
\begin{aligned}
i\,\partial_\tau u &= -\frac{1}{2}\omega_0''\partial_X^2 u + k_0^3|u|^2 u \\
&\quad + \varepsilon\Big(\frac{i}{6}\omega_0'''\partial_X^3 u - 3ik_0^2|u|^2\partial_X u - k_0^2 u|D_X||u|^2\Big).
\end{aligned}
$$

Numerical validation of the approximation

a. Reconstruction of the free surface

- Because Dysthe equation only describes *the wave envelope*, another step is required to reconstruct the free surface.
- *In modulation theory*, this reconstruction is usually done *perturbatively*, based on an Ansatz similar to *Stokes' expansion*, by adding contributions from higher harmonics of the wave spectrum.
- Here the reconstruction procedure inverts the transformation associated with the 3rd order normal form that eliminates non-resonant cubic terms in Hamiltonian

Euler system:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix} .$$

Initial conditions $\eta(x,0)$ and $\xi(x,0)$ obtained from initial condition specified for Dysthe equation

$$u(x,0) = B_0(1 + 0.1\cos(\lambda x))$$

Choice of parameters: $B_0 = 0.1; k_0 = 0.9; \varepsilon = 0.09; \lambda = 0.02 (\ll k_0)$

Ice Parameter: $\dfrac{\mathcal{P}}{\sqrt{g\mathcal{D}}} = 1.$

Comparison of surface elevation η predicted from:

- Hamiltonian Dysthe equation
- NLS equation
- Euler equations

Euler equations :

- The computational domain: $0 \leq x < L$ (L large) with periodic boundary conditions
- Discretization in space by a pseudo-spectral method via FFT.
- Dirichlet-Neumann operator approximated by its series expansion (about 6 terms in the expansion)
- Linear terms solved exactly by the integrating factor technique.

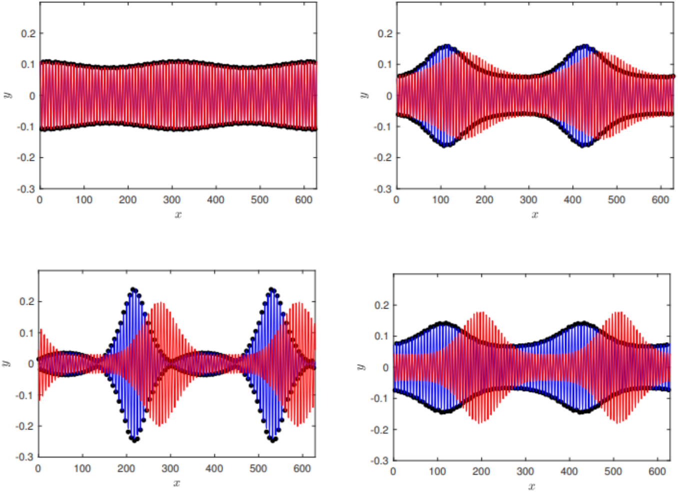


Figure: Times: $t = 0, 4000, 5000, 7000$.
Blue curve: Dysthe equation. Red curve: NLS equation.
Black dots: Euler system.

Thank you !