Formalising Noncommutative Geometry in Lean 4: First Steps Towards Connes' Reconstruction Theorem

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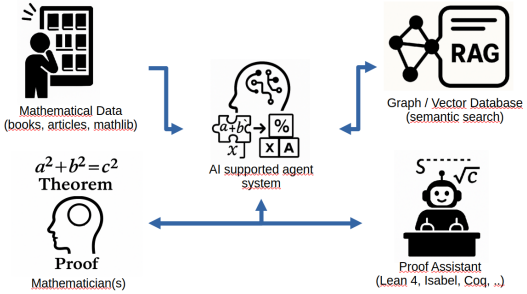
Mathematical Data, Proof Assistants, and AI



- Explosion of accessible mathematical data (arXiv, electronic books, Lean mathlib).
- Proof assistants becoming more user-friendly (e.g., Lean 4).
- AI could help organising mathematics and translating human proofs into machine-readable formats.

Overview of Proof Assistants

Proof assistants have been developed in computer science to check validity of algorithms and proofs.

What is a Proof Assistant?

- Interactive theorem prover
- Assists in developing formal proofs
- Checks correctness of logical steps
- Used in mathematics, computer science, formal verification
- Combines automated and interactive techniques

Examples of Popular Proof Assistants

- **Lean** - Calculus of Constructions (type theory), developed by Microsoft Research, user-friendly, powerful features
- **Coq** - Calculus of Constructions (type theory), developed by INRIA, formalizes mathematical proofs, verifies software
- **Isabelle** - Higher-order logic, developed by Tobias Nipkow and Lawrence Paulson, supports various logics, formal verification
- **HOL Light** - Higher-order logic, developed by Mike Gordon at Cambridge University, verifies mathematical proofs, hardware/software

Importance in Mathematics

- **Verification of complex proofs**
  - Four-color theorem (1976, verified 2005 in Coq by Gonthier)
  - Liquid Tensor Experiment by Scholze (Lean)
- **Formalizing mathematics** (focus on Lean 4)
  - Perfectoid spaces by Scholze, Buzzard, Commelin, Massot
  - Cap-set problem by Dahmen, Hölzl, Lewis
  - Continuum hypothesis by Han, van Doorn
  - PFR conjecture by Tao, Dillies, Mehta
  - Fermat's Last Theorem by Buzzard, Taylor
- **Reducing human error in proofs**
- **Facilitating collaboration**

Lean 4

Technical features of Lean 4:

- Lean 4 is a functional programming language and interactive proof assistant.
- Lean 4 is strongly typed (every variable has a fixed type, think of integers)
- Lean 4 is based on the **Calculus of Constructions** (dependent type theory)
- Fully extensible: users can modify the parser, elaborator, tactics, decision procedures, and more.

Practical features of Lean 4:

- Easily used in VS Code (via extension).
- Has a large library of formalized mathematics (mathlib).
- Has young and active community.
- Lean 4 is open source.

Lean 4 has been developed primarily by Leonardo de Moura at Microsoft Research.

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Curry-Howard-Lambek correspondence: Mathematical statements and proofs can be equivalently formulated in:

- Set theory + first-order logic,
- (Dependent) type theory, or
- Category theory.

Important for Lean 4:
Curry-Howard correspondence ensures
set theory + first-order logic $\Leftrightarrow$ dependent type theory

Dependent Type Theory II

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- $\mathbf{0}/\mathbf{1}$ denote the generic type with no/one element.
  Note: there are more types with no/one element

Dependent Type Theory III

Constructing types from types (a few examples)

- Function types:
  Given types A, B, the function type is denoted $A \to B$.
  Construction: if $f : A \to B$ and $a : A$ then $f(a) : B$.
  Note: $A \to B$ is not inhabited iff A is inhabited and B is not inhabited.
- Product types:
  Given types A, B, the procduct type is denoted by $A \times B$.
  Construction: if $a : A$ and $b : B$ then $(a, b) : A \times B$.
- Dependent function types:
  Given a type A and a universe $\mathcal{U}$ and $B : A \to \mathcal{U}$.
  The dependent function type is denoted by $\Pi_{(x:A)} B(x)$.
  Construction: if $f : \Pi_{(x:A)} B(x)$ and $a : A$ then $f(a) : B(a)$.

Propositions and Proofs in Type Theory

Propositions as types

- Prop is the universe of all propositions (true or false).
- Propositions are types $P : \text{Prop}$ with no or one representative.
  Example: $P = \text{Fermat's last theorem}$
- $P : \text{Prop}$ is true if it is inhabited, otherwise it's false.
- Proofs are terms $\text{proof} : P$ but not types.
  Example: $P = \text{Fermat's last theorem}$ is inhabited, i.e. there exists a proof.

Overview of Dependent Type Theory

Types	Logic	Sets
A	proposition	set
$a : A$	proof	element
$B(x)$	predicate	family of sets
$b(x) : B(x)$	conditional proof	family of elements
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$
$A + B$	$A \vee B$	disjoint union
$A \times B$	$A \wedge B$	set of pairs
$A \to B$	$A \Rightarrow B$	set of functions
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$

Figure: Overview from the HoTT Book

Dependent Type Theory and Lean 4

Generically Lean 4 requires constructive proofs.

Lean's Calculus of Constructions:
Calculus of Constructions is equivalent to ZF with intuitionistic logic.

Lean 4 can also work non-constructively, i.e. with classical logic.

Lean library for standard logic: `Classical`
Provides full ZFC with first order logic.

A "Simple" Lean 4 Example: There are infinitely many primes

- Lean 4 uses **tactics** to construct proofs.
- Tactic **by** starts a proof.
- Tactic **show** allows to show a goal (helpful for large proofs).
- Tactic **have** allows to introduce a new hypothesis.
- Tactic **sorry** declares a proof without providing it.

```
theorem primes_infinite : ∀ n, ∃ p > n, Nat.Prime p := by
  intro n
  have : 2 ≤ Nat.factorial (n + 1) + 1 := by
    sorry
  rcases exists_prime_factor this with ⟨p, pp, pdvd⟩
  refine ⟨p, ?_, pp⟩
  show p > n
  by_contra ple
  push_neg at ple
  have : p ∣ Nat.factorial (n + 1) := by
    sorry
  have : p ∣ 1 := by
    sorry
  show False
  sorry
```

[0]Formalisation from "Mathematics in Lean".

Connes' Reconstruction Theorem

Why formalising Noncommutative Geometry in Lean 4?

- Algebraic fields of mathematics are easier to formalise.
- NCG may provide back door to formalising Differential Geometry.
- Because I like NCG and want to formalise it.

The Aim: Reconstruction Theorem

Let (A, H, D) be a p-dimensional commutative spectral triple. There exist a compact oriented Riemannian p-manifold X, a Hermitian vector bundle $E \to X$, and an essentially self-adjoint Dirac-type operator D_E on E, such that

$$(A, H, D) \cong (C^\infty(X), L^2(X, E), D_E).$$

Where we stand: Functional Analysis & C^*-algebras

Definition/Theorem	**In mathlib**
Hilbert space and bounded/self-adjoint/compact operators	Yes
Compact operator and compact resolvent	Yes
*-algebra and *-representation on Hilbert space	Partial
GNS construction from a state on a C*-algebra	No
C*-algebra and C*-norm identity	Yes
Gelfand–Naimark theorem for C*-algebras	Partial
Gelfand duality for commutative C*-algebras	Yes
Compact operators $K(H)$	Yes
Fredholm operators and Fredholm modules	Partial
Functional calculus for self-adjoint operators	Partial

Where we stand: Smooth Manifolds & Vector Bundles

Definition/Theorem	**In mathlib**
Smooth manifold (Hausdorff, second-countable, C^∞-atlas)	Yes
Algebra $C^\infty(X)$ of smooth functions	Yes
Complex vector bundle and smooth sections	Partial
Serre–Swan theorem	No
Riemannian metric on a manifold	Partial
Orientation and oriented manifold	No
Spin and spin^c structures	No
Spinor bundle from spin^c structure	No
Dirac-type operator on Hermitian bundle	No
Spectral theorem for self-adjoint elliptic operators	Partial

Where we stand: Spectral Triples

Definition/Theorem	**In mathlib**
Spectral triple (A, H, D) with bounded commutators	No
Summability: $\mathrm{Tr}(\lvert D\rvert^{-s}) < \infty$ for $s > p$	No
Dimension spectrum of a spectral triple	No
Regularity: commutators with $\lvert D\rvert$ densely defined	No
Smooth domain of D	No
$H_\infty = \bigcap \mathrm{Dom}(D^k)$	No
Finiteness: H_∞ is finitely generated projective A-module	No
Orientability: Hochschild p-cycle with $\pi_D(c) = \gamma$	No
Hochschild complex and Hochschild homology	No
Grading γ with $\gamma D = -D\gamma$	No
Real structure J on H	No
First-order condition $[[D, a], Jb^*J^{-1}] = 0$	No

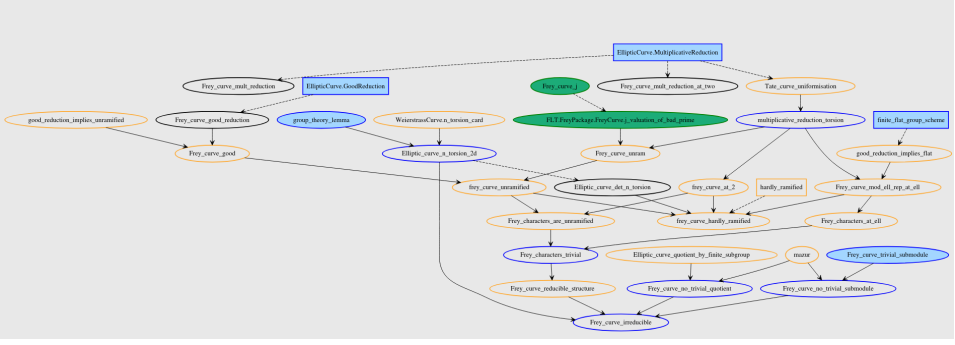
Challenges with Lean 4

- **Learning curve:** Lean 4 has a steep learning curve, especially for beginners.
- **Precision:** Everything has to be spelled out. No "put it into coordinates and you see..."', "as one easily sees...", "trivially...", etc.
- **Complexity:** Lean 4's syntax and semantics can be complex and difficult to understand (although it is improving and better than in Lean 3).
- **Limited libraries:** Many mathematical concepts are not yet formalized in Lean 4.
- **Documentation:** Documentation for Lean 4 is still evolving, making it challenging to find information.
- **Searchability:** Finding specific definitions or theorems in **mathlib** can be difficult due to the large amount of data and naming conventions.

The iPhone-moment for Lean 4 is still to come.

Graphical visualisation of Mathematics and Lean 4 code I

Lean blueprints: plasTeX plugin to generate graph representations of proofs

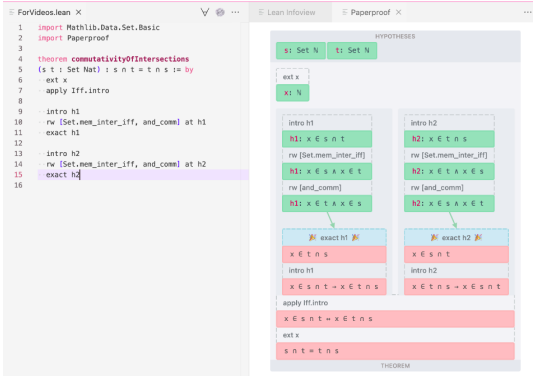


Graphical blueprint representation of the proof of Fermat's Theorem. (K. Buzzard, R. Taylor)

Graphical visualisation of Mathematics and Lean 4 code II

Paperproof:

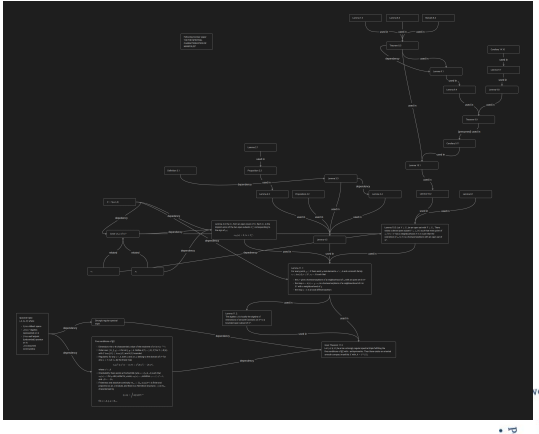
- VS Code extension for Lean 4
- Graph representation of Lean 4 code
- Not interactive yet



Current IT Project

Aim: Graph representation of mathematical texts and Lean 4 code.

- Interactive graph of definitions, theorems & proofs
- Bidirectional sync: graph ↔ Lean code
- Nodes/edges show proof status & presence in Lean
- Works with both math text and Lean code
- VS Code plugin for Lean 4
- AI assistance for graph/code generation



Graph of the reconstruction theorem.

Resources

Lean Game Server

Lean Main

Lean Community

Lean mathlib

Lean blueprint

Paperproof

People working on NCG in Lean 4

C. Stephan (Potsdam)

J. Taylor (Potsdam)

L. Ronge (Potsdam)

F. Hanisch (Potsdam)