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The BV quantization in NCG: the case of finite spectral triple

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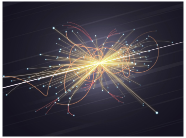
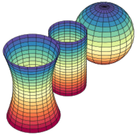
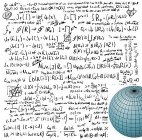
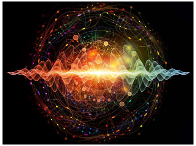


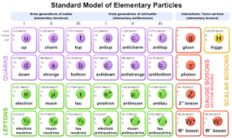
Applications of NonCommutative Geometry to Gauge Theories, Field Theories, and Quantum Space-Time, CIRM

7 April 2025





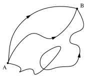




The BV construction: where it was discovered

Context: quantization of a gauge theory (X_0, S_0) via a path integral approach $\rightsquigarrow Z := \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$

Z ↓ partition function



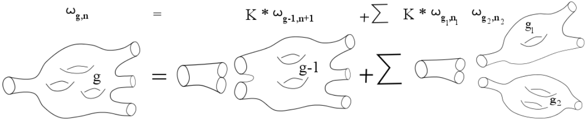
path integral

Problem 1: the measure is not well-defined

- *Approach 1*: Functorial QFT, that is, try to define the integral by implementing Fubini's theorem

TQFT = Functor of symmetric monoidal categories

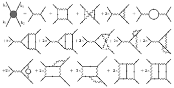
$$\mathcal{C}ob_n \longrightarrow Vect_{\mathbb{C}}$$



- *Approach 2*: Perturbative QFT, that is, try to define the integral by implementing the principle of stationary phase appearing in the finite-dimensional setting also in the infinite dimensional context

$$\int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu] \underset{\hbar \to 0}{\backsim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar} S_0(x_0)} \, |\det S_0''(x_0)|^{-\frac{1}{2}} \; e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|} \Phi_{\Gamma}.$$

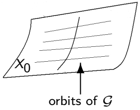
where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, order of its automorphism group $|Aut(\Gamma)|$ and weight Φ_Γ.



Feynman diagrams

The introduction of ghost fields

⚠ To apply the perturbative approach the critical points of S_0 have to be isolated and regular



Problem 2: for a gauge invariant action functional, critical points appear in orbits
$\rightsquigarrow$ a path integral quantization of gauge theories is not straightforward

How to eliminate these redundant symmetries without changing the underlying physical theory?

- take the quotient w.r.t. the action of the group $\rightsquigarrow$ get orbifolds or even more complicated objects
- add extra auxiliary variables $\rightsquigarrow$ ghost fields ✓

$$\int_{-\infty}^{+\infty} e^{-x^2} dx \rightsquigarrow \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy$$

Def. A ghost field φ is characterized by: ghost degree: $\deg(\varphi) \in \mathbb{Z}$ & parity: $\epsilon(\varphi) \in \{0,1\}$
where $\epsilon(\varphi) = 0$ is bosonic/real and $\epsilon(\varphi) = 1$ is fermionic/Grassm. s.t. $\deg(\varphi) \equiv \epsilon(\varphi) \ \ mod \ \mathbb{Z}/\mathbb{Z}2$

A bit of history:

- Faddeev - Popov [1967]: to construct the perturbative path integral for the Yang-Mills theory, they proposed to eliminate the divergences of the integrand by introducing fermionic ghost fields of degree 1

The BV construction: the key idea

- Becchi, Rouet, Stora and Tyutin [1975]: observed the need of introducing ghostfields of higher ghost degree for theories with non-independent gauge symmetry generators. Moreover, they discovered the BRST complex.

$$\boxed{\textit{BRST cohomology} = \textit{Chevalley-Eilenberg cohomology}}$$

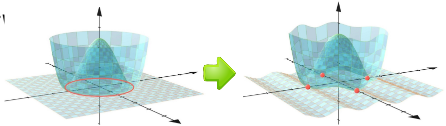
with only ghost fields of degree 1

- Zinn-Justin [1975]: enriched the structure on the ghost sector by the introduction of antibracket $\{\ ,\ \}$
- Batalin - Vilkovisky [1981/1983]: they suggested the introduction of antifields/antighost fields

Def. For a ghost φ, its antighost φ^* has $\deg(\varphi^*) = -\deg(\varphi) - 1$ & $\epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (mod\ \mathbb{Z}/2\mathbb{Z})$

Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \qquad \int_{X_0} e^{\frac{i}{\hbar} S_0}\, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}] \subset X_t} e^{\frac{i}{\hbar} S_q}\, d\mu_{BV}$$



The goal: To find
- $\mathcal{L}$ Lagrangian $\subset X_t$ ghost sector &
- $S_q \in \mathcal{C}^\infty(X_t)[[\hbar]]$, sol. quant. master eq.

s.t. $S_q|_{\mathcal{L}}$ has *isolated* and *regular* critical points.

The (classical) BV-extension, in the algebraic geometric approach

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\tilde{X}, \tilde{S})$$

Initial data: a gauge theory

- X_0: vector sp $\cong \mathbb{A}^{n^2}_{\mathbb{R}}$
- $S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \dots, x_{n^2}]$
- $\mathcal{G} = U(n)$

BV extended theory

- $\tilde{X} = \bigoplus_{i \in \mathbb{Z}} [\tilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\underset{[1]}{\tilde{X}} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\tilde{X}]^0 = X_0$ [2]
  - $\mathcal{F}$: graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank
- $\tilde{S} \in [\mathcal{O}_{\tilde{X}}]^0$, s.t. $\underset{[2]}{\tilde{S}|_{X_0}} = S_0$ & $\{\tilde{S}, \tilde{S}\} = 0$ [3] sol. to the classical master equation
  - $\{\ ,\ \}$: 1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$, $\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$, $\{\varphi^*_i, \varphi_j\} = \delta_{ij}$

Note:

[1] While $\mathcal{F}$ accounts for the ghost field sector, $\mathcal{F}^*[1]$ describes the anti-ghost content $\rightsquigarrow$ for each ghost field introduced we also include the corresponding anti-ghost field.

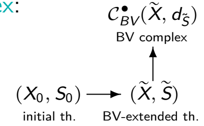
[2] In degree 0 we have only the initial (physical) fields. If we restrict to X_0, we get back the initial (physically relevant) theory.

[3] Each BV-extended theory naturally induces a cohomology complex: the BV-complex.

Step 2: The BV-complex

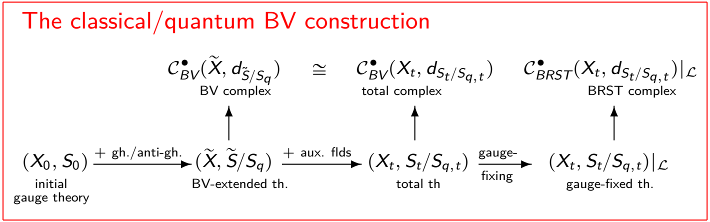
Step 2: Any BV-extended theory $(\widetilde{X}, \widetilde{S})$, with $\{\widetilde{S}, \widetilde{S}\} = 0$ induces a BV cohom. complex:

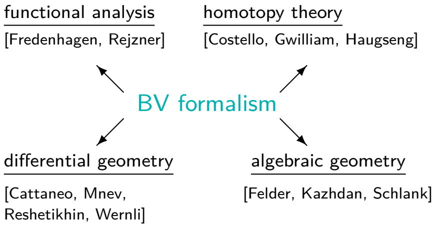
- Cochain spaces: $\mathcal{C}^i(\widetilde{X}, d_{\tilde{S}}) = [\mathcal{O}_{\widetilde{X}}]^i$
- Coboundary op.: $d_{\tilde{S}} := \{\widetilde{S}, -\} : \mathcal{C}^\bullet(\widetilde{X}, d_{\tilde{S}}) \to \mathcal{C}^{\bullet+1}(\widetilde{X}, d_{\tilde{S}})$, $\quad d_{\tilde{S}}^2 = 0$



➡ The BV construction ⇝ cohomological approach to the study of gauge symmetries. These cohomology groups capture relevant physical information about (X_0, S_0):

$$H^0_{BV}(\widetilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}$$





From spectral triples to gauge theories

Def. A gauge theory $(X_0, S_0, \mathcal{G})$ is a physical theory with

$$X_0 = \text{field configuration space} \qquad S_0 : X_0 \to \mathbb{R}, \text{ action functional}$$

and $\mathcal{G}$ a group acting on X_0 through an action $F : \mathcal{G} \times X_0 \to X_0$, such that it holds that

$$S_0(F(g,\varphi)) = S_0(\varphi) \qquad \forall \varphi \in X_0, \forall g \in \mathcal{G}.$$

spectral triple $(\mathcal{A}, \mathcal{H}, D)$	➡	gauge theory $(X_0, S_0, \mathcal{G})$
▸ $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$		▸ $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\}$ ⇝ conf. sp = inner fluctuations
▸ $\mathcal{H}$ = Hilbert space		▸ $S[D+\varphi] := Tr(f(\frac{D+\varphi}{\Lambda}))$ ⇝ action func. = spectral action
▸ $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator		▸ $\mathcal{G} = \mathcal{U}(\mathcal{A})$ ⇝ gauge group = unitary elements in $\mathcal{A}$

Spectral action: $S[D+\varphi] = Tr\big(f(\frac{D+\varphi}{\Lambda})\big)$;

- for f a regular funct. (good decay, ...)
- for Λ = cut off;
- for φ a self-adjoint, with $\varphi = \sum_j a_j[D, b_j]$, $a_j, b_j \in \mathcal{A}$

Fermionic action: $S[\psi] = \frac{1}{2}\langle (J)\psi, D\psi\rangle$,

- for $\langle\ ,\ \rangle$ the inner product structure on $\mathcal{H}$;
- for $\psi \in \mathcal{H}_f \subseteq \mathcal{H}$

➡ Grassmannian nature of $\mathcal{H}_f$

Questions and goals

partially with W.D. van Suijlekom

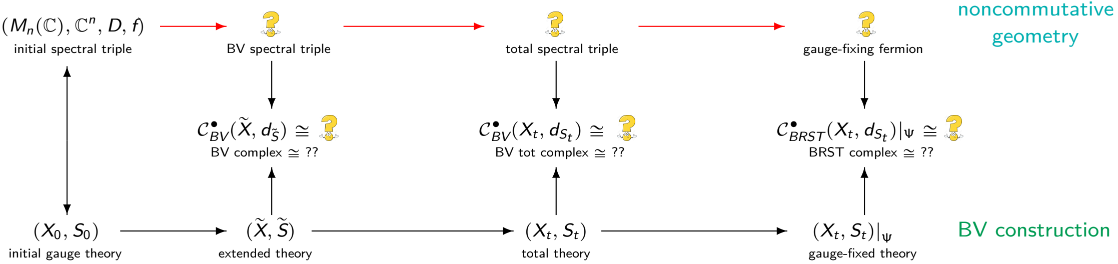
Note: the finite case might look mathematically simpler but physically less interesting setting to consider. However, several results showed that the finite term is the one which encodes the particle content.

➡ We want to study the BV construction for gauge theories induced by finite spectral triples

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Questions and goals:

- Can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?



The model

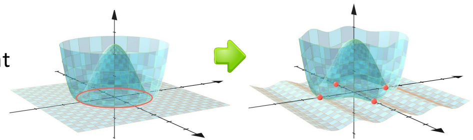
Given the spectral triple $(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$, the induced gauge theory has:

- $X_0 := \{M \in M_n(\mathbb{C}) \text{ s.t. } M^* = M\} \cong \mathbb{A}_{\mathbb{R}}^{n^2} := \{(x_1, \ldots, x_{n^2})\}$ — affine configuration space
- $S_0[M] := Tr(f(M + D_0))$ with $S_0 \in \mathcal{O}_{X_0} \cong \mathbb{R}[x_1, \ldots, x_{n^2}]$ — polynomial spectral action
- $\mathcal{G} = U(n)$, acting by the adjoint action on X_0 — $U(n)$-gauge theory

To be invariant under the adjoint action of $U(n)$, the functional S_0 should be a polynomial in the Casimir operators of order k, $2 \leqslant k \leqslant n$.

Example: the quadratic case $\quad S_0 = \sum_{k=0}^{r} (x_1^2 + \cdots + x_{n^2-1}^2)^k g_k(x_{n^2})$

Critical locus:
- the origin is the only trivial orbit
- any critical point different from the origin determines an orbit of non-isolated critical point

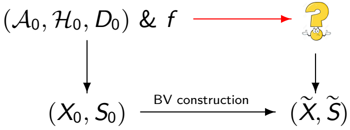


Need of BV to determine:
- $\mathcal{L}$ Lagrangian $\subset X_t$ ghost sector

&
- $S_q \in \mathcal{C}^\infty(X_t)[[\hbar]]$, sol. quant. master eq.

Step 1: the BV extension in NCG [1]

Question 1: Can the BV extended theory $(\widetilde{X}, \widetilde{S})$ be described as a new BV-spectral triple? Can we encode the BV-extension process in the language of NCG?



Note:

- finite spectral triple are naturally defined over $\mathbb{C}$
  - to go from $\mathbb{C}$ to $\mathbb{R}$ we introduce a real structure $J : \mathcal{H} \to \mathcal{H}$
- in $S_{BV} := \widetilde{S} - S_0$ there appear Grassmannian variables
  - include S_{BV} as fermionic action of the new spectral triple:

$$S[\psi] = \frac{1}{2}\langle J\psi, D\psi\rangle, \quad \psi \in \mathcal{H}_f \subseteq \mathcal{H},$$ we can impose a Grassmannian nature to the elements in $\mathcal{H}_f$

$$(\mathcal{A}_0, \mathcal{H}_0, D_0) \ \& \ f \xrightarrow{\text{BV construction}} (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$$

How to extract the information from the initial spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0)$?

ghost fields: Which mathematical structure in $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ determines the ghost sector? Which role are the ghost fields going to play in the BV-spectral triple?

extended action: how can we determine S_{BV} starting from (D_0, f)?

Step 1: the BV extension in NCG [2]

- We introduce the notion of BV-spectral triple.

Def. Let $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ be a spectral triple with induced gauge theory (X_0, S_0) and let $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ denote a real spectral triple with fermionic action $S_{ferm} : \mathcal{H}_{BV,f} \to \mathcal{H}_{BV,f}$, where $\mathcal{H}_{BV,f} \cong Q_f^*[1] \oplus Q_f$, for Q_f is a $\mathbb{Z}$-graded vector space. Then $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ is a BV-spectral triple associated to $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ if:

$$\widetilde{X} := (\mathcal{Q}_f^*[1] + X_0^*[1]) \oplus (X_0 + \mathcal{Q}_f) \quad \& \quad \widetilde{S}[\Psi^*, \varphi^*, \varphi, \Psi] := S_0[\varphi] + \tfrac{1}{2} S_{ferm}[\Psi^*, \Psi]$$

is a BV-theory associated to (X_0, S_0).

initial spectral triple ⇓ induced gauge theory

$(\mathcal{A}_0, \mathcal{H}_0, \underbrace{D_0) \;\&\; f}$ → $X_0 = \Omega^1(\mathcal{A}_0)$, $S_0 = Tr(f(D_0 + \varphi))$

BV spectral triple ⇓ BV-extended theory

$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, \underbrace{D_{BV}, J_{BV}})$ → $\tilde{X} = X_0 + X_0^*[1] + \mathcal{H}_{BV,f}$, $S_{BV} = \frac{1}{2} S_{ferm}$

Step 1: the BV extension in NCG [3]

For the model:

$\mathcal{A}_{BV} = \mathcal{A}_0 = M_n(\mathbb{C})$ ➡ one can prove, a posteriori, that this algebra is the maximal $*$-algebra completing $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ to a spectral triple

The algebra: stays unchanged as it describes the physical field-content of the theory.

$\mathcal{H}_{BV} = \mathcal{Q}^*[1] \oplus \mathcal{Q}$ where $\mathcal{Q} := [M_n(\mathbb{C})]_0 \oplus [M_n(\mathbb{C})]_1$ ➡ The grading corresponds to the ghost degree in the ghost sector

$$\mathcal{H}_0 = \mathbb{C}^n \xrightarrow{+\text{ ghost/anti-ghost fields}} \boxed{\mathcal{H}_{BV} = [M_n(\mathbb{C})]_{-2} \oplus [M_n(\mathbb{C})]_{-1}} \oplus \boxed{[M_n(\mathbb{C})]_0 \oplus [M_n(\mathbb{C})]_1}$$

where

$$\mathcal{H}_{BV,f} = [i\mathfrak{su}(\mathfrak{n})]_{-2} \oplus [i\mathfrak{su}(\mathfrak{n})]_{-1} \oplus [i\mathfrak{su}(\mathfrak{n})]_1 \oplus [i\mathfrak{su}(\mathfrak{n})]_2$$

➡ fully determined by $\mathfrak{su}(n) = \mathfrak{u}(\mathcal{A}_0)/\mathcal{Z}(\mathfrak{u}(\mathcal{A}_0))$

Explicitly:

$$\Psi^* = \Big(\underbrace{[C_1^*, \ldots, C_{n^2-1}^*, 0]_{-2}}_{\text{bosonic antighost fields}}, \underbrace{[x_1^*, \ldots, x_{n^2-1}^*, 0]_{-1}}_{\text{fermionic antifields}}\Big) \qquad \Psi = \Big(\underbrace{[x_1, \ldots, x_{n^2-1}, 0]_0}_{\text{bosonic (initial) fields}}, \underbrace{[C_1, \ldots, C_{n^2-1}, 0]_1}_{\text{fermionic ghost fields}}\Big)$$

The Hilbert space: describes the ghost sector of the BV-extended theory.

Step 1: the BV extension in NCG [4]

$$D_{BV} = \begin{pmatrix} 0 & R \\ R^* & S \end{pmatrix} \qquad \begin{matrix} R : \mathcal{Q} \to \mathcal{Q}^*[1] \\ S : \mathcal{Q} \to \mathcal{Q} \end{matrix}$$

The linear operators R and S are represented, as block matrices, by

$$R := \frac{1}{2}\begin{pmatrix} 0 & -ad(C) \\ ad(C) & -ad(x) \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & ad(x^*) \\ ad(x^*) & ad(C^*) \end{pmatrix}$$

where $ad(z) : M_n(\mathbb{C}) \to M_n(\mathbb{C});$
$\varphi \mapsto [\alpha(z), \varphi]_-.$

Explicitly, the matrix representation of these linear operators has in position (p, r) the term: $-\sum_q i \cdot f_{pqr}\, z_q$

→ Structure constants of $\mathfrak{su}(n)$

The self-adjoint operator D_{BV} is completely obtained by:

- linearity in the antifields, which enforces the zero-block matrix in position $(1, 1)$ in D_{BV};
- degree condition, that is, the induced fermionic action has to have total ghost degree 0, which determines the variables to insert in each block;

] Conditions of the BV construction

- structure constants of $\mathfrak{su}(n) = \mathfrak{u}(\mathcal{A}_0)/\mathcal{Z}(\mathfrak{u}(\mathcal{A}_0))$, which dictate the entries in each block matrix.

] by the gauge symmetries

The operator D_{BV} determines the BV-action $S_{BV} := \widetilde{S} - S_0$ as induced fermionic action.

The real structure: $J_{BV} : \mathcal{H}_{BV} \to \mathcal{H}_{BV}$ with $J_{BV}(\varphi) = \varphi^\dagger$

Step 1: the BV extension in NCG [5]

Prop. $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) := (M_n(\mathbb{C}), M_n(\mathbb{C})^{\oplus 4} = \mathcal{Q}^*[1] \oplus \mathcal{Q}, D_{BV}, J_{BV})$ is a finite spectral triple of KO-dimension 1.

Proof. One has to check that all conditions defining a real spectral triple are satisfied, including the anticommutativity of D_{BV} and J_{BV} required by the KO-dimension 1 case.

Theorem: Let $(\mathcal{A}_0, \mathcal{H}_0, D_0) := (M_n(\mathbb{C}), \mathbb{C}^n, D_0)$ be a finite spectral triple with induced gauge theory *I.* (X_0, S_0). Then, a BV-spectral triple associated to it is given by:

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) := (M_n(\mathbb{C}), \mathcal{Q}^*[1] \oplus \mathcal{Q}, D_{BV}, J_{BV})$$

Proof. It consists in showing that:

$$\widetilde{X} := (\mathcal{Q}_f + X_0) \oplus (X_0^* + \mathcal{Q}_f^*[1]) \quad \& \quad \widetilde{S} := S_0 + \tfrac{1}{2} S_{ferm}$$

is a BV-theory associated to $(X_0 := [\Omega^1(\mathcal{A}_0)]_{s.a.}, S_0 := Tr(f(D_0)))$.

Step 1: $(\mathcal{A}_0, \mathcal{H}_0, D_0) \longrightarrow (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ ✓

Any BV-theory induces a BV complex: $(\widetilde{X}, \widetilde{S}) \longrightarrow \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\widetilde{S}})$. What about a BV-spectral triple?

Step 2: the BV cohomology in NCG [1]

Step 2: $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \dashrightarrow$ 

Idea: to look at cohomology theories naturally appearing in the context of NCG ➡ Hochschild cohomology complex

Aim: To construct a graded isomorphism of cochain complexes

$$\Phi : \mathcal{C}^{\bullet}_{BV}(\widetilde{X}, d_{\tilde{S}}) \longrightarrow \mathcal{C}^{\bullet}_{H,\Delta}(\mathcal{M},\mathcal{B}) \quad \text{that is} \quad \Phi^k : \mathcal{C}^{k}_{BV}(\widetilde{X}, d_{\tilde{S}}) \longrightarrow \mathcal{C}^{k}_{H,\Delta}(\mathcal{M},\mathcal{B}) \quad \text{s.t.}$$

$$d^k_H \circ \Phi^k(\varphi) = \Phi^{k+1} \circ d^k_{\tilde{S}}(\varphi), \quad \forall \varphi \in \mathcal{C}^k_{BV}(\widetilde{X}, d_{\tilde{S}}), \forall k$$

where:

- $(\mathcal{B}, \Delta)$ = 1-shifted graded coalgebra:
  $\mathcal{B} = \bigoplus_{n\in\mathbb{Z}} \mathcal{B}_n$ $\mathbb{Z}$-graded vector space & $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ linear map s.t.
  $\Delta(\varphi_a) \in \bigoplus_{i+j=a+1} \mathcal{B}_i \otimes \mathcal{B}_j$ 1-shifted $\quad (\Delta \otimes Id)(\Delta(\varphi)) = (-1)^{|z^{(1)}|+1}(Id \otimes \Delta)(\Delta(\varphi))$ graded coassociative
- $(\mathcal{M}, \omega)$: degree-1 right comodule over $(\mathcal{B}, \Delta)$
  $\mathcal{M}$ = vector sp. & $\omega : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}_1$ linear map s.t. $\quad (\omega \otimes Id) \circ \omega = -(Id \otimes \Delta) \circ \omega$ graded compatibility

Def. The graded Hochschild complex is given by: $\mathcal{C}^q_H(\mathcal{M},\mathcal{B}) := \mathcal{M} \otimes T^q(\mathcal{B})$ super-graded tensor algebra

$$d_H(\varphi) := \omega(f) \otimes \varphi_1^{i_1} \otimes \cdots \otimes \varphi_k^{i_k} + \sum_{j=1}^k (-1)^{i_1+\cdots+i_{j-1}} f \otimes \varphi_1^{i_1} \otimes \cdots \otimes \Delta(\varphi_j^{i_j}) \otimes \cdots \otimes \varphi_k^{i_k})$$

➡ To determine: $(\mathcal{B}, \Delta)$ & $(\mathcal{M}, \omega)$ from $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$

Step 2: the BV cohomology in NCG [2]

$$\mathcal{B} := \Big(\underbrace{\mathcal{Q}_f}_{B_{m>0}} + \underbrace{[\mathcal{O}_{X_0}]^{\leqslant (deg(S_0)-1)}}_{B_0} \Big) \oplus \Big(\underbrace{X_0^*[1]}_{B_{-1}} + \underbrace{\mathcal{Q}_f^*[1]}_{B_{-m},\, m>1} \Big)$$

Coproduct: $\Delta(\varphi_a) := \{S_0 + \frac{1}{2} S_{ferm}, \varphi_a\}$ for $\{-,-\}$ the antibracket structure induced by the pairing fields/anti-fields on $\mathcal{H}_{BV,f}$.

$$\mathcal{M} := \langle \Omega^1(\mathcal{A}_{BV}) \rangle \cong \mathcal{O}_{X_0}$$

Coaction: $\omega(f) := \{S_0 + \frac{1}{2} S_{ferm}, f\}$

Lemma $(\mathcal{B}, \Delta)$ is a 1-shifted graded coalgebra and $(\mathcal{M}, \omega)$ is a degree-1 comodule over $(\mathcal{B}, \Delta)$

Theorem: Let $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ be a BV-spectral triple associated to $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ and corresponding *I.* to a BV-theory $(\tilde{X}, \tilde{S})$. Given $(\mathcal{B}, \Delta)$ and $(\mathcal{M}, \omega)$ as defined above, it holds that:

$$\mathcal{C}^\bullet_{BV}(\tilde{X}, d_{\tilde{S}}) \cong \mathcal{C}^\bullet_{H,\Delta}(\mathcal{M}, \mathcal{B})$$

BV-spectral triple

↓

Hochschild complex

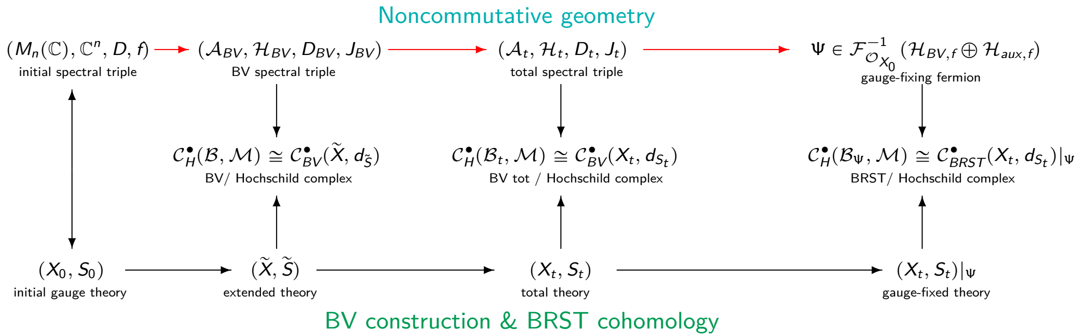
$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, \underbrace{D_{BV}, J_{BV}})$$

$\mathcal{A}_{BV} \to \mathcal{M}$, $\mathcal{H}_{BV} \to \mathcal{B}$, $D_{BV}, J_{BV} \to$ *deg* $0 : \omega$, *deg* $\neq 0 : \Delta$

Step 1 & 2:

$$(\mathcal{A}_0, \mathcal{H}_0, D_0) \to (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \to (\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}), d_H) \to \dots$$

Where we are:



Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
- Established the (noncom.) geometrical role played by ghost/anti-ghost fields in the BV spectral triple
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

What is next? Some interesting open problems

Project 1: The BV formalism for Chern-Simons theory in NCG

Idea: To extend the BV construction for the Chern-Simons theory from classical differential forms to universal forms induced by cyclic cocycles.

T. Krajewski

Project 2: the BV formalism for fuzzy geometries

Idea: To apply the previous result to a fuzzy geometry, which induces a Yang-Mills matrix model:

$S_{\mathrm{YM}} = -\frac{1}{2} Tr_{N \otimes n}([D_\mu, D_\nu][D_\mu, D_\nu])$ ➡ compute $\int_{M_N(\mathbb{C})^4_{\text{skew-adj}}} e^{-S_{YM}[D]}$, towards quantum gravity

C. Perez-Sanchez

Project 3: Spectral triples and higher-groups

Idea: To extend the notion of spectral triple to have induced gauge theory with a higher-group as gauge group

A. Frabetti

Project 4: The BV formalism for noncommutative manifolds

Idea: To rethink the BV formalism in a purely noncommutative and infinite dimensional setting.

R. Nest