Introduction to Noncommutative Physics

A (Biased) Panoramic Overview

Richard Szabo

Applications of Noncommutative Geometry to Gauge Theories,
Field Theories, and Quantum Space-Time

Marseille, 7 April 2025

Wikipedia on Noncommutative Geometry/Physics

"**Noncommutative geometry** is a branch of mathematics concerned with a geometric approach to **noncommutative algebras**, and with the construction of spaces that are locally presented by noncommutative algebras of functions, possibly in some generalized sense. A noncommutative algebra is an associative algebra in which the multiplication is not commutative ..."

Applications in mathematical physics:

"There is an influence of physics on noncommutative geometry, on the other hand there is **no** application of noncommutative geometry in physics.[3]"

[3] Connes, Douglas & Schwarz '97

"In mathematical physics, **noncommutative quantum field theory** (or quantum field theory on noncommutative spacetime) is an application of noncommutative mathematics to the spacetime of quantum field theory that is an outgrowth of noncommutative geometry and index theory in which the coordinate functions are noncommutative."

Outline

In the spirit of *ChatGPT*, not *Wikipedia*!

- Spacetime quantization
- Mathematical aspects: Spectral triples, Morita equivalence, . . .
- Noncommutative field theory: Quantum mechanics $\longrightarrow$ low-energy effective field theories
- String theory: Noncommutative Yang-Mills theory, T-duality
- Deformation quantization: Seiberg-Witten map
- Drinfel'd twist deformation: Noncommutative gravity
- Noncommutative QFT: UV/IR mixing
- Matrix models: Emergent gravity, Fuzzy field theories

Spacetime Quantization

- **Quantum gravity:** Classical general relativity breaks down at Planck scale $\ell_{\mathrm{P}} = \sqrt{\hbar\, G/c^3} \sim 1.6 \times 10^{-33}$ cm

  Quantum $+$ *Gravity* $=$ *?*

- **Emergent phenomenon:** Quantize fundamental degrees of freedom from which gravity emerges

  e.g. AdS/CFT holography: Bulk spacetime geometry and gravity emerge from entanglement of quantum states on boundary

- **Quantum geometry:** Apply principles of quantum mechanics to spacetime itself

  Since gravity affects spacetime geometry, quantum gravity should quantize spacetime (in some way):

  $$\big[x^\mu \,,\, x^\nu\big] \;=\; \mathrm{i}\,\theta^{\mu\nu} \qquad \Longleftrightarrow \qquad \Delta x^\mu\,\Delta x^\nu \;\geqslant\; \tfrac{1}{2}\,\big|\theta^{\mu\nu}\big|$$

  $\theta^{\mu\nu}$ like $\hbar$ in phase space quantization $\big[x^\mu \,,\, p_\nu\big] \;=\; \mathrm{i}\,\hbar\,\delta^\mu{}_\nu$

- **"Pointless" Geometry** relevant for quantum gravity phenomenology

  (Addaz *et al.* '21; Alves Batista *et al.* '25)

Spacetime Quantization

- **Noncommutativity of Spacetime:** To probe physics at Planck scale $\ell_{\rm P}$, Compton wavelength of probe $\leqslant \ell_{\rm P}$

  $\Longrightarrow$ Huge mass $m \geqslant \hbar/\ell_{\rm P}\, c$ concentrated in tiny volume $\ell_{\rm P}^3$ forms a **black hole**, whose event horizon (Schwarzschild radius $\sim m$) hides measurement

  (Doplicher, Fredenhagen & Roberts '95)

- **Fundamental Uncertainties** are common in many **models**:
  - Loop Quantum Gravity (minimal area/volume)
  - String Theory (minimal length uncertainty relations from high-energy scattering) (Amati, Ciafaloni & Veneziano '89)
  - Spin Foam Models (effective noncommutative QFTs)
  - Group Field Theory, Causal Dynamical Triangulations, Causal Set Theory, Hořava-Lifshitz Gravity, ...

- Quantum spacetimes may come with quantum symmetries

  e.g. deformations of Poincaré symmetry, quantum groups, ...

- **Noncommutative Geometry:** framework for understanding certain regimes of quantum gravity as well as structure of (effective) theories

  (Connes '85; Woronowicz '87; Majid '88; ...)

κ-Minkowski Space

(Lukierski *et al.* '91, Majid & Ruegg '94; ...)

$$\begin{aligned} \left[x^\mu\,,\,x^\nu\right] &= \tfrac{\mathrm{i}}{\kappa}\left(x^\mu\,\xi^\nu - x^\nu\,\xi^\mu\right) \\ \left[x^\mu\,,\,p_\nu\right] &= \mathrm{i}\,\hbar\,\delta^\mu{}_\nu + \tfrac{\mathrm{i}}{\kappa}\left(p^\mu\,\xi_\nu + p_\nu\,\xi^\mu\right) \\ \left[p_\mu\,,\,p_\nu\right] &= 0 \end{aligned}$$

$\xi^0 = 1$, $\xi^i = 0$, $\kappa =$ mass scale $(\sim \hbar/\ell_{\mathrm{P}})$

κ-Poincaré symmetry, dispersion relations: $E^2 = \boldsymbol{p}^2\,c^2 - m^2\,c^4\left(1 - \frac{\xi\cdot p}{\hbar\,\kappa}\right)^2$

Doubly special relativity:
Speed of a photon depends on its energy E (through $\ell_{\mathrm{P}}\,E$)
— astrophysical γ-ray bursts
(Amelino-Camelia '01;
Kowalski-Glikman & Nowak '02;
Maguejo & Smolin '02; ...)

3D Quantum Gravity

('t Hooft '96; Matschull & Welling '98; Freidel & Livine '05; . . .)

- Integrate out gravitational degrees of freedom in $3D$ gravity coupled to spinless matter (spin foam model) $\Longrightarrow$ effective scalar field theory on $SO(1,2)$ Lie algebra noncommutative spacetime:

$$\begin{aligned}
\left[x^\mu\,,\,x^\nu\right] &= \mathrm{i}\,\ell_{\mathrm{P}}\,\epsilon^{\mu\nu\lambda}\,x^\lambda \\
\left[x_\mu\,,\,p_\nu\right] &= \mathrm{i}\,\sqrt{\hbar^2-\ell_{\mathrm{P}}^2\,p^2}\;\delta_{\mu\nu}-\mathrm{i}\,\ell_{\mathrm{P}}\,\epsilon_{\mu\nu\lambda}\,p_\lambda \\
\left[p_\mu\,,\,p_\nu\right] &= 0
\end{aligned}$$

- Dispersion relations:

$$E^2 = \boldsymbol{p}^2\,c^2-\left(\frac{\sinh(\ell_{\mathrm{P}}\,\hbar^{-1}\,m\,c^2)}{\ell_{\mathrm{P}}\,\hbar^{-1}}\right)^2$$

- Doubly special relativity arises in low energy limit of quantum gravity

Quantum/Geometry: Spectral Description

- Distance: Riemannian manifold (M, g)

$$d(x,y) = \inf_{\gamma : x \to y} \int_\gamma \mathrm{d}s \quad , \quad \mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$$

- Duality: $M = \mathrm{Spec}(\mathcal{A})$, $\mathcal{A} = C(M)$ is a *commutative* (C^*-)algebra

  Gel'fand-Naimark Theorem: Every *Hausdorff* space arises in this way (characters $\chi : \mathcal{A} \longrightarrow \mathbb{C}$, $\chi_x(f) = f(x)$)

- Generalised Dirac operator: Represent $\mathcal{A}$ on a Hilbert space $\mathcal{H}$

$$d(x,y) = \sup_{f \in \mathcal{A}} \big\{ |\chi_x(f) - \chi_y(f)| \, \big| \, \|[D,f]\| \leqslant 1 \big\} \quad , \quad \mathrm{d}s^{-1} = D : \mathcal{H} \longrightarrow \mathcal{H}$$

- Spectral triple: $\mathcal{T} = (\mathcal{A}, \mathcal{H}, D)$ (+ "real structure" + . . .)

  Connes' Reconstrution Theorem for compact spin manifolds

  (Connes '95; Connes '08)

- Noncommutative differential calculus

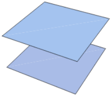
Morita Equivalence

- Gauge group: $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^\dagger u = u u^\dagger = \mathbb{1}\}$
  Gauge transformations: $g_u : \mathcal{A} \longrightarrow \mathcal{A} \quad , \quad g_u(a) = u\, a\, u^\dagger$
  generate $\mathrm{Inn}(\mathcal{A})$ with $\mathrm{Aut}(\mathcal{A}) = \mathrm{Inn}(\mathcal{A}) \rtimes \mathrm{Out}(\mathcal{A})$
- Consider $\mathcal{A} = C^\infty(M)$ $\qquad \mathrm{Inn}(\mathcal{A}) = \{\mathbb{1}\}\ ,\ \mathrm{Out}(\mathcal{A}) = \mathrm{Diff}(M)$
  $\mathcal{A}_N = C^\infty(M) \otimes \mathrm{Mat}(N)$
  $\mathrm{Inn}(\mathcal{A}_N) = C^\infty(M, U(N))\ ,\ \mathrm{Out}(\mathcal{A}_N) = \mathrm{Diff}(M)$
- $\mathrm{Mat}(N)$ has only one non-trivial irrep (as an associative algebra), so $\mathcal{A}$ and $\mathcal{A}_N$ give (at the level of topology) the same space
- $\mathcal{A}$ and $\mathcal{A}_N$ are **Morita equivalent**

  $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$ if $\mathcal{B} \simeq \mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ for some $\mathcal{A}$-module $\mathcal{E}$
- Morita equivalent algebras have equivalent representation theories, in particular their K-theories are the same: $K(\mathcal{A}) \simeq K(\mathcal{B})$
- Duality: Sections of vector bundle $E \longrightarrow M$ form module over $\mathcal{A} = C(M)$
  Serre-Swan Theorem: (fgp) $\mathcal{A}$-modules $\equiv$ vector bundles

Spectral Standard Model

- Almost commutative geometry $C(M) \otimes \mathcal{A}_\mathsf{F}$: spacetime M is augmented by a discrete 'internal' space $F = \mathrm{Spec}(\mathcal{A}_\mathsf{F})$; for $\mathcal{A}_\mathsf{F} = \mathbb{C} \oplus \mathbb{C}$

$M \times F =$ 

- Standard Model on a 2-sheeted spacetime: Higgs field is the noncommutative gauge potential in the discrete direction (Connes & Lott '89)
- Geometrical interpretation of the full Standard Model:

$$\mathcal{A}_\mathsf{F} = \mathbb{C} \oplus \mathbb{H}_\mathsf{R} \oplus \mathbb{H}_\mathsf{L} \oplus \mathrm{Mat}(3)$$

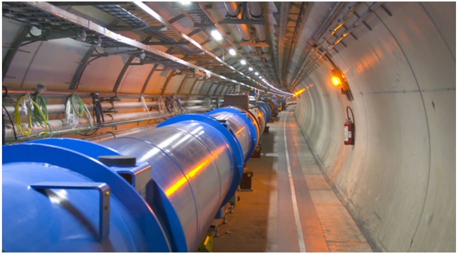
  Fibre space $F = \mathrm{Spec}(\mathcal{A}_\mathsf{F})$ has classical dimension 0, KO-dimension 6 (mod 8), so $M \times F$ is a noncommutative Kaluza-Klein compactification of a spacetime of KO-dimension $4 + 6 = 10$

  (Connes '06; Barrett '06; Chamseddine, Connes & Marcolli '06; . . .)

- Spectral action: Fluctuations of Dirac operator couple gravity to the Standard Model (Connes '96; Chamseddine & Connes '96; . . .)

Field Theory on Quantized Spacetime

Quantum field theory works well at least down to LHC scale
$\ell_{\mathrm{LHC}} \sim 2 \times 10^{-18}$ cm — What happens below this scale?



Noncommutative quantum field theory may be relevant at scales in between ℓ_{P} and ℓ_{LHC}

Noncommutativity alternative to supersymmetry, string theory

Physics beyond Standard Model / standard physics in strong external fields

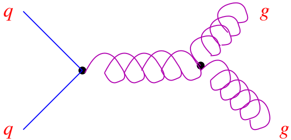
Relates field theory to gravity (easier to quantize?)

Field Theory on Quantized Spacetime

Foundational issues in quantum field theory

New phenomena:

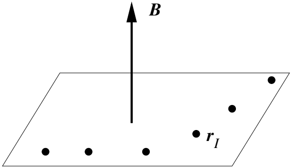
- Controlled Lorentz violation (of localized field configurations within a fixed observer inertial frame): Well-constrained experimentally (Bolmont *et al.* '22)
- Violations of causality (no sharply localized light-cone) — "light-wedges" (Grosse & Lechner '08)
- UV/IR mixing (non-renormalizable!): Spacetime coarse-graining does not generally tame UV divergences of quantum field theory
- New/forbidden interactions



- New phenomenology: Experiment?

(Chaichian, Sheikh-Jabbari & Tureanu '00; Carroll *et al.* '01; Calmet *et al.* '01; Amelino-Camelia *et al.* '03; Balachandran *et al.* '05; Helling & You '07; ...)

Electrons in Strong Magnetic Fields



$$\mathcal{L}_m = \frac{m}{2}\,\dot{\boldsymbol{x}}^2 - \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A} - V(\boldsymbol{x})$$

$$A_i = -\frac{B}{2}\,\varepsilon_{ij}\,x^j$$

In strong field limit $e\,B \gg m$ (lowest Landau level):

$$\mathcal{L}_0 \;=\; -\frac{e\,B}{2c}\,\dot{x}^i\,\varepsilon_{ij}\,x^j - V(\boldsymbol{x})$$

Canonical quantization gives **noncommuting coordinates**:

$$[x^i, x^j] \;=\; \mathrm{i}\,\theta^{ij} \;=\; \frac{\mathrm{i}\,\hbar\,c}{e\,B}\,\varepsilon^{ij}$$

Peierls substitution: First order energy shift due to impurity potential $V\big(x^i - \frac{\mathrm{i}}{2}\,\theta^{ij}\,\frac{\partial}{\partial x^j}\big)$ in perturbation theory of lowest Landau level (Peierls '33)

Magnetic Sources

(Jackiw '85; Günaydin & Zumino '85; Mickelsson '85)

- More generally, electrons experience magnetic field $\boldsymbol{B}$ (with sources) via Lorentz force $\dot{\boldsymbol{p}} = \frac{e}{m\,c}\,\boldsymbol{p}\times\boldsymbol{B} + \boldsymbol{\nabla} V(\boldsymbol{x})$ for kinematical momentum $\boldsymbol{p} = m\,\dot{\boldsymbol{x}}$

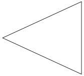
- Hamiltonian $\mathrm{H} = \frac{1}{2m}\,\boldsymbol{p}^2 + V(\boldsymbol{x})$ generates Lorentz force $-\,\mathrm{i}\,\hbar\,\dot{\boldsymbol{p}} = [\mathrm{H},\boldsymbol{p}]$ only for noncommuting momenta:

$$[x^i,x^j] = 0 \quad , \quad [x^i,p_j] = \mathrm{i}\,\hbar\,\delta^i{}_j \quad , \quad [p_i,p_j] = \frac{\mathrm{i}\,\hbar\,e}{c}\,\varepsilon_{ijk}\,B^k$$

- Magnetic translations $U(\boldsymbol{a}) = \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\,\boldsymbol{a}\cdot\boldsymbol{p}}$ do not commute:

$$U(\boldsymbol{a}_1)\,U(\boldsymbol{a}_2) = \mathrm{e}^{\,\frac{\mathrm{i}\,e}{\hbar\,c}\,\Phi_{12}}\,U(\boldsymbol{a}_1+\boldsymbol{a}_2) = \mathrm{e}^{\,\frac{2\,\mathrm{i}\,e}{\hbar\,c}\,\Phi_{12}}\,U(\boldsymbol{a}_2)\,U(\boldsymbol{a}_1)$$

$\Phi_{12} =$ magnetic flux through

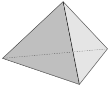


Magnetic Sources

- Nonassociating momenta:

$$[p_1, p_2, p_3] \; := \; [p_1,[p_2,p_3]]+[p_2,[p_3,p_1]]+[p_3,[p_1,p_2]] \; = \; \frac{\hbar^2 \, e}{c} \, \boldsymbol{\nabla} \cdot \boldsymbol{B}$$

$$\big(U(\boldsymbol{a}_1)\, U(\boldsymbol{a}_2)\big)\, U(\boldsymbol{a}_3) \; = \; \mathrm{e}^{\frac{\mathrm{i}\, e}{\hbar\, c}\, \Phi_{123}}\, U(\boldsymbol{a}_1)\, \big(U(\boldsymbol{a}_2)\, U(\boldsymbol{a}_3)\big)$$

$\Phi_{123} =$ magnetic charge enclosed by 

- $\boldsymbol{\nabla} \cdot \boldsymbol{B} \; = \; 0$: no sources, no flux, associativity, $\boldsymbol{p} \; = \; -\,\mathrm{i}\,\hbar\,\boldsymbol{\nabla} - \frac{e}{c}\,\boldsymbol{A}$
- $\boldsymbol{\nabla} \cdot \boldsymbol{B} \; \neq \; 0$: nonassociativity unless $\dfrac{e\,\Phi_{123}}{2\pi\,\hbar\,c} \; = \;$ integer (Jackiw '85)
  (Point) magnetic monopoles , Dirac quantization condition
- Smooth distributions of magnetic charge
  $\Longrightarrow$ **Nonassociative quantum mechanics needed!**

Phase Space Quantum Mechanics

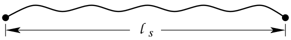
(Grönewold; Moyal; Weyl; Wigner; . . .)

The operator-state formulation of quantum mechanics cannot handle nonassociative structures:

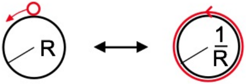
- Treat position and momentum on equal footing
- Observables and states $\longmapsto$ real functions on phase space
- Algebraic structure: star-product ; Trace: integration
- State function (“density matrix”): $S_\rho = \psi^* \star \psi$, $\mathrm{Tr}\, S_\rho = 1$
- Expectation values: $\langle \mathcal{O} \rangle = \mathrm{Tr}\, \mathcal{O} \star S_\rho$
- Schrödinger equation: $\mathrm{H} \star \psi = \mathrm{i}\, \hbar \, \frac{\partial \psi}{\partial t}$

Nonassociative quantum mechanics: Reality, positivity, minimal volume, GNS construction, . . . (Mylonas, Schupp & Sz ’13; Schupp & Sz ’23)

String Geometry



- Strings see geometry in different ways than particles do,
  e.g. **T-duality** $R \longmapsto \ell_s^2/R$ is a string symmetry



- Spacetime geometry is an approximate notion: Valid at sizes $R \gg \ell_s$, but breaks down at $R \sim \ell_s$ due to non-locality
- Isolate geometry from non-locality: Geometry makes sense in decoupling limit $\alpha' = \ell_s^2 \longrightarrow 0$ with R finite; Use effective field theories as probes
- Not all spacetime geometries are ordinary geometric spaces,
  e.g. noncommutative spaces can arise as decoupling limits
- Isometries in spectral geometry: For torus backgrounds with constant NS–NS B-fields, T-duality is an isomorphism between spectral triples
  (Fröhlich & Gawedzki '93; Lizzi & Sz '97; Fröhlich, Grandjean & Recknagel '97; Roggenkamp & Wendland '03)
- Noncommutative T-duality covariant formalism (Landi, Lizzi & Sz '98; Freidel, Leigh & Minic '17; Kodzoman & Lescano '23)

Open String Dynamics in Constant B-Fields

(Douglas & Hull '97; Ardalan, Arfaei & Sheikh-Jabbari '98; Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; . . .)

$$S = \int_\Sigma \frac{1}{4\pi\,\alpha'}\,g_{ij}\,\partial x^i\cdot\partial x^j - \frac{\mathrm{i}}{2}B_{ij}\,\partial x^i\wedge\partial x^j$$

- 2-point function on boundary of disk: ordering

$$\big\langle x^i(t)\,x^j(t')\big\rangle = -\alpha'\,G^{ij}\log(t-t')^2 + \tfrac{\mathrm{i}}{2}\,\theta^{ij}\,\mathrm{sgn}(t-t')$$

- Open-closed string relation $(g,B)\longrightarrow(G,\theta)$: In decoupling limit $\alpha'\sim\epsilon^{1/2}$, $g_{ij}\sim\epsilon$ with $\epsilon\longrightarrow 0$:

$$G = -(2\pi\,\alpha')^2\,B\,g^{-1}\,B \qquad , \qquad \theta = B^{-1}$$

- Extends to curved backgrounds and non-constant B, possibly with H-flux $H=\mathrm{d}B\neq 0$ (Cornalba & Schiappa '01; Herbst, Kling & Kreuzer '01)

  Examples: D-branes in WZW models (Alekseev, Recknagel & Schomerus '99)

  Holographic duals to integrable deformations of $\mathrm{AdS}_5\times S^5$ σ-models

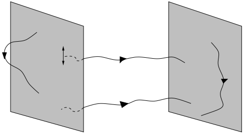
  (van Tongeren '15; Araujo *et al.* '17; Meier & van Tongeren '23; . . .)

Noncommutative Yang-Mills Theory

- Open string interactions in scattering amplitudes captured by Moyal-Weyl star-product:

$$f \star g = \cdot \, \exp\left(\tfrac{\mathrm{i}}{2}\,\theta^{ij}\,\partial_i \otimes \partial_j\right)(f \otimes g)$$

More generally, noncommutative star-product quantizes θ



Massless bosonic modes:
A_i, X^a gauge and scalar fields,
low-energy dynamics described by
noncommutative gauge theory
(Connes & Rieffel '87)

- Dependent on regularization scheme: ordinary gauge theory (Pauli-Villars) vs noncommutative gauge theory (point-splitting)

- **Seiberg-Witten map:** Field redefinition to ordinary "dual" gauge theory:

$$\widehat{A}^{\,i}[A,\theta] \;=\; \theta^{ij}\,A_j + \tfrac{1}{2}\,\theta^{kl}\,A_l\,\big[\partial_k(\theta^{ij}\,A_j) - \theta^{ij}\,F_{jk}\big] + \cdots$$

Similarly $\widehat{\lambda}[\lambda, A]$ with $\widehat{A}[A+\delta_\lambda A] \;=\; \widehat{A}[A] + \widehat{\delta}_{\widehat{\lambda}[\lambda,A]}\widehat{A}[A]$

Morita Duality of Noncommutative Gauge Theory

- Open string T-duality on a p-torus T^p acts on Dp-brane charges

- $SO(p,p;\mathbb{Z})$ T-duality group acts on $\mathcal{E} = \frac{1}{\alpha'}\,(g + 2\pi\,\alpha'\,B)$ as
  $\mathcal{E}' = (A\,\mathcal{E} + B)\,\frac{1}{C\,\mathcal{E} + D}$ for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(p,p;\mathbb{Z})$

- In decoupling limit:

$$G' = (C\,\theta + D)\,G\,(C\,\theta + D)^\top \quad , \quad \theta' = (A\,\theta + B)\,\frac{1}{C\,\theta + D}$$

  and $\quad g'_{\rm YM} = g_{\rm YM}\,\big|\det(C\,\theta + D)\big|^{1/4}$

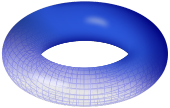
- Noncommutative gauge theory inherits this T-duality symmetry

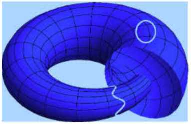
- Refinement of topological T-duality via Morita equivalence of noncommutative tori: $K(T^p_\theta) = K(T^p_{\theta'})$

(Connes, Douglas & Schwarz '97; Schwarz '98; Rieffel & Schwarz '98;
Brace, Morariu & Zumino '98; Pioline & Schwarz '99; ...)

Morita Duality and Non-Geometric Backgrounds

- New features when $H = \mathrm{d}B \neq 0$: "non-geometric" backgrounds





$$\begin{tikzcd} \text{Torus bundle} \arrow[d] \arrow[r, "\text{T-duality}"] & \text{T-fold} \arrow[d] \\ S^1_x & S^1_x \end{tikzcd}$$

Patching: Diffeomorphisms

Patching: T-duality

- Topological T-duality: If $M = \mathrm{Spec}(\mathcal{A})$, T-dual is $\widehat{\mathcal{A}} = \mathcal{A} \rtimes_\alpha \widehat{\mathbb{R}}^p$ where $\alpha : \mathbb{R}^p \longrightarrow \mathrm{Aut}(\mathcal{A})$ with $K(\widehat{\mathcal{A}}) = K(\mathcal{A})$

- T-folds: Rigorous description as a bundle of noncommutative tori $T^p_{\theta(x)}$

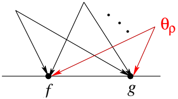
(Mathai & Rosenberg '04; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '07; Brodzki, Mathai, Rosenberg & Sz '08; Bouwknegt & Pande '08; Hull & Sz '19; Aschieri & Sz '20)

Deformation Quantization and Kontsevich Formality

- Formality maps U_n : multivector fields $\longrightarrow$ differential operators define L_∞-quasi-isomorphisms of dg-Lie algebras relating Schouten brackets to Gerstenhaber brackets (Kontsevich '97; Cattaneo & Felder '99)

- Star-product for a bivector $\theta \xrightarrow{\text{quantization}} \star$:

$$f \star g = \Phi(\theta)(f,g) := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_n(\theta,\dots,\theta)(f,g)$$



- Formality condition $[\Phi(\theta),\star]_G \;=\; \mathrm{i}\,\hbar\,\Phi([\theta,\theta]_S)$:

  $\star$ associative $\Longleftrightarrow$ θ Poisson (nonassociative $\star$ is possible)

- Up to gauge equivalence:

$$\begin{aligned} f \star g \;=\; & f\cdot g + \tfrac{\mathrm{i}\,\hbar}{2}\,\theta^{ij}\,\partial_i f\cdot\partial_j g - \tfrac{\hbar^2}{4}\,\theta^{ij}\,\theta^{kl}\,\partial_i\partial_k f\cdot\partial_j\partial_l g \\ & - \tfrac{\hbar^2}{6}\,\theta^{ij}\,\partial_j\theta^{kl}\,(\partial_i\partial_k f\cdot\partial_l g - \partial_k f\cdot\partial_i\partial_l g) + \cdots \end{aligned}$$

- Exponentiates like Moyal-Weyl formula when $\theta^{ij}\,\partial_j\theta^{kl} \;=\; 0$

Seiberg-Witten Maps

- Poisson θ , quantum Moser lemma $\equiv$ covariantizing map $\mathcal{D}$

  (Jurčo, Schupp & Wess '00):

$$\begin{tikzcd}
B\ : \arrow[d, "\text{Moser}"', "\rho"] & \theta \arrow[r, "\text{quantization}"] \arrow[d, "\rho"'] & \star \arrow[d, "\mathcal{D}"] \\
B+F\ : & \theta' \arrow[r, "\text{quantization}"'] & \star'
\end{tikzcd}$$

$\theta' = \theta\,(1+\hbar\,F\,\theta)^{-1}$ where $F = \mathrm{d}A$

$\rho =$ flow generated by $\xi = \theta^{ij}\,A_j\,\partial_i$

- $\Phi(\xi) =$ deformed diffeomorphisms, integrate to a "flow" $\mathcal{D}$
- Covariant coordinates: $X^i = \mathcal{D}x^i =: x^i + \theta^{ij}\,\widehat{A}_j$
- Equivalent star-products: $\mathcal{D}(f\star g) = \mathcal{D}f\star'\mathcal{D}g$
- $\delta_\lambda A = \mathrm{d}\lambda \Longrightarrow \widehat{\delta}_{\widehat{\lambda}}\mathcal{D}f = \mathrm{i}\,[\,\widehat{\lambda}\overset{\star}{,}\mathcal{D}f]$ and $\widehat{\delta}_{\widehat{\lambda}}\widehat{A} = \mathrm{d}\widehat{\lambda} + \mathrm{i}\,[\,\widehat{\lambda}\overset{\star}{,}\widehat{A}\,]$

  Noncommutative gauge transformations realised as inner automorphisms

Hopf Cocycle Twist Quantization

(Majid '95)

- Drinfel'd twist for a quasi-triangular Hopf algebra $H(\Delta, S, R, \varepsilon, \cdot)$:

$$F \;=\; F_{(1)} \otimes F_{(2)} \;\in\; H \otimes H$$

$$(F \otimes 1)\,\Delta_1 F \;=\; (1 \otimes F)\,\Delta_2 F \qquad \text{2-cocycle condition}$$

- Maps H to new quasi-triangular Hopf algebra $H_F(\Delta_F, S_F, R_F, \varepsilon, \cdot)$ with $\Delta_F \;=\; F\,\Delta\,F^{-1}$

- Quantizes any H-module algebra $\mathcal{A}$ to an associative noncommutative algebra $\mathcal{A}_F$ carrying representation of twisted Hopf algebra H_F:

$$f \star g \;=\; \cdot\big(F^{-1}(f \otimes g)\big) \;=\; F^{-1}_{(1)} f \cdot F^{-1}_{(2)} g$$

- If $\mathcal{A}$ is commutative, then $\mathcal{A}_F$ is braided-commutative

- Simultaneously quantizes all H-covariant constructions as functorial isomorphism of braided monoidal categories of left modules:

$$\mathcal{Q}_F \,:\, {}^{H}\mathcal{M} \;\longrightarrow\; {}^{H_F}\mathcal{M}$$

Hopf Cocycle Twist Quantization

- Example: $\mathfrak{g} \; = \;$ Lie algebra of symmetries of a manifold M, $H \; = \; U(\mathfrak{g}) \;\; , \;\; \mathcal{A} \; = \; C^\infty(M) \; , \; \Omega^\bullet(M) \; , \; \dots$
- Let $\mathfrak{g} \; = \; U\Gamma(TM) \; =$ enveloping algebra of vector fields on M
  e.g. Moyal-Weyl twist $F = \exp\big(-\frac{\mathrm{i}}{2}\,\theta^{ij}\,\partial_i \otimes \partial_j\big)$
- If $\mathcal{A}$ is a $U\Gamma(TM)$-module algebra (functions, forms, tensors on M), then $\Gamma(TM)$ acts on $\mathcal{A}$ via Lie derivative and Leibniz rule
- Braiding given by triangular R-matrix $R_F \; = \; F^{-2}$
- Twisted symmetries: $\xi \in H$ does not “see” the $\star$-product:

$$\delta_\xi(f \star g) \; = \; \delta_{\xi_{(1)}} f \star \delta_{\xi_{(2)}} g \quad , \quad \Delta_F \xi \; = \; \xi_{(1)} \otimes \xi_{(2)}$$

- More generally, a cochain twist F maps H to a quasi-Hopf algebra H_F and quantizes $\mathcal{A}$ to a “quasi-associative” algebra $\mathcal{A}_F$

(Beggs & Majid ’05; Barnes, Schenkel & Sz ’14; Blumenhagen & Fuchs ’16;
Aschieri, Dimitrijević Ćirić & Sz ’17; . . .)

Noncommutative Gravity

- Massless bosonic modes of closed strings give background geometry and gravity: g_{ij} , B_{ij} , Φ

- Worldsheet conformal invariance $\Longrightarrow$ Closed string effective action:

$$S = \int R - \frac{1}{12}\,\mathrm{e}^{\,-\Phi/3}\,H_{ijk}\,H^{ijk} - \frac{1}{6}\,\partial_i\Phi\,\partial^i\Phi + \cdots$$

- Noncommutative version? Duality?

- **General relativity on noncommutative spacetime:** (Aschieri *et al.* '05; . . .)
  Twisted tensor calculus/diffeomorphism symmetry, deformed Einstein equations, (exact) noncommutative black hole solutions, . . .

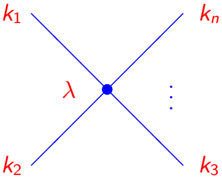
- Twisted diffeomorphisms **do not** arise as physical symmetries of string theory (Álvarez-Gaumé, Meyer & Vázquez-Mozo '06)

- Twisted symmetries **do** arise in AdS/CFT: integrable Yang-Baxter deformations of $\mathrm{AdS}_5 \times S^5$ string theory with Drinfel'd twisted symmetry, dual to noncommutative Yang-Mills with twisted symmetries (Vicedo '15; van Tongeren '15; Borsato, Driezen & Miramontes '21; Meier & van Tongeren '23)

UV/IR Mixing – The Problem

- Noncommutative quantum field theories are typically plagued by the problem of **UV/IR mixing**:

$$\widetilde{\phi}(k)\,\widetilde{\phi}(q) \;\longrightarrow\; \widetilde{\phi}(k)\,\widetilde{\phi}(q)\ \mathrm{e}^{\,\mathrm{i}\,k\times q} \quad , \quad k\times q \;=\; \tfrac{1}{2}\,k_i\,\theta^{ij}\,q_j$$



$$=\;\lambda\ \mathrm{e}^{\,\mathrm{i}\,\sum_{I<J} k_I\times k_J}$$

with $k_1+k_2+\cdots+k_n \;=\; 0$; effective at energies E with $E\,\sqrt{\theta}\ll 1$

- **Non-planar graphs:** UV cutoff Λ $\Longrightarrow$ Effective IR cutoff $\Lambda_0 \;=\; \dfrac{1}{\theta\,\Lambda}$

(Minwalla, Van Raamsdonk & Seiberg ’99)

- **The field theory cannot be renormalized!!!**

UV/IR Mixing – The Physics

- **Non-locality:** ϕ, ψ supported in region of small size $\Delta \ll \sqrt{\theta}$
  $\Longrightarrow$ $\phi \star \psi$ non-zero in large region of size $\frac{\theta}{\Delta}$

- $\mathrm{e}^{\,\mathrm{i}\,k\cdot x} \star \phi(x) \star \mathrm{e}^{\,-\,\mathrm{i}\,k\cdot x} \ = \ \phi(x^i - \theta^{ij}\,k_j)$

  IR dynamics: "dipoles" with dipole moment $\Delta x^i \ = \ \theta^{ij}\,k_j$
  (like electron-hole bound state in strong magnetic field)

  (Sheikh-Jabbari '99; Bigatti & Susskind '99)

- UV dynamics: Elementary quantum fields ϕ, pointlike momenta k_i

- Non-renormalizable gravitational sector: For $k < \Lambda < \Lambda_{\rm NC} \ = \ \frac{1}{\sqrt{\theta}}$,
  1-loop effective action for noncommutative gauge fields (involving $U(1)$ sector UV/IR mixing terms) gives induced Einstein-Hilbert action (Λ related to G)

  (Grosse, Steinacker & Wohlgenannt '08)

- Modified dispersion relations: $E^2 \ = \ \boldsymbol{p}^2 + m^2 + \Delta M^2\big(\frac{1}{p\,\theta}\big)$
  Compare with experiments for $\Lambda_0 < E < \Lambda \ = \ \frac{1}{\theta\,\Lambda_0}$

  (Amelino-Camelia *et al.* '03; Helling & You '07; . . .)

UV/IR Mixing – A Cure

- Covariant version makes UV and IR regimes indistinguishable

- Make UV/IR "duality" symmetric by introducing a "magnetic" background B_{ij}: (Langmann & Sz '02; Langmann, Sz & Zarembo '03)

  $$k_i \longmapsto K_i = k_i + B_{ij}\, x^i \qquad (\text{“Landau” momenta})$$

  "Noncommutative momentum space": $[K_i, K_j] = 2\,\mathrm{i}\, B_{ij}$

- **Grosse-Wulkenhaar model:** Real Euclidean scalar $\lambda\, \phi_{2d}^{\star 4}$-theory in background harmonic oscillator potential:

  $$\Delta \longmapsto \Delta + \tfrac{1}{2}\, \omega^2\, \widetilde{x}^2 \quad , \quad \widetilde{x} = 2\, \theta^{-1} \cdot x$$

- QFT is symmetric under Fourier transformation of fields: $k_i \leftrightarrow \widetilde{x}_i$
  Covariant model is renormalizable to all orders in λ
  (Grosse & Wulkenhaar '04; Rivasseau *et al.* '05; Disertori *et al.* '06; . . .)

- Physical origin of "magnetic background" ω?

  Coupling to curvature of a noncommutative space (Burić & Wohlgenannt '09)

Another Cure – Braided Quantum Field Theory

- Another approach to renormalizable noncommutative QFT is by modifying the *path integral* directly (rather than the classical theory) — this is called braided quantum field theory
- Renormalization properties of braided QFT very different — UV/IR mixing seems far less severe and maybe even absent (Oeckl '00; Balachandran *et al.* '06; Bu *et al.* '06; Fiore & Wess '07; . . .)
- Oeckl's algebraic approach to braided QFT is based on braided versions of Wick's Theorem and Gaussian integration (Oeckl '99; Sasai & Sasakura '07)
- A recent systematic approach uses the modern purely algebraic formalism of Batalin-Vilkovisky (BV) quantization (à la Costello-Gwilliam)

  Braided generalization is dual to the **braided L_∞-algebras** that construct braided field theories equivariant under a triangular Hopf algebra action, with braided noncommutative fields (Dimitrijević Ćirić, Giotopoulos, Radovanović & Sz '21; Nguyen, Schenkel & Sz '21; . . .)
- **Note:** Notion of braided gauge symmetry is not new — kinematical aspects of this idea appeared long before (Brzezinski & Majid '92; . . .)
- Explicit realizations as effective theories, e.g. in string theory?

Matrix Models

- Noncommutative Yang-Mills action for $U(N)$ gauge field $A_i(x)$:

$$S = -\frac{1}{4g^2} \int \mathrm{Tr}\, F_{ij}^2$$

with field strength tensor $F_{ij} = \partial_i A_j - \partial_j A_i - \mathrm{i}\,[A_i \overset{\star}{,} A_j]$

$U(\infty)$ gauge symmetry $\equiv$ "deformed" canonical transformations

(Lizzi, Sz & Zampini '01)

- In "covariant coordinates" $X_i = \theta_{ij}^{-1}\, x^j + A_i$ the noncommutative gauge theory becomes a matrix model (Aoki *et al.* '99; Madore *et al.* '00):

$$S = -\frac{1}{4g^2}\, \mathrm{Tr}\big(-\mathrm{i}\,[X_i, X_j] + \theta_{ij}^{-1} \big)^2$$

Spacetime disappears!

- Twisted reduced model: Dimensional reduction of ordinary Yang-Mills theory to a point (**IKKT model** for non-perturbative IIB string theory)

  Compactification (quotient conditions) without the twist gives Yang-Mills theory on a noncommutative torus (Connes, Douglas & Schwarz '97)

- Equations of motion: $[X_i, [X_i, X_j]] = 0$

Matrix Models

- Vacuum: $[X_i, X_j] = -\mathrm{i}\,\theta^{-1}_{ij}$ (no finite $N \times N$ solutions)

- Non-perturbative finite N definition: (Ambjørn *et al.* '99)

$$S = -\frac{1}{4g^2} \sum_{i \neq j} \mathrm{e}^{-2\pi\,\mathrm{i}\,Q_{ij}/N}\ \mathrm{Tr}\big(U_i\,U_j\,U_i^\dagger\,U_j^\dagger\big)$$

  $U_i = \mathrm{e}^{\,\mathrm{i}\,a\,X_i}$ unitary $N \times N$ matrices, $\theta^{-1}_{ij} = \dfrac{2\pi\,Q_{ij}}{N\,a^2}$

  Twisted Eguchi-Kawai model: One-plaquette reduction of Wilson's lattice gauge theory with 't Hooft flux (planar limit of Yang-Mills theory) (Gonzalez-Arroyo & Okawa '83)

  Noncommutative version of lattice gauge theory, $N \sim$ size of lattice

- Non-vacuum solutions $\big[X^i\,,\,X^j\big] = \mathrm{i}\,\theta^{ij}(x)$ (e.g. fuzzy spaces)

  Emergent gravity: Dynamical quantum spacetime $\Longrightarrow$ Gravity related to quantum fluctuations X^i of spacetime at Planck scale

  $U(1)$ "photon" is really a graviton, defining a non-trivial geometric background coupled to $SU(N)$ gauge fields

  (Rivelles '02; Yang '06; Steinacker '07; ...)

Field Theory on the Fuzzy Sphere

(Hoppe '82; Madore '91; . . .)

- Fuzzy sphere: Take the spin $\alpha = \frac{N-1}{2}$ irrep of $su(2)$, with generators

$$[X_i, X_j] = \mathrm{i}\, r_N\, \epsilon_{ijk}\, X_k \ , \ X_i\, X_i = \mathbb{1} \ , \ X_i^* = X_i$$

$$\mathcal{A} = (\alpha) \otimes (\alpha)^* \simeq \mathrm{Mat}(N)$$

- Free scalar field theory: $S_0 = \frac{4\pi}{N} \,\mathrm{Tr}\, \Phi\, (\Delta + m^2)\, \Phi$

(Grosse, Klimcik & Presnajder '95; . . .)

$$\Delta(\Phi) = \frac{1}{r_N^2}\, [X_i, [X_i, \Phi]] \qquad \text{(fuzzy Laplacian)}$$

- Fuzzy spherical harmonics: $Y_j^J \in \mathcal{A}$ $(0 \leqslant J \leqslant N \ , \ -J \leqslant j \leqslant J)$ satisfy

$$\Delta(Y_j^J) = J\,(J+1)\, Y_j^J \quad , \quad \frac{4\pi}{N}\, \mathrm{Tr}(Y_j^{J\,*}\, Y_{j'}^{J'}) = \delta_{JJ'}\, \delta_{jj'}$$

$$Y_i^I\, Y_j^J = \sum_{K,k} \pm \sqrt{(2I+1)\,(2J+1)\,(2K+1)} \begin{pmatrix} I & J & K \\ i & j & -k \end{pmatrix} \begin{Bmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{Bmatrix} Y_k^K$$

Field Theory on the Fuzzy Sphere

- Interactions: expressed in terms of Wigner 3j and 6j symbols of $su(2)$

  E.g. For $\mathrm{Tr}\,\Phi^4$: $$I^{J_0\cdots J_3}_{j_0\cdots j_3} = \prod_{i=0}^{3}\sqrt{2J_i+1}\,\sum_{J,j}(-1)^j\,(2J+1) \times \begin{pmatrix} J_0 & J_1 & J \\ j_0 & j_1 & j \end{pmatrix}\begin{pmatrix} J_2 & J_3 & J \\ j_2 & j_3 & -j \end{pmatrix}\begin{Bmatrix} J_0 & J_1 & J \\ \alpha & \alpha & \alpha \end{Bmatrix}\begin{Bmatrix} J_2 & J_3 & J \\ \alpha & \alpha & \alpha \end{Bmatrix}$$

- 2-point function at 1-loop in Φ^4-theory receives planar and non-planar loop corrections — scaling limits (Chu, Madore & Steinacker '01)

- Fuzzy sphere has well-known $su(2)$-equivariant 3D differential calculus on $\mathcal{A} = (\alpha)\otimes(\alpha)^*$ given by Chevalley-Eilenberg dg-algebra of $su(2)$

- Enables construction of fuzzy field theories with gauge symmetries (Chern-Simons, Yang-Mills, etc.) (Madore '91; Klimcik '97; Grosse & Presnajder '98; Carow-Watamura & Watamura '98; . . .)

- Low-energy effective field theory on D-branes in $SU(2)$ WZW model (S^3 with H-flux at large radius) (Alekseev, Recknagel & Schomerus '99)

Field Theory on the Fuzzy Torus

(Weyl '31; Barrett & Gaunt '19; . . .)

- Fuzzy torus $\mathcal{A} \simeq \mathrm{Mat}(N)$: $a = \sum_{i,j\in\mathbb{Z}_N} a_{ij}\, U^i\, V^j$ with:

  $$U\,U^* = V\,V^* = \mathbb{1} \quad , \quad U\,V = q\,V\,U \quad , \quad U^N = V^N = \mathbb{1}$$

  where $q = \mathrm{e}^{\,2\pi\,\mathrm{i}\,/N}$; $\mathrm{Tr}(a) = a_{00}$ defines a trace on A

- Free scalar field theory:

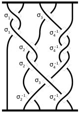
  $$\Delta(\Phi) = -\tfrac{1}{(q^{1/2}-q^{-1/2})^2}\,\big(\big[U,\big[U^*,\Phi\big]\big]+\big[V,\big[V^*,\Phi\big]\big]\big)$$

- Fuzzy plane waves $e_k = U^{k_1}\,V^{k_2} \in \mathcal{A}$ satisfy

  $$\Delta(e_k) = \big([k_1]_q^2+[k_2]_q^2\big)\,e_k \quad , \quad \mathrm{Tr}(e_k^*\,e_l) = \delta_{k,l}$$

  where $[n]_q = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$ (q-numbers)

- Group Hopf algebra $H = \mathbb{C}[\mathbb{Z}_N^2]$ acts on $\mathcal{A}$:



Can be quantized as a braided scalar field theory —
extension to gauge theories (differential calculus)?

(Nguyen, Schenkel & Sz '21)