Quantum gravity from non-commutative geometry

John Barrett

Outline

- Finite real spectral triples
- Matrix geometries
- Fuzzy sphere
- Quantum Gravity
- Fermion integration

Spacetime

Georges Seurat. *A Sunday on La Grande Jatte — 1884*. The Art Institute of Chicago.

Real spectral triple

Connes 1995

- $\mathscr{A}$ a $*$-algebra
- $\mathscr{H}$ a bimodule over $\mathscr{A}$
- $D\colon \mathscr{A} \to \mathscr{A}$, self-adjoint
- $J\colon \mathscr{H} \to \mathscr{H}$ antilinear
- $\Gamma\colon \mathscr{H} \to \mathscr{H}, \quad \Gamma^2 = 1$

$$J\, l(a)^* = r(a)\, J$$

$$D\Gamma = \pm\, \Gamma D \quad \text{odd/even}$$

First order: $[[D, l(a)], r(b)] = 0$ for $a, b \in \mathscr{A}$

KO dimension

$$J^2 = \pm 1$$

$$JD = \pm' DJ$$

$$J\Gamma = \pm'' \Gamma J$$

S	0	1	2	3	4	5	6	7
$\pm$	+	+	-	-	-	-	+	+
$\pm'$	+	-	+	+	+	-	+	+
$\pm''$	+	+	-	+	+	+	-	+

Real spectral triples

- $M =$ Spin manifold ($\mathscr{A}$ commutative)
- $F =$ SM internal space ($\mathscr{H}$ finite-dimensional)
- $M \times F =$ SM vacuum

Project: replace M with finite NC real spectral triple

Matrix Geometry

JWB 2015

- $\mathscr{A} = M(n, \mathbb{R})$ or $M(n/2, \mathbb{H})$
- $\mathscr{H} = \mathbb{C}^k \otimes M(n, \mathbb{C})$

$\mathbb{C}^k$: Type (p,q) Clifford module

- $l(a)\big(v \otimes m\big) = v \otimes am$
- $J(v \otimes m) = (Cv) \otimes m^*$
- $\Gamma(v \otimes m) = (\gamma v) \otimes m$

$\mathbb{C}^k$ are global spinors

Dirac operator

by Clifford type

- (0,1) $D = i\,[L, \cdot\,]$
- (1,0) $D = \{H, \cdot\,\}$
- (0,2) $D = \gamma^1 \otimes [L_1, \cdot\,] + \gamma^2 \otimes [L_2, \cdot\,]$
- (1,1) $D = \gamma^1 \otimes \{H, \cdot\,\} + \gamma^2 \otimes [L, \cdot\,]$
- (2,0) $D = \gamma^1 \otimes \{H_1, \cdot\,\} + \gamma^2 \otimes \{H_2, \cdot\,\}$
- (0,3) $D = \{H, \cdot\,\} + \gamma^1 \otimes [L_1, \cdot\,] + \gamma^2 \otimes [L_2, \cdot\,] + \gamma^3 \otimes [L_3, \cdot\,]$

etc.

Fuzzy sphere

As a matrix geometry

$\mathscr{H} = \mathbb{C}^2 \otimes M(n, \mathbb{C})$

$d_{GP} = \gamma^i \otimes [L_i, \cdot\,] + 1$ type (0,3) **Grosse, Prešnajder 1995**

$\mathscr{A} = M(n, \mathbb{R})$ n odd

$M(n/2, \mathbb{H})$ n even

KO $s = 3$. No chirality operator

Spectrum: cutoff S^2

Doubled sphere

JWB 2015

$\mathscr{H} = \mathbb{C}^4 \otimes M(n, \mathbb{C})$

$$D_S = \begin{pmatrix} 0 & d_{GP} \\ d_{GP} & 0 \end{pmatrix}$$

$\mathscr{A}$ same

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

KO $s = 2$.

$$UD_SU^{-1} = \begin{pmatrix} d_{GP} & 0 \\ 0 & -d_{GP} \end{pmatrix}$$

Spectrum: cutoff doubled S^2

Sphere spinor bundles

Grosse, Klimčík, Prešnajder 1996

$\mathcal{O}$: Hilbert space of 2 H.O.

$N = a_1^* a_1 + a_2^* a_2 = a_i^* a_i$, number operator

$N = n$ defines $V_n \subset \mathcal{O}$, irrep of SU(2)

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R = V_n \otimes V_{n+1} \oplus V_{n+1} \otimes V_n$$

$$D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} \qquad d = a_i^* \otimes a_i$$

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J = \tau(C \otimes C)$$

- KO $s = 2$
- Spectrum: cutoff S^2
- No algebra

Doubled spinor bundles

$$D \oplus -D_{n-1} = \begin{pmatrix} 0 & d^* \oplus -d_{n-1} \\ d \oplus -d^*_{n-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix} \begin{pmatrix} 0 & d_{GP} \\ d_{GP} & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = U D_S U^{-1}$$

$\alpha \colon V_n \otimes V_{n+1} \oplus V_n \otimes V_{n-1} \to V_n \otimes V_1 \otimes V_n$ β similar

$\mathscr{A} = \mathrm{End}(V_n)$

Fluctuation formula

$K_i \in \mathscr{A}$

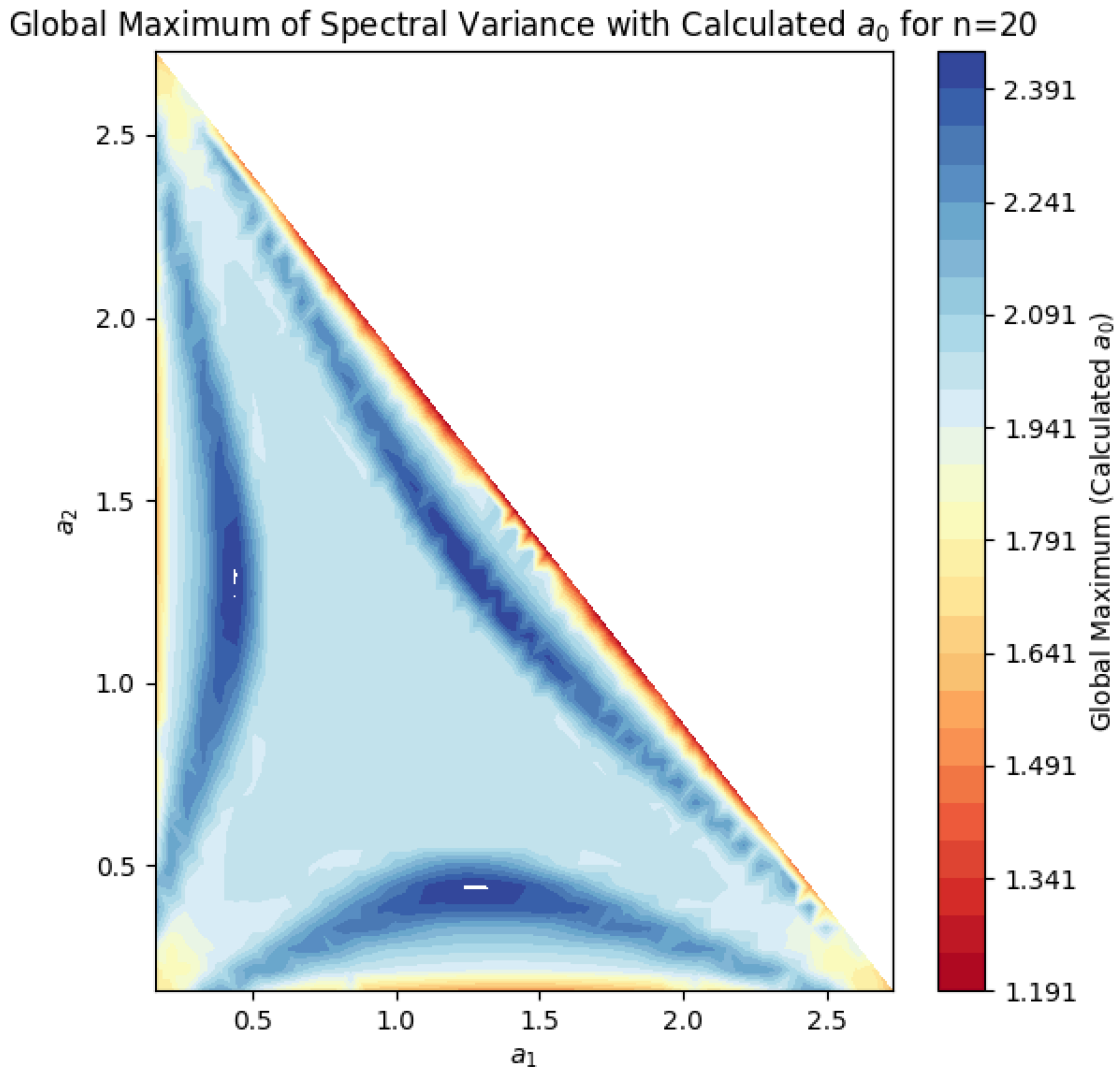
$$\begin{pmatrix} d & 0 \\ 0 & -d_{n-1}^* \end{pmatrix} \to \begin{pmatrix} d & 0 \\ 0 & -d_{n-1}^* \end{pmatrix} + F + JF^*J^{-1}$$

$$F = \frac{1}{n+1} \left[\begin{pmatrix} d & -\imath \\ \imath^* & d_{n-1}^* \end{pmatrix} \begin{pmatrix} K_i \otimes 1 & 0 \\ 0 & K_i \otimes 1 \end{pmatrix}, \begin{pmatrix} 1 \otimes L_i & 0 \\ 0 & 1 \otimes L_i \end{pmatrix} \right]$$

$$\imath \colon V_{n-1} \otimes V_n \to V_n \otimes V_{n+1}, \qquad \imath\, d_{n-1} \to d\, \imath$$

Deformed fuzzy sphere

$$d = a_0 + a_1\gamma^1 \otimes [L_1, \cdot\,] + a_2\gamma^2 \otimes [L_2, \cdot\,] + (1 - a_1 - a_2)\gamma^3 \otimes [L_3, \cdot\,]$$



Data visualisation by Qi Gong

Spectral variance =

heat capacity of fermion

Quantum gravity

Partition function for Euclidean QG + SM:

$$Z(f) = \int_{D\in\mathscr{G},\ \psi\in\mathscr{H}_+} e^{-S(D)+i\langle J\psi, D\psi\rangle}\ f(D,\psi)\ \mathrm{d}D\,\mathrm{d}\psi$$

For Lorentzian QG, rotate contour of integration in $\mathscr{G}\otimes\mathbb{C}$

GGI talk (YouTube) JWB: "The Euclidean contour rotation in quantum gravity"

Phase transition

$$S(D) = \mathrm{tr}\, V(D)$$

$$V(D) = D^4 + g_2 D^2$$

$$S = \sum_{\lambda} \lambda^4 + g_2 \lambda^2$$

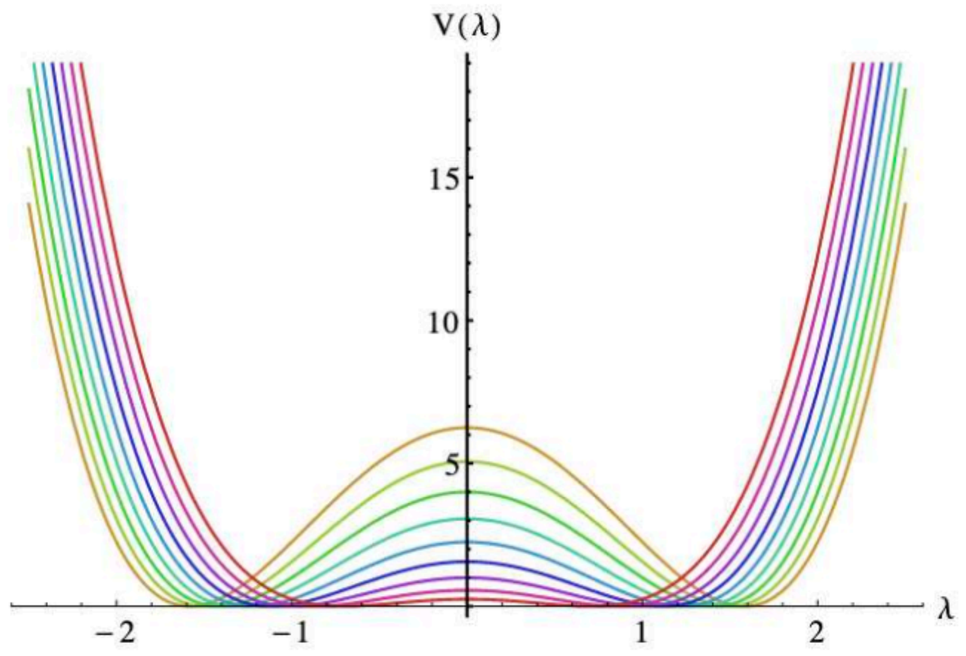
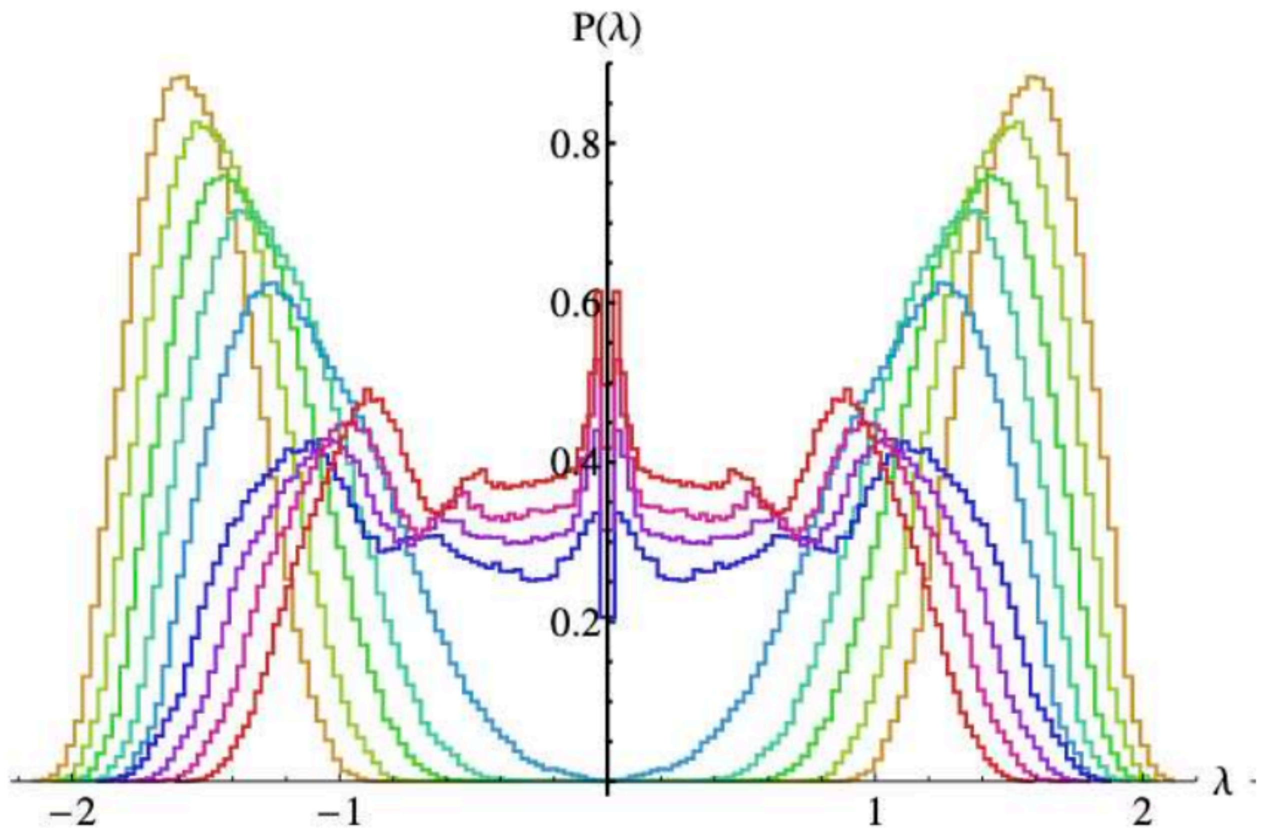


Figure 11: The potential $V = \lambda^4 + g_2\lambda^2$ for $g_2 = -1, -1.5, -2, -2.5, -3, -3.5, -4, -4.5, -5$. The lines are coloured from red ($g_2 = -1$) through to yellow ($g_2 = -5$).



(c) Type (2, 0)

Monte Carlo
Eigenvalue distribution

JWB + L. Glaser 2016

3d models

Numerical simulation of random

Dirac operators

Thesis submitted to the University of Nottingham for the degree of

Doctor of Philosophy, March 2022.

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Type $(p, q) = (3,0)$:

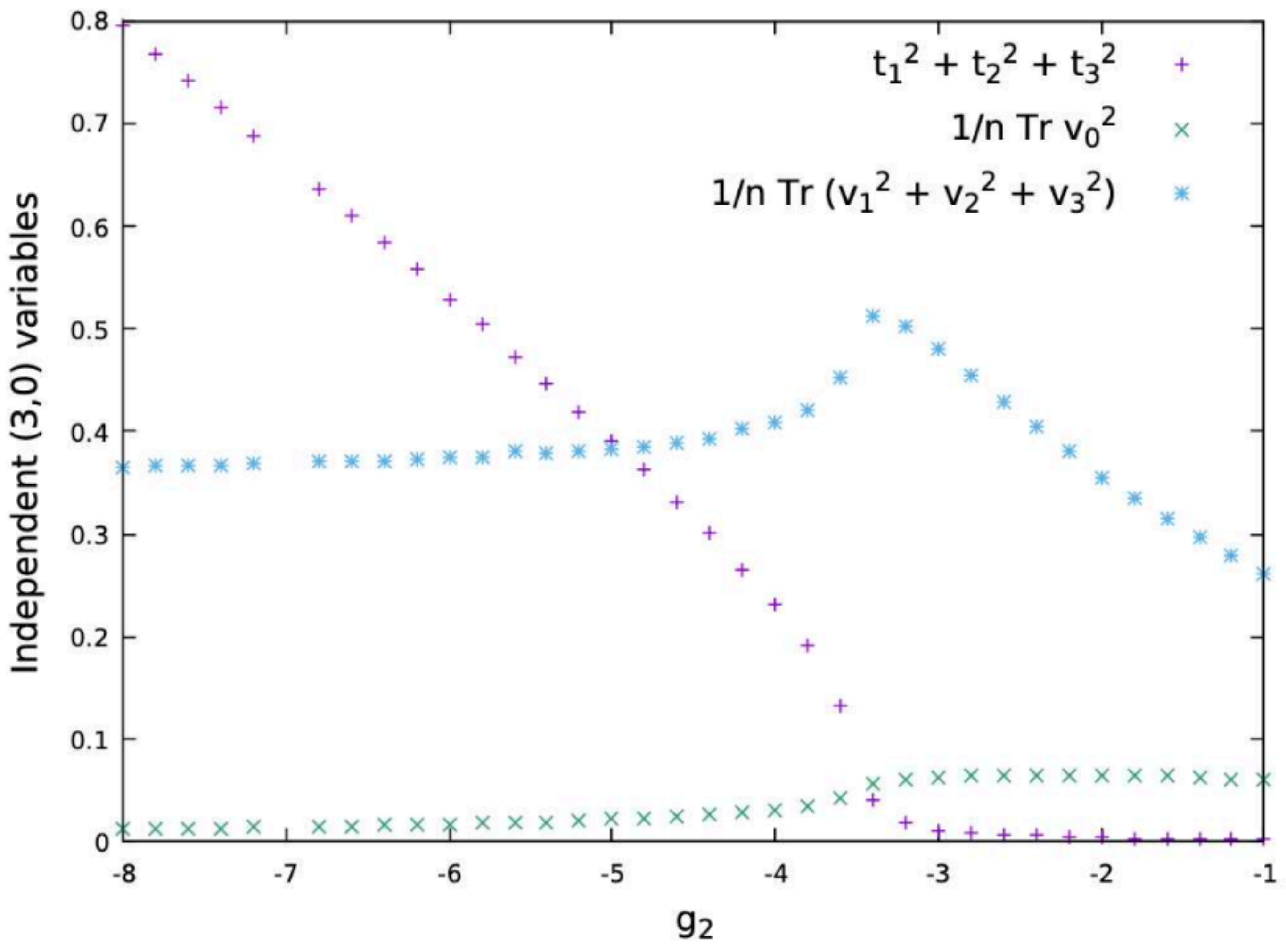
$$D = 1 \otimes [m_0, \cdot] + \sum_1^3 \sigma_i \otimes \{m_i, \cdot\}$$

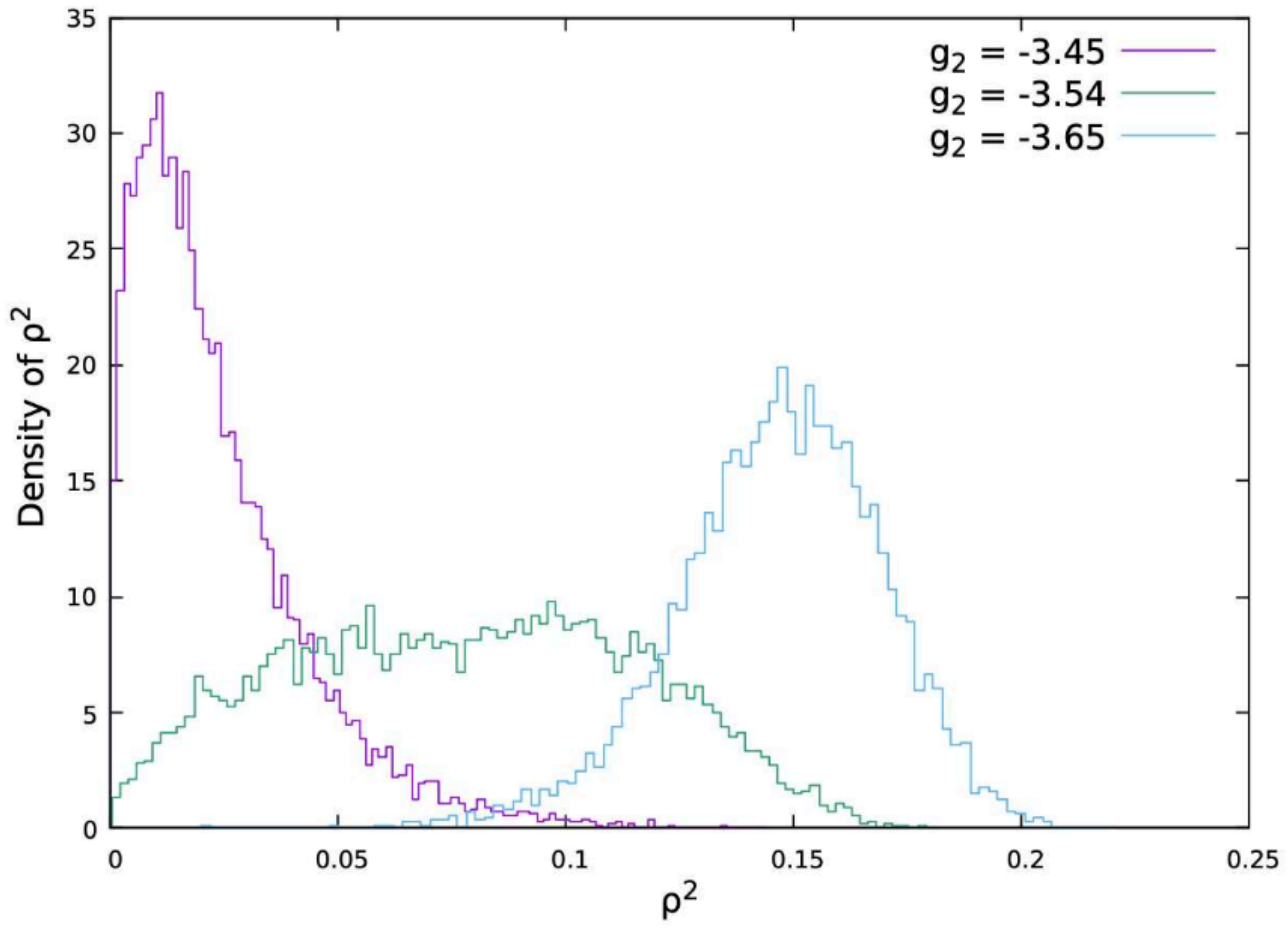
Type $(p, q) = (0,3)$:

$$D = 1 \otimes \{m_0, \cdot\} + \sum_1^3 \sigma_i \otimes [m_i, \cdot]$$

Decompose $m_\mu = t_\mu 1 + v_\mu$
with $\operatorname{tr} v_\mu = 0$

Type (3,0)





$$D = 1 \otimes [m_0,\cdot] + \sum_{1}^{3} \sigma_i \otimes \{m_i,\cdot\}$$

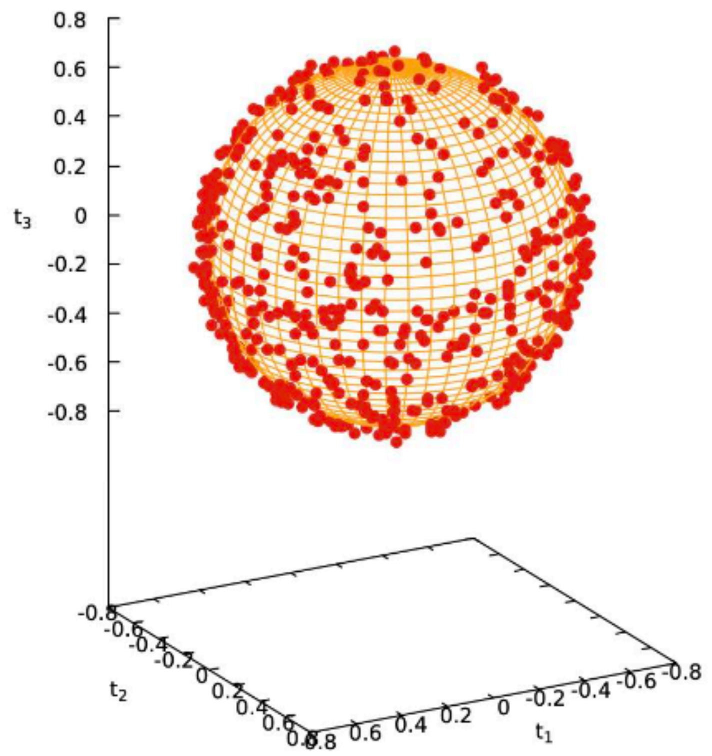
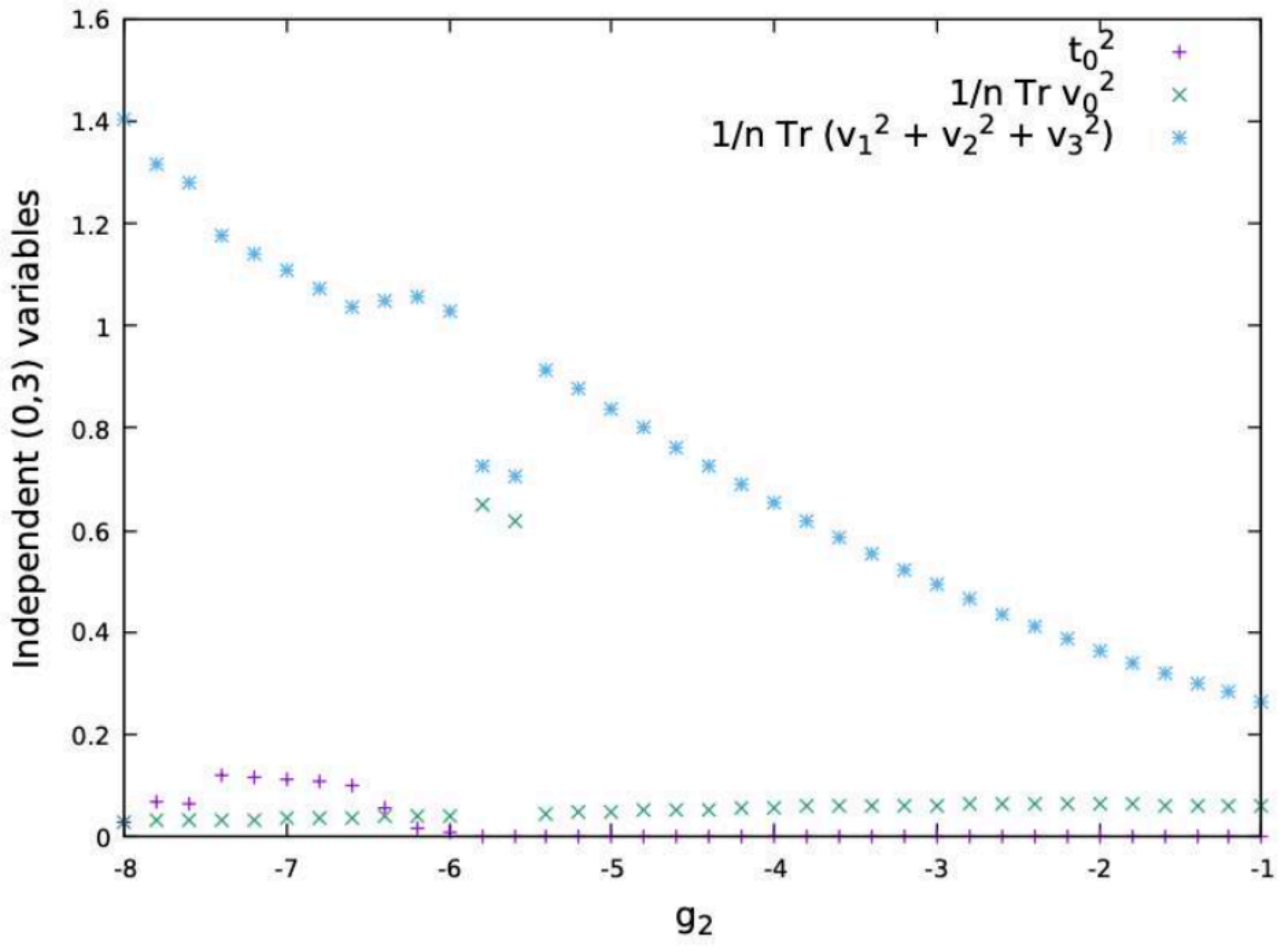


Figure 6.8: Monte Carlo history of t_1, t_2 and t_3 in region II of the (3,0) model at $g_2 = -6$, $n = 8$. The solid orange sphere is a guide for the eyes.

2nd order transition to commutative phase

Type (0,3)



$$D = 1 \otimes \{m_0,\cdot\} + \sum_1^3 \sigma_a \otimes [m_a,\cdot]$$

g_2	Chain 1	Chain 2	Chain 3	Chain 4	Fuzzy sphere	$-g_2/16$
-300	**18.6946(3)**	**18.6946(2)**	**18.6945(2)**	**18.6946(2)**	18.6951	18.75
-150	**9.3465(3)**	9.3740(3)	**9.3465(2)**	9.3739(2)	9.3476	9.375
-100	**6.2301(2)**	**6.2301(3)**	**6.2301(2)**	**6.2301(3)**	6.2317	6.25

Fuzzy sphere $v_a = R l_a,\ a = 1,2,3$

$[l_a, l_b] = \sum_c\ i\epsilon_{abc} l_c,$ irreducible

$$\frac{1}{n}\operatorname{Tr} v_c^2 = -\frac{g_2}{8}\frac{n^2-1}{2n^2-1} \approx -\frac{g_2}{16}, \quad c = 1,2,3$$

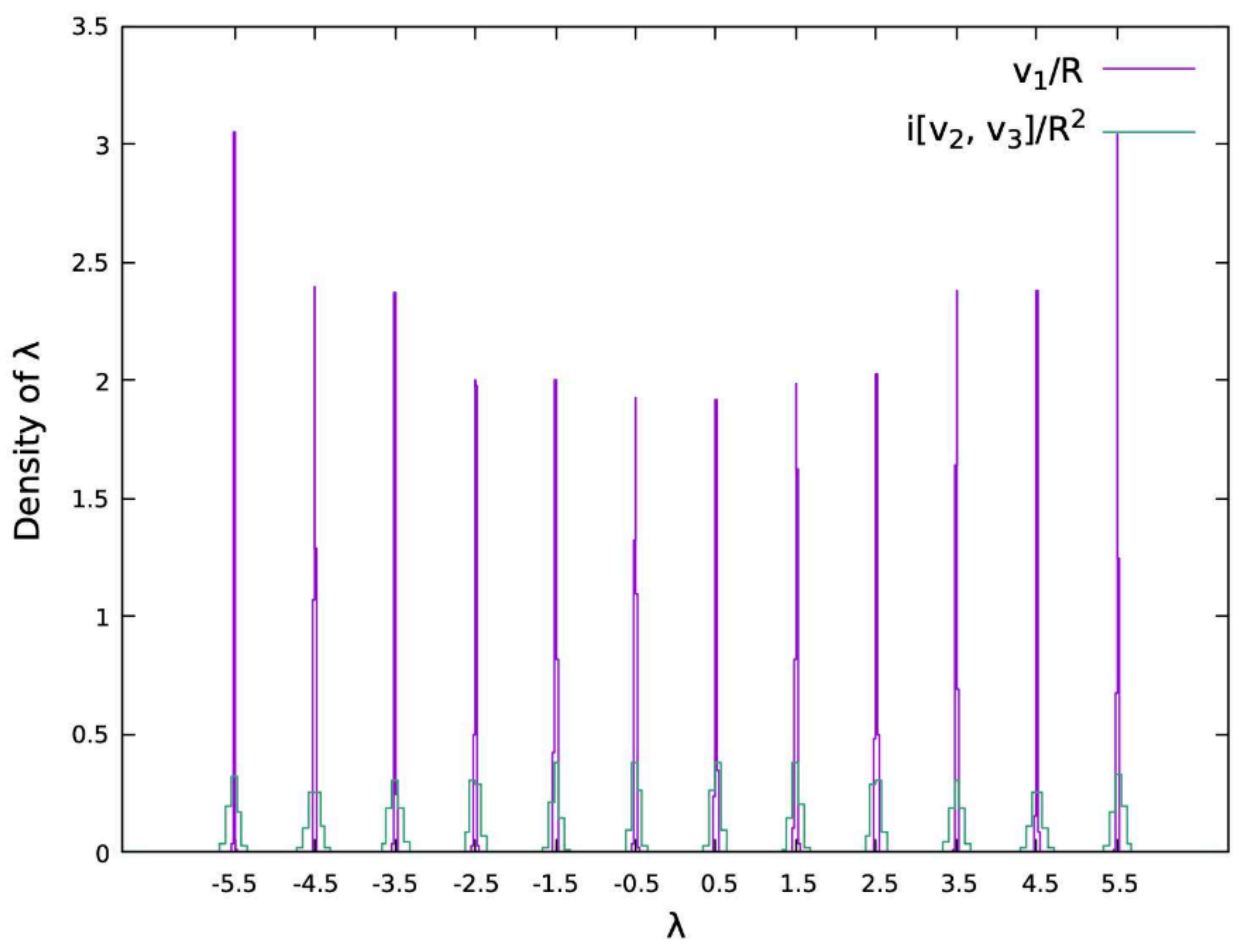


Figure 6.6: Model (0,3), eigenvalue density of v_1/R (purple) and $i[v_2, v_3]/R^2$ (green) for $n = 12$, $g_2 = -300$. The spectrum is compatible with an $su(2)$ solution.

Fermion integration

JWB 2024

Start with $(\mathscr{A}, \mathscr{H}, \mathscr{D}, J, \Gamma)$ with $s = 4$

Double to $s = 2$ spectral triple

$$\mathscr{H}' = \mathscr{H} \oplus \mathscr{H} \quad (= \mathscr{H}_+ \oplus \mathscr{H}_-)$$

$$D' = \begin{pmatrix} D & \bar{\mu}\Gamma \\ \mu\Gamma & D \end{pmatrix}, \mu \in \mathbb{C}$$

$$J' = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad \Gamma' = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}$$

$$\mathscr{A}' = \mathscr{A}$$

Then

$$\int_{\psi \in \mathscr{H}_+} e^{i\langle J\psi, D\psi\rangle} \, \mathrm{d}\psi$$

$$= e^{i\theta I_D/2} \sqrt[4]{\det(D^2 + |\mu|^2)}$$

$$e^{i\theta} = i\mu/|\mu|$$

Applications

Integrating over 1 generation → Induced SM action

JWB 2011, Stephan ~2011

Or, integrate over RH neutrinos → Induced SM action

Alex Nagen PhD thesis 2025

Dirac ensembles

Khalkhali, Pagliaroli, Verhoeven 2024