Continuum Calogero–Moser models

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The CCM models

- Continuum Calogero–Moser (CCM) equations:

$$i\frac{d}{dt}q = -q'' \pm 2iqC_+\left(|q|^2\right)'$$

for $q(t)$ in the Hardy space

$$L_+^2 := \{f \in L^2 : \widehat{f}(\xi) = 0 \text{ for } \xi < 0\}$$

- Cauchy–Szegő projection C_+:

$$\widehat{C_+ f}(\xi) := 1_{[0,\infty)}(\xi)\widehat{f}(\xi)$$

Physical origins

- Focusing CCM: continuum limit of Calogero–Moser particle system
  [Abanov–Bettelheim–Wiegmann '09]

$$x_1(t), \ldots, x_n(t) \quad \text{solve CM:} \qquad \frac{d^2 x_j}{dt^2} = \sum_{k \neq j} \frac{1}{(x_j - x_k)^3}$$

$$\rightsquigarrow \quad \rho(t,x) = \sum \delta(x - x_j(t)), \ \ v(t,x) \quad \text{solve} \quad \partial_t \rho + (\rho v)' = 0$$

$$\rightsquigarrow \quad q(t,x) = \sqrt{\rho} e^{i \int^x (v + \pi\rho)\, dx} \quad \text{solves focusing CCM}$$

- Defocusing CCM: modulation theory for the setting of the Benjamin–Ono equation [Pelinovsky '95]

$$u(t,x) = \epsilon^{-1} q(t,x) e^{i\theta(t,x;\epsilon)} + O(1) \quad \text{solves ILW}$$

$$\overset{\epsilon\to 0}{\Longrightarrow} \qquad q(t,x) \quad \text{solves Intermediate NLS}$$

$$\overset{\text{depth}\to\infty}{\Longrightarrow} \qquad q(t,x) \quad \text{solves defocusing CCM}$$

Conserved quantities

- Mass:

$$M(q) = \int |q|^2 \, dx$$

- Momentum:

$$P(q) = \int -i\overline{q}q' \mp \tfrac{1}{2}|q|^4 \, dx$$

- Hamiltonian:

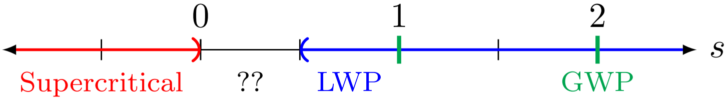
$$H(q) = \frac{1}{2} \int \left| q' \mp iqC_+\left(|q|^2\right) \right|^2 dx$$

$$\vdots$$

- Completely integrable: infinite sequence of conserved quantities

Defocusing case

- Well-posedness in H^s spaces on $\mathbb{R}$ and $\mathbb{T}$: [Gérard–Lenzmann '22]

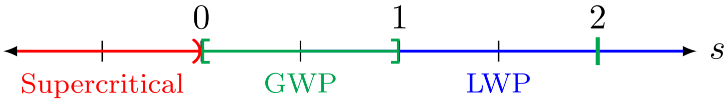


- Scaling symmetry:

$$q(t,x) \quad \mapsto \quad q_\lambda(t,x) = \sqrt{\lambda} q(\lambda^2 t, \lambda x) \quad \text{for } \lambda > 0$$

- Critical H^s regularity is $s = 0$

Theorem 1 (Killip–L.–Vișan '23). Fix $0 \leq s < 1$. The defocusing CCM equation is globally well-posed in $H^s_+(\mathbb{R})$.



- Analogous result on $\mathbb{T}$ already known [Badreddine '23]

Focusing case

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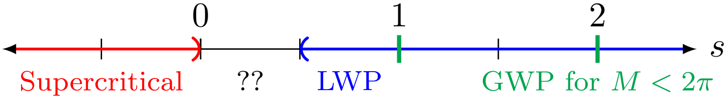


- Solitons:

$$q(t,x) = \sqrt{\lambda} R(\lambda x + x_0), \qquad R(x) = \frac{\sqrt{2}}{x+i}$$

- Mass is $\|q\|_{L^2}^2 = \|R\|_{L^2}^2 = 2\pi$ for all $\lambda > 0$

- N-soliton resembles N interacting solitons

$$q(t,x) = \sum_{j=1}^{N} \frac{a_j(t)}{x - z_j(t)}$$

- The poles $z_j(t) \in \mathbb{C}_-$ solve a complexified CM particle system: [Gérard–Lenzmann '22]

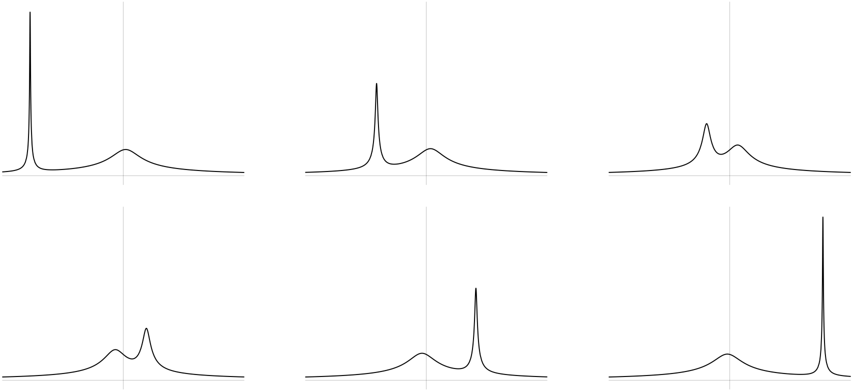
$$\frac{d^2 z_j}{dt^2} = \sum_{k \neq j} \frac{8}{(z_j - z_k)^3}$$

- Mass of an N-soliton is $2\pi N$

- N-solitons with $N \geq 2$ exhibit turbulent behavior:

$$\|q(t)\|_{H^s} \sim |t|^{2s} \quad \text{as } t \to \pm\infty$$

for any $s > 0$ [Gérard–Lenzmann '22]



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  for any $s > 0$ [Gérard–Lenzmann '22]

- Mass of an N-soliton is $2\pi N$

- There exist solutions with mass $2\pi + \epsilon$ that either exhibit similar behavior, or blow up in finite time [Hogan–Kowalski '24]

- There exist smooth solutions with mass $2\pi + \epsilon$ so that $\|q(t)\|_{H^1}$ blows up in finite time [Kim–Kim–Kwon '24]

Theorem 2 (Killip–L.–Vişan '23). Fix $0 \leq s < 1$. The focusing CCM equation is globally well-posed in the space

$$\left\{ q \in H^s_+(\mathbb{R}) : \|q\|^2_{L^2} < 2\pi \right\}.$$

- Analogous result on $\mathbb{T}$ already known [Badreddine '23]
- There exist smooth solutions with mass $2\pi + \epsilon$ so that $\|q(t)\|_{H^1}$ blows up in finite time [Kim–Kim–Kwon '24]

Equicontinuity

- $\mathcal{F} \subseteq L^2_+$ is equicontinuous

$$\iff \quad \sup_{q\in\mathcal{F}} \sup_{|y|<\delta} \|q(\cdot + y) - q(\cdot)\|_{L^2} \to 0 \quad \text{as } \delta \to 0$$

$$\iff \quad \sup_{q\in\mathcal{F}} \int_\kappa^\infty |\widehat{q}(\xi)|^2 \, d\xi \to 0 \quad \text{as } \kappa \to \infty$$

- At critical regularity,

Equicontinuity of orbits $\rightsquigarrow$ No concentration in physical space

$\rightsquigarrow$ No blowup

Theorem 3 (Defocusing case). If $\mathcal{F} \subseteq L^2_+(\mathbb{R})$ is bounded and equicontinuous, then the set of orbits

$$\mathcal{F}^* := \{q(t) : q(0) \in \mathcal{F},\ t \in \mathbb{R}\}$$

for defocusing CCM is also bounded and equicontinuous.

Theorem 4 (Focusing case). If $\mathcal{F} \subseteq L^2_+(\mathbb{R})$ is equicontinuous and

$$\sup_{q\in\mathcal{F}} \|q\|_{L^2}^2 < 2\pi,$$

then the set of orbits $\mathcal{F}^*$ for focusing CCM is bounded and equicontinuous.

- The constant 2π here is sharp

Lax pair

- Lax pair:

$$q(t) \text{ solves CCM} \qquad \Longleftrightarrow \qquad \frac{d}{dt}L = [P, L],$$

$$L := -i\partial \mp qC_+\overline{q}, \qquad P := \dots \qquad \text{on } L^2_+$$

- L is symmetric, P is antisymmetric
- Formally,

$$U(t) \quad \text{solves} \quad \frac{d}{dt}U = PU$$

$$\Longrightarrow \qquad L(t) = U(t)L(0)U(t)^* \quad \text{with } U(t) \text{ unitary}$$

- So spectrum of $L(t)$ is conserved

Proof of equicontinuity

Goal: If $\mathcal{F} \subseteq L^2_+$ is equicontinuous and

$$\sup_{q \in \mathcal{F}} \|q\|^2_{L^2} < \begin{cases} 2\pi & \text{(focusing)} \\ \infty & \text{(defocusing)} \end{cases}$$

then the set of orbits $\mathcal{F}^*$ is also equicontinuous.

- The spectrum of

$$L_q := -i\partial \mp q C_+ \overline{q} \quad \text{on } L^2_+$$

is conserved in time (formally)

Proposition. The quantity

$$\operatorname{tr}(R_q - R_0), \quad \text{where} \quad R_q = (L_q + \kappa)^{-1},$$

is finite for $q \in L^2_+$. Moreover, it is conserved for H^∞_+ solutions.

- Expand as a series in q:

$$\operatorname{tr}(R_q - R_0) = \sum_{\ell \geq 1} (\pm 1)^\ell \operatorname{tr}\left\{ \left(R_0 q C_+ \overline{q} \right)^\ell R_0 \right\}$$

- The leading order term is

$$\operatorname{tr}\left(R_0 q C_+ \overline{q} R_0 \right) = \frac{1}{2\pi} \int_0^\infty \frac{|\widehat{q}(\xi)|^2}{\xi + \kappa} \, d\xi$$

- Build a conserved quantity to estimate the high frequencies:

$$\beta(\kappa, q) := \|q\|_{L^2}^2 \mp 2\pi\kappa \, \mathrm{tr}(R_q - R_0)$$

- Quadratic term is

$$\beta^{[2]} := \|q\|_{L^2}^2 - 2\pi\kappa \, \mathrm{tr}\left(R_0 q C_+ \overline{q} R_0\right) = \int_0^\infty \frac{\xi}{\xi + \kappa} |\widehat{q}(\xi)|^2 \, d\xi$$

- For $\mathcal{F} \subseteq L^2_+$ bounded,

$$\begin{aligned} \mathcal{F} \text{ is equicontinuous} \quad &\iff \quad \sup_{q \in \mathcal{F}} \beta^{[2]}(\kappa, q) \to 0 \quad \text{as } \kappa \to \infty \\ &\implies \quad \sup_{q \in \mathcal{F}} \beta(\kappa, q) \to 0 \quad \text{as } \kappa \to \infty \end{aligned}$$

Defocusing case

$$L_q = -i\partial + qC_+\overline{q}, \qquad qC_+\overline{q} \geq 0$$

$$\implies \quad -\operatorname{tr}\left\{\frac{1}{L_q+\kappa} - \frac{1}{L_0+\kappa}\right\} = \operatorname{tr}\left\{\frac{1}{\sqrt{L_0+\kappa}}\,\frac{qC_+\overline{q}}{L_0+qC_+\overline{q}+\kappa}\,\frac{1}{\sqrt{L_0+\kappa}}\right\}$$

$$\leq \operatorname{tr}\left\{\frac{1}{\sqrt{L_0+\kappa}} \quad \frac{qC_+\overline{q}}{L_0+\kappa} \quad \frac{1}{\sqrt{L_0+\kappa}}\right\}$$

$$\implies \quad \beta(\kappa, q) \geq \beta^{[2]}(\kappa, q)$$

$$\implies \quad \beta^{[2]}(\kappa, q(t)) \leq \beta(\kappa, q(t)) \equiv \beta(\kappa, q(0)) \overset{\kappa\to\infty}{\longrightarrow} 0$$

Focusing case

$$L_q = -i\partial - qC_+\overline{q}, \qquad qC_+\overline{q} \leq \tfrac{M}{2\pi} L_0$$

$$\Longrightarrow \qquad L_q \geq (1 - \tfrac{M}{2\pi}) L_0$$

$$\Longrightarrow \qquad (L_q + \kappa)^{-1} \leq \tfrac{2\pi}{2\pi - M} (L_0 + \kappa)^{-1}$$

$$\Longrightarrow \qquad \ldots$$

Explicit formula

Theorem 5 (Killip–L.–Vișan '23). If $q^0 \in L^2_+$ (and $\|q^0\|^2_{L^2} < 2\pi$ in the focusing case), then the global solution $q(t)$ to CCM satisfies

$$q(t,z) = \tfrac{1}{2\pi i} I_+\big\{\big(X + 2tL_{q^0} - z\big)^{-1} q^0\big\} \quad \forall \operatorname{Im} z > 0.$$

- Here, X is an extension of multiplication-by-x on $L^2_+(\mathbb{R})$
- I_+ is an extension of $q \mapsto \int q\,dx$ on $L^2_+(\mathbb{R})$
- Cubic Szegő equation: [Gérard–Grellier '15, Gérard–Pushnitski '24]
- Benjamin–Ono equation: [Gérard '23]

Proof of well-posedness

Goal: If $\{q_n^0\}_{n\geq 1} \subseteq \mathcal{S}$ converges in L^2_+ and

$$\sup_n \|q_n^0\|_{L^2}^2 < \begin{cases} 2\pi & \text{(focusing)} \\ \infty & \text{(defocusing)} \end{cases}$$

then the solutions $q(t)$ converge in $C_t([-T, T]; L^2_+)$.

Steps:

1. The explicit formulas for q_n^0 converge for fixed t, z
2. The solutions $q_n(t)$ converge weakly to $q(t)$ in L^2_+ for fixed t
3. The solutions $q_n(t)$ are precompact in $C_t([-T, T]; L^2_+)$

1. The explicit formulas for q_n^0 converge for fixed t, z

- The operator $X + 2tL_0 = X - 2it\partial$ on L^2_+ is maximally accretive
- $A_0 := (X + 2tL_0 - z)^{-1}$ is bounded $L^2_+ \to L^2_+$ for $\operatorname{Im} z > 0$
- A_0 is also bounded $L^2_+ \to L^\infty_+$ and $L^1_+ \to L^2_+$, provided $t \neq 0$
- The series

$$(X + 2tL_q - z)^{-1} = \underset{L^2 \leftarrow L^2}{A_0} \pm \underset{L^2 \leftarrow L^1}{A_0} \underset{\longleftarrow}{2tqC_+\overline{q}} \underset{L^\infty \leftarrow L^2}{A_0} + \dots$$

converges for fixed t, z

2. The solutions $q_n(t)$ converge weakly to $q(t)$ in L^2_+ for fixed t

- Fix t, and pass to any subsequence of $\{q_n(t)\}_{n\geq 1}$
- Conservation of mass $\Rightarrow q_n(t) \rightharpoonup \widetilde{q}(t)$ in L^2_+ along a subsequence
- Poisson integral formula: For $\operatorname{Im} z > 0$,

$$q_n(t,z) = \int \frac{\operatorname{Im} z}{\pi|x-z|^2} q_n(t,x)\,dx$$

- So $\widetilde{q}(t,z)$ is given by the explicit formula for all $\operatorname{Im} z > 0$
- This does not depend on the subsequence

3. The solutions $q_n(t)$ are precompact in $C_t([-T, T]; L^2_+)$

- Equicontinuity in space: already have
- Equicontinuity in time:

$$\|q_n(t+h) - q_n(t)\|_{C_t L^2_x} \leq |h| \left\| \tfrac{d}{dt} P_{\leq N} q_n \right\|_{L^\infty_t L^2_x} + 2 \|P_{>N} q_n\|_{C_t L^2_x}$$

- Tightness in space:
  - For fixed $b > 0$, the functions $x \mapsto q_n(t, x + ib)$ are tight
  - Poisson integral formula: $q_n(t, x+ib) = [e^{-b|\partial|} q_n](t, x)$
  - $\|q_n(t)\|_{L^2(|x| \geq R)} \leq \|P_{\leq N} q_n(t)\|_{L^2(|x|\geq R)} + 2 \|P_{>N} q_n(t)\|_{L^2}$

Dispersive decay

- Dispersive estimate for the free Schrödinger flow:

$$\|e^{it\Delta}q\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}} \|q\|_{L^1}$$

Theorem 6 (Killip–L.–Vișan '25+). There exists $\delta > 0$ so that for any initial data $q \in L^2_+(\mathbb{R})$ satisfying

$$\|q\|_{L^2} \leq \delta \quad \text{and} \quad q \in L^1,$$

the corresponding global solution $q(t)$ of CCM satisfies

$$\|q(t)\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}}$$

for all $t \neq 0$.

- Cubic NLS: Dispersive decay for...
  - small q in $H^{1,1}$ [Deift–Zhou '03, Kato–Pusateri '11]
  - small q in $H^{\frac{1}{2}+,\frac{1}{2}+}$ [Hayashi–Naumkin '98]
  - small q in $H^{0,\frac{1}{2}+}$ [Ifrim–Tataru '15]
- Benjamin–Ono: Dispersive decay fails for small q in $L^1 \cap L^2$

Proof sketch

- Explicit formula:

$$q(t,z) = \tfrac{1}{2\pi i} I_+\big\{\big(X + 2tL_q - z\big)^{-1} q\big\} \quad \forall\, \operatorname{Im} z > 0.$$

- Free resolvent:

$$A_0 = (X + 2tL_0 - z)^{-1}$$

$$[(X + 2tL_0)f]\widehat{\ }(\xi) = (i\partial_\xi + 2t\xi)\widehat{f}(\xi) = e^{it\xi^2}\, i\partial_\xi\, e^{-it\xi^2}\widehat{f}(\xi)$$

$$\implies \quad A_0 = e^{-it\Delta}(X - z)^{-1} e^{it\Delta}$$

$$I_+(A_0 f) = I_+\left[e^{-it\Delta}(X - z)^{-1} e^{it\Delta} f\right]$$

$$= I_+\left[(X - z)^{-1} e^{it\Delta} f\right]$$

$$= 2\pi i\left[e^{it\Delta} f\right](z)$$

- Expand as a series in q:

$$q(t, z) = \tfrac{1}{2\pi i}\, I_+\left[(X + 2tL_q - z)^{-1} q\right]$$

$$= \tfrac{1}{2\pi i}\, I_+\, A_0 q \;\pm\; \tfrac{1}{2\pi i}\, I_+\, A_0\, 2tqC_+\overline{q}\, A_0 q \;+\; \ldots$$

$$= \left[e^{it\Delta} q \;\pm\; e^{it\Delta}\, 2tqC_+\overline{q}\, A_0 q \;+\; \ldots\right](z)$$

- $A_0 := (X + 2tL_0 - z)^{-1}$ is bounded $L^2_+ \to L^2_+$ for $\operatorname{Im} z > 0$
- A_0 is also bounded $L^1_+ \to L^\infty_+$ for $t \neq 0$:

$$\|A_0\|_{L^1_+ \to L^\infty_+} \lesssim |t|^{-1}$$

$$q(t,z) = \Big[\underset{L^\infty \leftarrow L^1}{e^{it\Delta}\, q} \pm \underset{L^\infty \leftarrow L^1}{e^{it\Delta}}\, \underset{\longleftarrow}{2tqC_+\overline{q}}\, \underset{L^\infty \leftarrow L^1}{A_0\, q} + \dots \Big](z)$$

- Poisson integral formula:

$$f(z) = \int \frac{\operatorname{Im} z}{\pi |x - z|^2} f(x)\, dx \quad \Longrightarrow \quad \|f(x+ib)\|_{L^\infty_x} \lesssim \|f\|_{L^\infty}$$

- Conclude:

$$\|q(t, x+ib)\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{2}} \quad \forall b > 0$$

Thank you!