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CIRM

T-Minkowski noncommutative spacetimes

$$[\hat a^\mu, \hat a^\nu] = i\,\theta^{\rho\sigma}\left(\delta^{[\mu}{}_\rho\,\delta^{\nu]}{}_\sigma - \hat\Lambda^{[\mu}{}_\rho\,\hat\Lambda^{\nu]}{}_\sigma\right) + i\,c^{\mu\nu}{}_\rho\,\hat a^\rho\,, \quad \hat\Lambda^\mu{}_\rho\hat\Lambda^\nu{}_\sigma\eta_{\mu\nu} = \eta_{\rho\sigma}\,,$$

$$[\hat\Lambda^\mu{}_\rho, \hat\Lambda^\nu{}_\sigma] = 0\,, \quad [\hat\Lambda^\mu{}_\rho, \hat a^\nu] = i\,f^{\alpha\beta}{}_\gamma\left(\hat\Lambda^\mu{}_\alpha\,\hat\Lambda^\nu{}_\beta\,\delta^\gamma{}_\rho - \delta^\mu{}_\alpha\,\delta^\nu{}_\beta\,\hat\Lambda^\gamma{}_\rho\right),$$

$$[\hat x^\mu, \hat x^\nu] = i\,\theta^{\mu\nu} + i\,c^{\mu\nu}{}_\rho\,\hat x^\rho\,, \qquad [\hat x^\mu, d\hat x^\nu] = i\,f^{\nu\mu}{}_\rho\,d\hat x^\rho\,,$$

$$[\hat x^\mu_a, \hat x^\nu_b] = i\,\theta^{\mu\nu} - i\,(f^{\mu\nu}{}_\gamma + f^{\nu\mu}{}_\gamma)\,(\hat x^\rho_a - \hat x^\rho_b) + \tfrac{i}{2}\,c^{\mu\nu}{}_\rho\,(\hat x^\rho_a + \hat x^\rho_b)\,.$$

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Applications of NonCommutative Geometry to
Gauge Theories, Field Theories, and Quantum Space-Time

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Based on:

- F. Mercati,
  "T-Minkowski Noncommutative Spacetimes I: Poincaré Groups, Differential Calculi, and Braiding",
  Prog. Theor. Exp. Phys. ***2024***, *073B06 (2024) arXiv:2404.08729*
- F. Mercati,
  "T-Minkowski Noncommutative Spacetimes II: Classical Field Theory"
  Prog. Theor. Exp. Phys. ***2024***, *123B05 (2024) arXiv:2311.16249*
- G. Fabiano, F. Mercati,
  "Multiparticle states in braided lightlike κ-Minkowski noncommutative QFT"
  Phys. Rev. ***D109***, *046011 (2024) arXiv:2310.15063*
- F. Lizzi, F. Mercati,
  "κ-Poincaré comodules, braided tensor products, and noncommutative quantum field theory"
  Phys. Rev. ***D103***, *126009 (2021) arXiv:2101.09683*

- I am interested in **quantum homogeneous spacetimes:** noncommutative spacetimes that are invariant under a quantum-group deformation of the Poincaré group.
- Noncommutativity, expressed through relations that break standard/commutative Poincaré invariance:

$$[\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i\, f^{\mu\nu}(\hat{\boldsymbol{x}})\,,$$

  actually appears the same in all inertial reference frames:

$$[\hat{\boldsymbol{x}}'^\mu, \hat{\boldsymbol{x}}'^\nu] = i\, f^{\mu\nu}(\hat{\boldsymbol{x}}')\,,$$

  $f^{\mu\nu}(\hat{\boldsymbol{x}})$, albeit having two spacetime indices, *is not a tensor:* its components do not rotate with a Lorentz transform.
- If $f^{\mu\nu}(\hat{\boldsymbol{x}})$ depends on some length scales, these will be the same to all (boosted) observers: new relativistic invariants besides c. **Doubly Special Relativity** principle.

[Amelino-Camelia, *Nature* ***418***, *34 (2002)*]

Algebraic formulation of Poincaré group

(Standard) Poincaré group as an Abelian Hopf algebra $\mathcal{P} = \mathrm{C}[\mathrm{ISO}(3,1)]$ generated by coordinates $\boldsymbol{a}^\mu$ and $\boldsymbol{\Lambda}^\mu{}_\nu$ s.t.:

$$\eta_{\mu\nu}\,\boldsymbol{\Lambda}^\mu{}_\rho\,\boldsymbol{\Lambda}^\nu{}_\sigma = \eta_{\rho\sigma}\,, \quad \eta^{\rho\sigma}\,\boldsymbol{\Lambda}^\mu{}_\rho\,\boldsymbol{\Lambda}^\nu{}_\sigma = \eta^{\mu\nu}\,, \quad \eta = \mathrm{diag}(-1,1,1,1)\,,$$

(commutative) pointwise product between functions:

$$(f\cdot g)(x) = f(x)g(x)\,, \qquad [\boldsymbol{a}^\mu, \boldsymbol{a}^\nu] = [\boldsymbol{\Lambda}^\mu{}_\rho, \boldsymbol{a}^\nu] = [\boldsymbol{\Lambda}^\mu{}_\rho, \boldsymbol{\Lambda}^\nu{}_\sigma] = 0\,,$$

group structure encoded via homomorphisms:

$$\Delta(\boldsymbol{\Lambda}^\mu{}_\nu) = \boldsymbol{\Lambda}^\mu{}_\rho \otimes \boldsymbol{\Lambda}^\rho{}_\nu\,, \qquad \epsilon(\boldsymbol{\Lambda}^\mu{}_\nu) = \delta^\mu{}_\nu\,, \quad S(\boldsymbol{\Lambda}^\mu{}_\nu) = (\boldsymbol{\Lambda}^{-1})^\mu{}_\nu\,,$$

$$\Delta(\boldsymbol{a}^\mu) = \boldsymbol{\Lambda}^\mu{}_\nu \otimes \boldsymbol{a}^\nu + \boldsymbol{a}^\mu \otimes 1\,, \quad \epsilon(\boldsymbol{a}^\mu) = 0\,, \qquad S(\boldsymbol{a}^\mu) = -(\boldsymbol{\Lambda}^{-1})^\mu{}_\nu\,\boldsymbol{a}^\nu\,,$$

(commutative) Minkowski space is a comodule $\mathcal{M} = \mathrm{C}[\mathbb{R}^{3,1}]$:

$$x'^\mu = \boldsymbol{\Lambda}^\mu{}_\nu \otimes x^\nu + \boldsymbol{a}^\mu \otimes 1\,.$$

Theta, Kappa and Zeta Poincaré

$$\eta_{\mu\nu}\,\mathbf{\Lambda}^\mu{}_\rho\,\mathbf{\Lambda}^\nu{}_\sigma = \eta_{\rho\sigma}\,, \qquad \eta^{\rho\sigma}\,\mathbf{\Lambda}^\mu{}_\rho\,\mathbf{\Lambda}^\nu{}_\sigma = \eta^{\mu\nu}\,, \qquad [\hat{\mathbf{\Lambda}}^\mu{}_\nu, \hat{\mathbf{\Lambda}}^\rho{}_\sigma] = 0\,,$$

	$[\hat{\boldsymbol{a}}^\mu, \hat{\boldsymbol{a}}^\nu] =$	$[\hat{\boldsymbol{a}}^\rho, \hat{\mathbf{\Lambda}}^\mu{}_\nu] =$
$\mathcal{P}_\theta$	$i\,\boldsymbol{\theta}^{\rho\sigma}\left(\delta^{[\mu}{}_\rho\,\delta^{\nu]}{}_\sigma\hat{\mathbf{1}} - \hat{\mathbf{\Lambda}}^{[\mu}{}_\rho\hat{\mathbf{\Lambda}}^{\nu]}{}_\sigma\right)$	0
$\mathcal{P}_\kappa$	$i\,(\boldsymbol{v}^\mu\,\hat{\boldsymbol{a}}^\nu - \boldsymbol{v}^\nu\,\hat{\boldsymbol{a}}^\mu)$	$i\,(\hat{\mathbf{\Lambda}}^\mu{}_\alpha\,\boldsymbol{v}^\alpha - \boldsymbol{v}^\mu)\,\hat{\mathbf{\Lambda}}^\rho{}_\nu +$ $i\,(\hat{\mathbf{\Lambda}}_{\beta\nu} - \eta_{\nu\beta}\hat{\mathbf{1}})\,\boldsymbol{v}^\beta\,\eta^{\mu\rho}$
$\mathcal{P}_\zeta$	$i\,\zeta^{[\mu}\left(\delta^{\nu]}_2\hat{\boldsymbol{a}}_1 - \delta^{\nu]}_1\hat{\boldsymbol{a}}_2\right)$ $\zeta^1 = \zeta^2 = 0$	$2\,i\,\zeta^\lambda\,\hat{\mathbf{\Lambda}}^\mu_\lambda\,\eta_{\rho[2}\,\hat{\mathbf{\Lambda}}^\nu{}_{1]} +$ $2\,i\,\zeta^\mu\delta^\nu{}_{[2}\hat{\mathbf{\Lambda}}_{1]\rho}$

$$\Delta(\hat{\mathbf{\Lambda}}^\mu{}_\nu) = \hat{\mathbf{\Lambda}}^\mu{}_\rho \otimes \hat{\mathbf{\Lambda}}^\rho{}_\nu\,, \qquad \epsilon(\hat{\mathbf{\Lambda}}^\mu{}_\nu) = \delta^\mu{}_\nu\,, \quad S(\hat{\mathbf{\Lambda}}^\mu{}_\nu) = (\hat{\mathbf{\Lambda}}^{-1})^\mu{}_\nu\,,$$

$$\Delta(\hat{\boldsymbol{a}}^\mu) = \hat{\mathbf{\Lambda}}^\mu{}_\nu \otimes \hat{\boldsymbol{a}}^\nu + \hat{\boldsymbol{a}}^\mu \otimes 1\,, \quad \epsilon(\hat{\boldsymbol{a}}^\mu) = 0\,, \qquad S(\hat{\boldsymbol{a}}^\mu) = -(\hat{\mathbf{\Lambda}}^{-1})^\mu{}_\nu\,\hat{\boldsymbol{a}}^\nu\,.$$

- **θ-Poincaré:** [Balachandran, Martone, *MPL**A24**, 1811 (2009)*]
- **κ-Poincaré:** [Lukierski, Nowicki, Ruegg, *PL**B271**, 321 (1991)*]
  [Lukierski, Ruegg *PL**B329**, 189 (1994)*]
  - ▶ v^μ space- and light-like:
    [Ballesteros, Herranz, del Olmo, Santander, *PL**B351**, 137 (1995)*]
- **ζ-Poincaré:** [Lukierski, Woronowicz, *PL**B633**, 116 (2006)*]
  - ▶ ζ^μ timelike (**ρ-Poincaré**):
    [Lizzi, Scala, Vitale, *PR**D106**, D106 (2022)*]
    [Fabiano, Gubitosi, Lizzi, Scala, Vitale, *JHEP **08**, 220 (2023)*]
  - ▶ ζ^μ spacelike (**λ-Poincaré**):
    [Gubitosi, Lizzi, Relancio, Vitale, *PR**D105**, 126013 (2022)*]

Theta, Kappa and Zeta Minkowski

$\mathcal{M}_\theta$	$[\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i\, \boldsymbol{\theta}^{\boldsymbol{\mu\nu}}\, \hat{\mathbf{1}}$
$\mathcal{M}_\kappa$	$[\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i \left(\boldsymbol{v}^{\boldsymbol{\mu}}\, \hat{\boldsymbol{x}}^\nu - \boldsymbol{v}^{\boldsymbol{\nu}}\, \hat{\boldsymbol{x}}^\mu \right)$
$\mathcal{M}_\zeta$	$[\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = 2\, i\, \boldsymbol{\zeta}^{[\boldsymbol{\mu}} \left(\eta^{\nu]2}\, \hat{\boldsymbol{x}}^1 - \eta^{\nu]1}\, \hat{\boldsymbol{x}}^2 \right)$

each covariant under

$$\hat{\boldsymbol{x}}'^\mu = \hat{\boldsymbol{\Lambda}}^\mu{}_\nu \otimes \hat{\boldsymbol{x}}^\nu + \hat{\boldsymbol{a}}^\mu \otimes 1\,.$$

- **θ-Minkowski:** 4D generalization of *Groenewold-Moyal plane*
  [Groenewold, *Physica* ***12****, 405 (1946)*]
  [Moyal, *Math. Proc. Camb. Philos. Soc.* ***45****, 45 (1949)*]
  [Connes, Douglas, Schwarz, *JHEP* ***003****, 9802 (1998)*]
  [Seiberg, Witten, *JHEP* ***032****, 9909 (1999)*]
  later ADS/CFT-motivated models θ-Poincaré invariant
  [Szabo, *CQG****23****, R199 (2006)*]
- **κ-Minkowski:** [Majid, Ruegg, *PL****B334****, 348 (1994)*]
- **ζ-Minkowski:** [Lukierski, Woronowicz, *PL****B633****, 116 (2006)*]

Meaning of comodule property

Notice that the standard coaction

$$\hat{\boldsymbol{x}}'^{\mu} = \hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu} \otimes \hat{\boldsymbol{x}}^{\nu} + \hat{\boldsymbol{a}}^{\mu} \otimes \hat{\mathbf{1}}\,,$$

makes $\mathcal{M}_\theta$ into a $\mathcal{P}_\theta$-comodule algebra:

$$[\hat{\boldsymbol{x}}'^{\mu}, \hat{\boldsymbol{x}}'^{\nu}] = \hat{\boldsymbol{\Lambda}}^{\mu}{}_{\rho} \hat{\boldsymbol{\Lambda}}^{\nu}{}_{\sigma} \otimes [\hat{\boldsymbol{x}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}] + [\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{a}}^{\nu}] \otimes \hat{\mathbf{1}} = i\, \boldsymbol{\theta}^{\boldsymbol{\mu\nu}}\, \hat{\mathbf{1}}\,.$$

Interpretation: the coordinate algebra looks the same in all inertial reference frames. For example, the uncertainty relations:

$$\delta(\hat{\boldsymbol{x}}^1)\delta(\hat{\boldsymbol{x}}^2) \geq \tfrac{1}{2}\,|\boldsymbol{\theta}^{\mathbf{12}}|\,, \qquad \delta(\hat{\boldsymbol{x}}'^1)\delta(\hat{\boldsymbol{x}}'^2) \geq \tfrac{1}{2}\,|\boldsymbol{\theta}^{\mathbf{12}}|\,,$$

look the same. $\boldsymbol{\theta}^{\boldsymbol{\mu\nu}}$ is not a tensor, **its components do not rotate under Lorentz transformations.**

Braided tensor product

We would like to have a notion of **multilocal** functions on spacetime, *i.e.* functions of more than one coordinate.

In the commutative case N-point functions belong to the tensor product algebra $\mathrm{C}[\mathbb{R}^{3,1}]^{\otimes N}$, generated by

$$x_1^\mu = x^\mu \otimes 1 \otimes \cdots \otimes 1\,, \qquad x_2^\mu = 1 \otimes x^\mu \otimes \cdots \otimes 1\,, \qquad \textit{etc.}$$

which is Abelian just like the 1-point algebra:

$$[x_a^\mu, x_b^\nu] = 0\,, \quad \forall a, b = 1, \dots, N\,.$$

In the noncommutative case the tensor product algebra is not covariant under the deformed Poincaré group action $\hat{\boldsymbol{x}}_a'^\mu = \hat{\boldsymbol{\Lambda}}^\mu{}_\nu \otimes \hat{\boldsymbol{x}}_a^\nu + \hat{\boldsymbol{a}}^\mu \otimes \hat{\mathbf{1}}^{\otimes N}$:

$$[\hat{\boldsymbol{x}}_a^\mu, \hat{\boldsymbol{x}}_b^\nu] = 0 \quad \Rightarrow \quad [\hat{\boldsymbol{x}}_a'^\mu, \hat{\boldsymbol{x}}_b'^\nu] \neq 0\,.$$

Easy to find a covariant generalization of the tensor product algebra:

θ) $[\hat{\boldsymbol{x}}^\mu_a, \hat{\boldsymbol{x}}^\nu_b] = i\, \boldsymbol{\theta^{\mu\nu}}\, \hat{\mathbf{1}}^{\otimes N}\,,$

[Aschieri, Dimitrijevic, Meyer, Wess, *CQG **23**, 1883 (2006)*]

[Fiore, Wess, *Phys. Rev.* ***D75**, 105022 (2007)*]

ζ) $[\hat{\boldsymbol{x}}^\mu_a, \hat{\boldsymbol{x}}^\nu_b] = -2i\eta_{\rho[1}\delta^{(\mu}{}_{2]}\boldsymbol{\zeta^{\nu)}} \left(\hat{\boldsymbol{x}}^\rho_a - \hat{\boldsymbol{x}}^\rho_b\right) + 2i\eta_{\rho[1}\delta^{[\nu}{}_{2]}\boldsymbol{\zeta^{\mu]}} \left(\hat{\boldsymbol{x}}^\rho_a + \hat{\boldsymbol{x}}^\rho_b\right)\,,$

[Mercati, *Prog. Theor. Exp. Phys.*, ***2024**, 073B06 (2024)*]

κ) $[\hat{\boldsymbol{x}}^\mu_a, \hat{\boldsymbol{x}}^\nu_b] = i\boldsymbol{v^{[\mu}}\,(\,\hat{\boldsymbol{x}}^{\nu]}_a + \hat{\boldsymbol{x}}^{\nu]}_b) + i(\boldsymbol{v^{(\mu}}\,\delta^{\nu)}{}_\rho - v_\rho\,\eta^{\mu\nu})\,\left(\hat{\boldsymbol{x}}^\rho_a - \hat{\boldsymbol{x}}^\rho_b\right)\,.$

However, Jacobi rules: $[\hat{\boldsymbol{x}}^\mu_a, [\hat{\boldsymbol{x}}^\nu_b, \hat{\boldsymbol{x}}^\rho_c]] + \textit{cyclic} = 0$ satisfied *iff* $\boldsymbol{v^\mu v_\mu} = 0$: the *lightlike* κ-Minkowski model of Ballesteros *et al.*

[Lizzi, Mercati, *Phys. Rev.* ***D103**, 126009 (2021)*]

Differential calculus

Covariant braided tensor product $\Rightarrow$ covariant 4D differential calculus.

In fact, a differential behaves exactly like the difference between the coordinates of two points, $\Delta\hat{\boldsymbol{x}}^\mu_{ab} = \hat{\boldsymbol{x}}^\mu_a - \hat{\boldsymbol{x}}^\mu_b$:

$$\Delta\hat{\boldsymbol{x}}'^\mu_{ab} = \hat{\boldsymbol{\Lambda}}^\mu{}_\nu \otimes \Delta\hat{\boldsymbol{x}}^\nu_{ab} \,,$$

$$\begin{aligned}
\text{ζ-Minkowski}: \quad [\hat{\boldsymbol{x}}^\mu_a, \Delta\hat{\boldsymbol{x}}^\nu_{bc}] \quad &= 0 \\
\text{ζ-Minkowski}: \quad [\hat{\boldsymbol{x}}^\mu_a, \Delta\hat{\boldsymbol{x}}^\nu_{bc}] \quad &= 2i\, \eta_{\rho[1}\delta^\nu{}_{2]} \boldsymbol{\zeta}^{\boldsymbol{\mu}}\, \Delta\hat{\boldsymbol{x}}^\rho_{bc} \,, \\
\text{κ-Minkowski}: \quad [\hat{\boldsymbol{x}}^\mu_a, \Delta\hat{\boldsymbol{x}}^\nu_{bc}] \quad &= i\, \boldsymbol{v}^{\boldsymbol{\nu}}\, \Delta\hat{\boldsymbol{x}}^\mu_{bc} - i\, \boldsymbol{v}_{\boldsymbol{\rho}}\, \eta^{\mu\nu}\, \Delta\hat{\boldsymbol{x}}^\rho_{bc} \,.
\end{aligned}$$

(iff $\boldsymbol{v}^{\boldsymbol{\mu}} \boldsymbol{v}_{\boldsymbol{\mu}} = 0$).

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Covariant braided tensor product $\Rightarrow$ covariant 4D differential calculus.

In fact, a differential behaves exactly like the difference between the coordinates of two points, $\Delta\hat{x}^\mu_{ab} = \hat{x}^\mu_a - \hat{x}^\mu_b$:

$$d\hat{x}'^\mu = \hat{\Lambda}^\mu{}_\nu \otimes d\hat{x}^\nu \,,$$

ζ-Minkowski : $[\hat{x}^\mu_a, d\hat{x}^\nu] = 0$

ζ-Minkowski : $[\hat{x}^\mu_a, d\hat{x}^\nu] = 2i\,\eta_{\rho[1}\delta^\nu{}_{2]}\boldsymbol{\zeta}^{\boldsymbol{\mu}}\, d\hat{x}^\rho \,,$

κ-Minkowski : $[\hat{x}^\mu_a, d\hat{x}^\nu] = i\,\boldsymbol{v}^{\boldsymbol{\nu}}\, d\hat{x}^\mu - i\,\boldsymbol{v}_{\boldsymbol{\rho}}\,\eta^{\mu\nu}\, d\hat{x}^\rho \,.$

(iff $\boldsymbol{v}^{\boldsymbol{\mu}}\boldsymbol{v}_{\boldsymbol{\mu}} = 0$).

R matrix formulation

All these models (θ, ζ, κ) can be written in terms of a 5D representation of an **R-matrix:**

$$\boldsymbol{R}^{\boldsymbol{AB}}_{\boldsymbol{EF}}\, \hat{\boldsymbol{T}}^{E}{}_{C}\, \hat{\boldsymbol{T}}^{F}{}_{D} = \hat{\boldsymbol{T}}^{B}{}_{G}\, \hat{\boldsymbol{T}}^{A}{}_{H}\, \boldsymbol{R}^{\boldsymbol{HG}}_{\boldsymbol{CD}}\,, \qquad \hat{\boldsymbol{x}}^{A}\, \hat{\boldsymbol{x}}^{B} = \boldsymbol{R}^{\boldsymbol{BA}}_{\boldsymbol{CD}}\, \hat{\boldsymbol{x}}^{C}\, \hat{\boldsymbol{x}}^{D}\,,$$

$$\hat{\boldsymbol{x}}^{A}_{a}\, \hat{\boldsymbol{x}}^{B}_{b} = \boldsymbol{R}^{\boldsymbol{BA}}_{\boldsymbol{CD}}\, \hat{\boldsymbol{x}}^{C}_{b}\, \hat{\boldsymbol{x}}^{D}_{a}\,, \qquad \hat{\boldsymbol{x}}^{A}\, d\hat{\boldsymbol{x}}^{B} = \boldsymbol{R}^{\boldsymbol{BA}}_{\boldsymbol{CD}}\, \hat{\boldsymbol{x}}^{C}\, d\hat{\boldsymbol{x}}^{D}\,,$$

where $\boldsymbol{R}^{\boldsymbol{AB}}_{\boldsymbol{CD}} \in \mathbb{R}$ and:

$$\hat{\boldsymbol{T}}^{\mu}{}_{\nu} = \hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}\,, \quad \hat{\boldsymbol{T}}^{\mu}{}_{4} = \hat{\boldsymbol{a}}^{\mu}\,, \quad \hat{\boldsymbol{T}}^{4}{}_{\mu} = 0\,, \quad \hat{\boldsymbol{T}}^{4}{}_{4} = \hat{\mathbf{1}}\,,$$

$$\hat{\boldsymbol{x}}^{A} = (\hat{\boldsymbol{x}}^{\mu}, \hat{\mathbf{1}})\,, \qquad \hat{\boldsymbol{x}}^{A}_{a} = (\hat{\boldsymbol{x}}^{\mu}_{a}, \hat{\mathbf{1}})\,, \qquad d\hat{\boldsymbol{x}}^{A} = (d\hat{\boldsymbol{x}}^{\mu}, 0)\,.$$

All of the above are covariant under:

$$\hat{\boldsymbol{x}}'^{A} = \hat{\boldsymbol{T}}^{A}{}_{B} \otimes \hat{\boldsymbol{x}}^{B}\,, \qquad \hat{\boldsymbol{x}}'^{A}_{a} = \hat{\boldsymbol{T}}^{A}{}_{B} \otimes \hat{\boldsymbol{x}}^{B}_{a}\,,$$

$$d\hat{\boldsymbol{x}}'^{A} = \hat{\boldsymbol{T}}^{A}{}_{B} \otimes d\hat{\boldsymbol{x}}^{B}\,, \qquad \hat{\boldsymbol{T}}'^{A}{}_{B} = \hat{\boldsymbol{T}}^{A}{}_{C} \otimes \hat{\boldsymbol{T}}^{C}{}_{B}\,.$$

Generalization: T-Poincaré models

Quantum Poicaré groups ($\hat{\mathbf{\Lambda}}^{\mu}{}_{\rho}\hat{\mathbf{\Lambda}}^{\nu}{}_{\sigma}\eta_{\mu\nu} = \eta_{\rho\sigma}$ and coalgebra undeformed):

$$[\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{a}}^{\nu}] = i\, \boldsymbol{\theta}^{\boldsymbol{\rho\sigma}} \left(\delta^{[\mu}{}_{\rho}\, \delta^{\nu]}{}_{\sigma} - \hat{\mathbf{\Lambda}}^{[\mu}{}_{\rho}\, \hat{\mathbf{\Lambda}}^{\nu]}{}_{\sigma}\right) + i\, (\boldsymbol{f}^{\boldsymbol{\nu\mu}}{}_{\boldsymbol{\rho}} - \boldsymbol{f}^{\boldsymbol{\mu\nu}}{}_{\boldsymbol{\rho}})\, \hat{\boldsymbol{a}}^{\rho},$$

$$[\hat{\mathbf{\Lambda}}^{\mu}{}_{\rho}, \hat{\mathbf{\Lambda}}^{\nu}{}_{\sigma}] = 0\,, \quad [\hat{\mathbf{\Lambda}}^{\mu}{}_{\rho}, \hat{\boldsymbol{a}}^{\nu}] = i\, \boldsymbol{f}^{\boldsymbol{\alpha\beta}}{}_{\boldsymbol{\gamma}} \left(\hat{\mathbf{\Lambda}}^{\mu}{}_{\alpha}\, \hat{\mathbf{\Lambda}}^{\nu}{}_{\beta}\, \delta^{\gamma}{}_{\rho} - \delta^{\mu}{}_{\alpha}\, \delta^{\nu}{}_{\beta}\, \hat{\mathbf{\Lambda}}^{\gamma}{}_{\rho}\right).$$

quantum Minkowski spaces:

$$[\hat{\boldsymbol{x}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}] = i\, \boldsymbol{\theta}^{\boldsymbol{\mu\nu}}\, \hat{\mathbf{1}}\,,$$

braided tensor product:

$$[\hat{\boldsymbol{x}}^{\mu}_a, \hat{\boldsymbol{x}}^{\nu}_b] = i\, \boldsymbol{\theta}^{\boldsymbol{\mu\nu}}\, \hat{\mathbf{1}} - i\, (\boldsymbol{f}^{\boldsymbol{\mu\nu}}{}_{\boldsymbol{\gamma}} + \boldsymbol{f}^{\boldsymbol{\nu\mu}}{}_{\boldsymbol{\gamma}})\, \left(\hat{\boldsymbol{x}}^{\rho}_a - \hat{\boldsymbol{x}}^{\rho}_b\right) + \tfrac{i}{2}\, \boldsymbol{c}^{\boldsymbol{\mu\nu}}{}_{\boldsymbol{\rho}} \left(\hat{\boldsymbol{x}}^{\rho}_a + \hat{\boldsymbol{x}}^{\rho}_b\right),$$

differential calculus:

$$[\hat{\boldsymbol{x}}^{\mu}, d\hat{\boldsymbol{x}}^{\nu}] = i\, \boldsymbol{f}^{\boldsymbol{\nu\mu}}{}_{\boldsymbol{\rho}}\, d\hat{\boldsymbol{x}}^{\rho}\,.$$

Jacobi identities

Imposing **Jacobi identities** ($c^{\mu\nu}{}_\rho = f^{\nu\mu}{}_\rho - f^{\mu\nu}{}_\rho$):

- Jacobi identities for structure constants $c^{\mu\nu}_\rho = f^{\nu\mu}{}_\rho - f^{\mu\nu}{}_\rho$:

$$c^{\mu\nu}{}_\lambda\, c^{\lambda\rho}{}_\sigma - c^{\mu\rho}{}_\lambda c^{\lambda\nu}{}_\sigma - c^{\nu\rho}{}_\lambda\, c^{\mu\lambda}{}_\sigma = 0\,,$$

- 2-cocycle condition for $\theta^{\mu\nu}$:

$$c^{\mu\nu}{}_\lambda\, \theta^{\rho\lambda} + c^{\rho\mu}{}_\lambda\, \theta^{\nu\lambda} + c^{\nu\rho}{}_\lambda\, \theta^{\mu\lambda} = 0\,,$$

- constraints on $f^{\mu\nu}{}_\rho$:

$$f^{\alpha\mu}{}_\lambda\, f^{\lambda\nu}{}_\beta - f^{\alpha\nu}{}_\lambda\, f^{\lambda\mu}{}_\beta = -c^{\mu\nu}{}_\rho\, f^{\alpha\rho}{}_\beta\,, \qquad \eta^{\theta(\sigma}\, f^{\varphi)\lambda}{}_\theta = 0\,,$$

Constraints on the 64 $f^{\mu\nu}{}_{\rho}$ parameters

The last constraints have a simple meaning:

- calling $(K^{\mu})^{\alpha}{}_{\beta} = f^{\alpha\mu}{}_{\beta}$, the linear constraint

$$\eta^{\theta(\sigma}\, f^{\varphi)\lambda}{}_{\theta} = 0\,, \quad \Rightarrow \quad (K^{\mu})^{\alpha\beta} = -(K^{\mu})^{\beta\alpha}\,,$$

  implies that $(K^{\mu})^{\alpha}{}_{\beta}$ are four **4D Lorentz algebra matrices.**

- The quadratic constraint

$$f^{\alpha\mu}{}_{\lambda}\, f^{\lambda\nu}{}_{\beta} - f^{\alpha\nu}{}_{\lambda}\, f^{\lambda\mu}{}_{\beta} = -c^{\mu\nu}{}_{\rho}\, f^{\alpha\rho}{}_{\beta}\,,$$

  implies that K^{μ} close a representation of the Lie algebra:

$$[K^{\mu}, K^{\nu}] = -c^{\mu\nu}{}_{\rho}\, K^{\rho}\,.$$

Characterization of the R-matrix

We get $\boldsymbol{R} = 1 \otimes 1 + i\,\boldsymbol{r}$, where $\boldsymbol{r}$ is a **triangular, nilpotent r-matrix:**

$$\boldsymbol{r} = -\theta^{\mu\nu}\, P_\mu \otimes P_\nu + K^\mu \wedge P_\mu\,,$$

where $(P_\mu)^A{}_B = \delta^A{}_\mu\, \delta^4{}_B$, $\quad (K_\mu)^A{}_B = \delta^A{}_\alpha\, \delta^\beta{}_B\, \boldsymbol{f}^{\alpha\mu}{}_{\beta}$.

Nilpotency of $\boldsymbol{r}$ $\Rightarrow$ the **Quantum Yang-Baxter Equation (QYBE)**:

$$\boldsymbol{R}^{AB}_{IJ}\, \boldsymbol{R}^{IC}_{LK}\, \boldsymbol{R}^{JK}_{MN} = \boldsymbol{R}^{BC}_{JK}\, \boldsymbol{R}^{AK}_{IN}\, \boldsymbol{R}^{IJ}_{LM}$$

reduces to the **Classical Yang-Baxter Equation (CYBE)**

$$[\boldsymbol{r}_{12}, \boldsymbol{r}_{13}] + [\boldsymbol{r}_{12}, \boldsymbol{r}_{23}] + [\boldsymbol{r}_{13}, \boldsymbol{r}_{23}] = 0\,,$$

which, in turn is equivalent to the **Jacobi identities** from before.

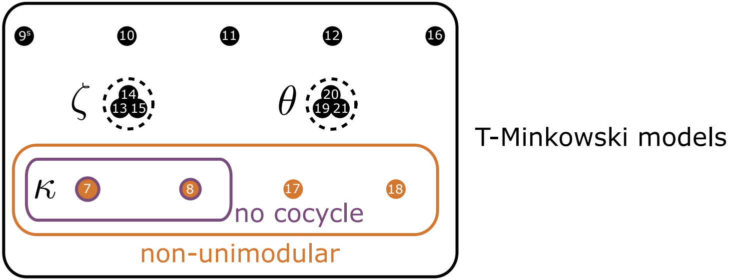
Classification of classical r-matrices

[Zakrzewski, *Commun. Math. Phys.* ***185***, *285 (1997), q-alg/9602001v1*]

classified the solutions of the CYBE for the Poincaré group, and found 16 automorphism classes (plus six solutions of the mCYBE).

Two mistakes amended in: [Tolstoy, *arXiv:0712.3962*]

11 multiparametric families of models, of which three have names:



...there are other 21 letters in standard Ionic alphabet 😉

The κ-lightlike model is generalized to a two-parameter family:

$$[\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^1] = -\frac{i}{\boldsymbol{\kappa}} \left(\hat{\boldsymbol{x}}^1 + \zeta^{\mathbf{(7)}}\, \hat{\boldsymbol{x}}^2\right), \quad [\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^2] = -\frac{i}{\boldsymbol{\kappa}} \left(\hat{\boldsymbol{x}}^2 - \zeta^{\mathbf{(7)}}\, \hat{\boldsymbol{x}}^1\right),$$

$$[\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2] = 0\,, \qquad [\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^3] = \frac{i}{\boldsymbol{\kappa}} \left(\hat{\boldsymbol{x}}^0 - \hat{\boldsymbol{x}}^3\right),$$

$$[\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^3] = \frac{i}{\boldsymbol{\kappa}} \left(\hat{\boldsymbol{x}}^1 + \zeta^{\mathbf{(7)}}\, \hat{\boldsymbol{x}}^2\right), \qquad [\hat{\boldsymbol{x}}^2, \hat{\boldsymbol{x}}^3] = \frac{i}{\boldsymbol{\kappa}} \left(\hat{\boldsymbol{x}}^2 - \zeta^{\mathbf{(7)}}\, \hat{\boldsymbol{x}}^1\right).$$

Quantum embeddable spaces and bicrossproduct structure

$\hat{\boldsymbol{x}}^A \, \hat{\boldsymbol{x}}^B = \boldsymbol{R}^{BA}_{CD} \, \hat{\boldsymbol{x}}^C \, \hat{\boldsymbol{x}}^D$ is a quantum homogeneous space in the sense of a **comodule algebra,** $\hat{\boldsymbol{x}}'^A = \hat{\boldsymbol{T}}^A{}_B \otimes \hat{\boldsymbol{x}}^B$.

Stronger requirement: **quantum embeddable space**

$$\hat{\boldsymbol{x}}^A = \hat{\boldsymbol{T}}^A{}_4 \, ,$$

see, *e.g.*, [Brzezinski, *J. Math. Phys.* ***37***, *(1996), q-alg/9509015*]

fulfilled *iff* $\boldsymbol{\theta^{\mu\nu} = 0}$.

As is the conditions for a **bicrossproduct structure.** T-Minkowski models generalize bicrossproduct.

Field theory on T-Minkowski spacetimes

$$\mathcal{M}_\ell \ : \quad [\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i\, \boldsymbol{\theta}^{\boldsymbol{\mu\nu}}\, \hat{\mathbf{1}} + i\, \boldsymbol{c}^{\boldsymbol{\mu\nu}}{}_{\boldsymbol{\rho}}\, \hat{\boldsymbol{x}}^\rho \,,$$

is a central extension of a Lie algebra $\mathfrak{g}$.

[Agostini, Lizzi, Zampini, *Mod. Phys. Lett.* ***A17***, *2105 (2002)*]

introduced **generalized Weyl systems,** extending the theory of Weyl maps from Moyal/θ to centrally-extended Lie algebras.

Choose a basis of **ordered plane waves**:

$$\hat{\boldsymbol{E}}[k] = :\ e^{i\, k_\mu \hat{\boldsymbol{x}}^\mu}\ := \sum_{n=0}^{\infty} \frac{1}{n!} : (i\, k_\mu \hat{\boldsymbol{x}}^\mu)^n\ : ,$$

: _ : = ordering/group factorization (*i.e.* a *Weyl map*).

$$\hat{\boldsymbol{E}}[k]\,\hat{\boldsymbol{E}}[q] = e^{i\,\theta_{\mu\nu}\,\Phi^{\mu\nu}(k)}\hat{\boldsymbol{E}}[\Delta(k,q)]\,, \quad \hat{\boldsymbol{E}}^*[k] = \hat{\boldsymbol{E}}\left[S(k)\right]\,, \quad \hat{\boldsymbol{E}}[\epsilon(k)] = \hat{\mathbf{1}}\,,$$

$(\mathbb{R}^4, \Delta, S, \epsilon)$ close the commutative Hopf algebra of functions on Lie group $\mathfrak{g}^*$. $\boldsymbol{\theta^{\mu\nu}}$ twists this algebra.

Ordering change = (nonlinear) coordinate changes on momentum space

$$\hat{\boldsymbol{E}}[k] = e^{i\,\theta_{\mu\nu}\,\varphi^{\mu\nu}(k')}\hat{\boldsymbol{F}}[k']\,, \qquad k'_\mu = k'_\mu(k)\,.$$

Left- and right-invariant Haar measures

$$d\mu^{\mathrm{L}}[\Delta(q,p)]\big|_q = d\mu^{\mathrm{L}}(p)\,, \qquad d\mu^{\mathrm{R}}[\Delta(q,p)]\big|_p = d\mu^{\mathrm{R}}(q)\,,$$

modular function

$$d\mu^{\mathrm{R}}(k) = \mathscr{T}(k)\,d\mu^{\mathrm{L}}(k)\,.$$

Noncommutative Fourier theory

Generic element of $\mathcal{M}_\ell$ (function on T-Minkowski space):

$$f(\hat{\boldsymbol{x}}) = \int d\mu^{\mathrm{L}}(k)\, \tilde{f}_{\mathrm{L}}(k)\, \hat{\boldsymbol{E}}[k] = \int d\mu^{\mathrm{R}}(k)\, \tilde{f}_{\mathrm{R}}(k)\, \hat{\boldsymbol{E}}[k]\,,$$

$\tilde{f}_{\mathrm{R}}(k) = \mathscr{T}(k)\, \tilde{f}_{\mathrm{L}}(k) \in$ a space of commutative functions (*i.e.* Schwarz).

Ordering change $\hat{\boldsymbol{E}}[k] \to \hat{\boldsymbol{F}}[k]$ = pull-back of diffeomorphism on $\tilde{f}_{\mathrm{L}}$ ($\tilde{f}_{\mathrm{R}}$).

Noncommutative integral

$\mathcal{M}_\ell$-linear map

$$\int d^4\hat{\boldsymbol{x}} : \mathcal{M}_\ell \to \mathbb{C}\,,$$

such that

$$\int d^4\hat{\boldsymbol{x}}\,\hat{\boldsymbol{E}}[k] = \int d^4\hat{\boldsymbol{x}}\,\hat{\boldsymbol{F}}[k] = \delta^{(4)}(k)\,, \quad \int d^4\hat{\boldsymbol{x}}\,f(\hat{\boldsymbol{x}}) = \tilde{f}_{\mathrm{L}}(o) = \tilde{f}_{\mathrm{R}}(o)\,,$$

invariant under T-Poincaré coaction:

$$\left(\mathrm{id} \otimes \int d^4\hat{\boldsymbol{x}}\right) \circ f(\hat{\boldsymbol{\Lambda}}^\mu{}_\nu \otimes \hat{\boldsymbol{x}}^\nu + \hat{\boldsymbol{a}}^\mu \otimes \hat{\mathbf{1}}) = \hat{\mathbf{1}} \otimes \int d^4\hat{\boldsymbol{x}}\,f(\hat{\boldsymbol{x}})\,,$$

In non-unimodular models, **twisted cyclic:**

$$\int d^4\hat{\boldsymbol{x}}\,f(\hat{\boldsymbol{x}})\,g(\hat{\boldsymbol{x}}) = \int d^4\hat{\boldsymbol{x}}\,[\mathscr{T} \triangleright g(\hat{\boldsymbol{x}})]\,f(\hat{\boldsymbol{x}}) = \int d^4\hat{\boldsymbol{x}}\,g(\hat{\boldsymbol{x}})\left[\mathscr{T}^{-1} \triangleright f(\hat{\boldsymbol{x}})\right]\,.$$

Differential calculus

$$d(\hat{\boldsymbol{x}}^\mu) = d\hat{\boldsymbol{x}}^\mu\,, \qquad d(\hat{f}\,\hat{g}) = d\hat{f}\,\hat{g} + \hat{f}\,d\hat{g}\,, \qquad d^2 = 0\,,$$

$$df(\hat{\boldsymbol{x}}) = i\,d\hat{\boldsymbol{x}}^\mu\,[\xi_\mu \triangleright f(\hat{\boldsymbol{x}})]\,, \qquad \xi_\mu \triangleright \hat{\boldsymbol{E}}[k] = i\,\xi_\mu(k)\,\hat{\boldsymbol{E}}[k]\,,$$

introduced in [Woronowicz, *Commun. Math. Phys.* ***122***, *125 (1989)*]

I have a closed-form formula for $\xi_\mu(k)$ for any T-Minkowski model.

k_μ transform nonlinearly under Lorentz transformations. $\xi_\mu(k)$ transforms covariantly, $\xi_\mu(k) \to \hat{\boldsymbol{\Lambda}}^\nu{}_\mu\,\xi_\nu(k)$.

The momentum-space coordinates ξ_μ are such that $d\mu^{\rm L}[k] = d^4\xi$. Momentum space is flat.

We all of this, following Woronowicz:

- I can define a (trivial) exterior algebra, Hodge-$*$, Lie and inner derivatives.
- I can define Klein-Gordon and Dirac fields modulo an ordering ambiguity of the action, which disappears in unimodular models.
- The spaces of classical solutions is invariant under changes of ordering choice.
- Gauge theories are problematic in non-unimodular models:

$$\int F' \wedge * F' = \int U^* \, F \wedge *(F) \, U = \int \mathscr{T}(U) \, U^* \, F \wedge *(F) \neq \int F \wedge *(F)$$

at least 5 solutions to this problem on the market:

[Aschieri, Castellani, *Gen. Rel. Grav.* ***45*** *45 (2013), arXiv:1205.1911*]
[Dimitrijevic, Jonke, Pachol, *SIGMA* ***10****, 10 (2014), arXiv:1403.1857*]
[Mathieu, Wallet, *JHEP* ***05****, 112 (2020) , arXiv:2002.02309*]
[Kupriyanov, Kurkov, Vitale, *JHEP* ***01****, 102 (2021) arXiv:2010.09863*]

Ordering independence and baby steps towards quantum theory

Braided tensor product algebra: coordinate differences

$$[\hat{\boldsymbol{x}}_a^\mu - \hat{\boldsymbol{x}}_b^\mu \,,\, \hat{\boldsymbol{x}}_c^\nu - \hat{\boldsymbol{x}}_d^\nu] = 0 \,, \qquad a,b,c,d = \{1,2,\dots,N\} \,,$$

close a commutative subalgebra. **Translation-invariant functions** (like N-point functions) belong to this subalgebra. Moreover:

$$\hat{\boldsymbol{E}}_a^*[k]\,\hat{\boldsymbol{E}}_b[k] = \hat{\boldsymbol{E}}_b[k]\,\hat{\boldsymbol{E}}_a^*[k] = e^{i\,\xi_\mu(k)(\hat{\boldsymbol{x}}_b^\mu - \hat{\boldsymbol{x}}_a^\mu)} \,, \quad \text{where } \hat{\boldsymbol{E}}_a[k] = :\, e^{i\,k_\mu \hat{\boldsymbol{x}}_a^\mu} \,:\,,$$

This expression can be obtained in any ordering:

$$\hat{\boldsymbol{E}}_a^*[k]\,\hat{\boldsymbol{E}}_b[k] = e^{i\,\xi_\mu(k)(\hat{\boldsymbol{x}}_b^\mu - \hat{\boldsymbol{x}}_a^\mu)} \,, \qquad \hat{\boldsymbol{F}}_a^*[q]\,\hat{\boldsymbol{F}}_b[q] = e^{i\,\xi_\mu(q)(\hat{\boldsymbol{x}}_b^\mu - \hat{\boldsymbol{x}}_a^\mu)} \,,$$

$q = q(k)$ is a (in general nonlinear) coordinate tranformation.

Define a **Pauli-Jordan function** (in unimodular models) as:

$$\Delta_{\rm PJ}(\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2) = \int d\mu_{\rm L}(k)\, \delta^{(4)}[\xi_\mu(k)\xi_\nu(k)\eta^{\mu\nu} - m^2]\, \hat{\boldsymbol{E}}_2^*[k]\, \hat{\boldsymbol{E}}_1[k]\,,$$

we can prove that

$$\Delta_{\rm PJ}(\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2) = \int d\mu_{\rm L}(q)\, \delta^{(4)}[\xi_\mu(q)\xi_\nu(q)\eta^{\mu\nu} - m^2]\, \hat{\boldsymbol{F}}_2^*[q]\, \hat{\boldsymbol{F}}_1[q]\,,$$

in fact, we can prove that $d\mu_{\rm L}(k) = d^4\xi = d\mu_{\rm L}(q)$, and both expressions are identical to:

$$\Delta_{\rm PJ}(\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2) = \int d^4\xi\, \delta^{(4)}[\xi_\mu\xi_\nu\eta^{\mu\nu} - m^2]\, e^{i\,\xi_\mu(\hat{\boldsymbol{x}}_1^\mu - \hat{\boldsymbol{x}}_2^\mu)}\,,$$

which is the standard/commutative Pauli-Jordan function. This expression is **ordering-independent.**

I hope to be able to define QFT in such an ordering-independent, T-Poincaré-invariant fashion.

Free scalar field theory **in unimodular models:** assuming **T-Poincaré covariance,** and field-independence of $[\hat{\phi}(\hat{\boldsymbol{x}}_1)\,,\,\hat{\phi}(\hat{\boldsymbol{x}}_2)]$, we get

$$[\hat{\phi}(\hat{\boldsymbol{x}}_1)\,,\,\hat{\phi}(\hat{\boldsymbol{x}}_2)] = i\,\Delta_{\mathrm{PJ}}(\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2)\,,$$

and we can prove a version of Wick's theorem.

$$\Downarrow$$

All N-point functions are T-Poincaré-invariant and commutative. However, they are **undeformed.**

[Fabiano, Mercati, *to appear soon*]

APPENDICES

“Physical” argument for the commutator ansatz

$$R^{AB}_{EF}\,\hat{\boldsymbol{T}}^{E}{}_{C}\,\hat{\boldsymbol{T}}^{F}{}_{D} = \hat{\boldsymbol{T}}^{B}{}_{G}\,\hat{\boldsymbol{T}}^{A}{}_{H}\,R^{HG}_{CD}\,,$$

$$\Downarrow$$

$$[\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{a}}^{\nu}] = i\,\Theta^{\mu\nu}(\hat{\boldsymbol{\Lambda}}) + i\,\boldsymbol{c}^{\mu\nu}{}_{\rho}\,\hat{\boldsymbol{a}}^{\rho}\,, \qquad [\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{\Lambda}}^{\nu}{}_{\rho}] = i\,\Sigma^{\mu\nu}_{\rho}(\hat{\boldsymbol{\Lambda}})\,,$$

$$[\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}, \hat{\boldsymbol{\Lambda}}^{\rho}{}_{\sigma}] = 0\,, \quad \eta^{\rho\sigma}\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\rho}\hat{\boldsymbol{\Lambda}}^{\nu}{}_{\sigma} = \eta^{\mu\nu}\,, \quad \left[\eta^{\rho\sigma}\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\rho}\hat{\boldsymbol{\Lambda}}^{\nu}{}_{\sigma}, \_\right] = 0\,,$$

"Physical" argument for the commutator ansatz

$$[\hat{a}, \hat{a}] = \ell^2 \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+\,\ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right]\hat{a} + (\dots)\,\hat{a}\,\hat{a}\,,$$

$$[\hat{a}, \hat{\Lambda}] = \ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+\left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right]\hat{a} + \ell^{-1}(\dots)\,\hat{a}\,\hat{a}\,,$$

$$[\hat{\Lambda}, \hat{\Lambda}] = \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+\,\ell^{-1}\left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right]\hat{a} + \ell^{-2}(\dots)\,\hat{a}\,\hat{a}\,,$$

"Physical" argument for the commutator ansatz

$$[\hat{a}, \hat{a}] = \ell^2 \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]$$

$$+ \ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} \right] \hat{a} + (\dots)\,\hat{a}\,\hat{a}\,,$$

$$[\hat{a}, \hat{\Lambda}] = \ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]$$

$$+ \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} \right] \hat{a} + \boxed{\ell^{-1}\,(\dots)\,\hat{a}\,\hat{a}}\,,$$

$$[\hat{\Lambda}, \hat{\Lambda}] = \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]$$

$$+ \boxed{\ell^{-1} \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} \right] \hat{a}} + \boxed{\ell^{-2}\,(\dots)\,\hat{a}\,\hat{a}}\,,$$

terms which need an inverse length scale

"Physical" argument for the commutator ansatz

$$[\hat{a}, \hat{a}] = \ell^2 \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+\,\ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right] \hat{a} + (\dots)\,\hat{a}\,\hat{a}\,,$$

$$[\hat{a}, \hat{\Lambda}] = \ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+ \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right] \hat{a} + \cancel{\ell^{-1}(\dots)\,\hat{a}\,\hat{a}}\,,$$

$$[\hat{\Lambda}, \hat{\Lambda}] = \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda}\right]$$

$$+\,\cancel{\ell^{-1}\left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda}\right]}\,\hat{a} + \cancel{\ell^{-2}(\dots)}\,\hat{a}\,\hat{a}\,,$$

have "bad commutative limit" (a bad large-scale regime, really)

“Physical” argument for the commutator ansatz

$$[\hat{a}, \hat{a}] = \ell^2 \left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} + (\dots)\, \hat{\Lambda}\, \hat{\Lambda} \right]$$

$$+ \ell \left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} \right] \hat{a} + \cancel{(\dots)\, \hat{a}\, \hat{a}},$$

$$[\hat{a}, \hat{\Lambda}] = \ell \left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} + (\dots)\, \hat{\Lambda}\, \hat{\Lambda} \right]$$

$$+ \cancel{\left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} \right]}\, \hat{a} + \cancel{\ell^{-1} (\dots)\, \hat{a}\, \hat{a}},$$

$$[\hat{\Lambda}, \hat{\Lambda}] = \cancel{\left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} + (\dots)\, \hat{\Lambda}\, \hat{\Lambda} \right]},$$

$$+ \cancel{\ell^{-1} \left[(\dots)\, \hat{\mathbf{1}} + (\dots)\, \hat{\Lambda} \right]}\, \hat{a} + \cancel{\ell^{-2} (\dots)}\, \hat{a}\, \hat{a}$$

dimensionless terms small at all scales

"Physical" argument for the commutator ansatz

$$[\hat{a}, \hat{a}] = \ell^2 \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]$$

$$+ \ell \left[(\dots)\,\hat{\mathbf{1}} + \cancel{(\dots)\,\hat{\Lambda}} \right] \hat{a} + \cancel{(\dots)\,\hat{a}\,\hat{a}},$$

$$[\hat{a}, \hat{\Lambda}] = \ell \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]$$

$$+ \cancel{\left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} \right] \hat{a}} + \cancel{\ell^{-1} (\dots)\,\hat{a}\,\hat{a}},$$

$$[\hat{\Lambda}, \hat{\Lambda}] = \cancel{\left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} + (\dots)\,\hat{\Lambda}\,\hat{\Lambda} \right]},$$

$$+ \cancel{\ell^{-1} \left[(\dots)\,\hat{\mathbf{1}} + (\dots)\,\hat{\Lambda} \right] \hat{a}} + \cancel{\ell^{-2} (\dots)\,\hat{a}\,\hat{a}}$$

bicrossproduct-like structure

List of T-Minkowski spacetimes

Case 7 ($\zeta^{(7)} \to 0 =$ κ-Minkowski lightlike) (non unimodular)

$$[\hat{x}^0, \hat{x}^1] = -i \left(\hat{x}^2 \zeta^{(7)} + \hat{x}^1 \right) , \quad [\hat{x}^0, \hat{x}^2] = i \left(\hat{x}^1 \zeta^{(7)} - \hat{x}^2 \right) , \quad [\hat{x}^1, \hat{x}^2] = 0 ,$$

$$[\hat{x}^0, \hat{x}^3] = i \left(\hat{x}^0 - \hat{x}^3 \right) , \quad [\hat{x}^1, \hat{x}^3] = i \left(\hat{x}^2 \zeta^{(7)} + \hat{x}^1 \right) , \quad [\hat{x}^2, \hat{x}^3] = i \left(\hat{x}^2 - \hat{x}^1 \zeta^{(7)} \right) .$$

Case 8 (non unimodular)

$$[\hat{x}^0, \hat{x}^1] = -i \left(\left(\hat{x}^0 - \hat{x}^3 \right) \zeta^{(8)} + \hat{x}^1 \right) , \quad [\hat{x}^0, \hat{x}^2] = -i \hat{x}^2 , \quad [\hat{x}^1, \hat{x}^2] = 0 ,$$

$$[\hat{x}^0, \hat{x}^3] = i \left(\hat{x}^0 - \hat{x}^3 \right) , \quad [\hat{x}^1, \hat{x}^3] = i \left(\left(\hat{x}^0 - \hat{x}^3 \right) \zeta^{(8)} + \hat{x}^1 \right) , \quad [\hat{x}^2, \hat{x}^3] = i \hat{x}^2 .$$

Cases $9^{(s)}$ (mistaken in Zakrzewski, corrected by Tolstoy)

$$[\hat{x}^0, \hat{x}^1] = i \left[s \left(\hat{x}^3 - \hat{x}^0 \right) + \hat{x}^1 \right] \zeta_1^{(9)} - i \, \hat{x}^2 \zeta_2^{(9)} , \quad [\hat{x}^0, \hat{x}^2] = -i \theta^{(9)} + i \, s \left(\hat{x}^0 - \hat{x}^3 \right) \zeta_2^{(9)} ,$$

$$[\hat{x}^1, \hat{x}^2] = i \left(\hat{x}^0 - \hat{x}^3 \right) \zeta_2^{(9)} , \quad [\hat{x}^0, \hat{x}^3] = i \left(\hat{x}^3 - \hat{x}^0 \right) \zeta_1^{(9)} ,$$

$$[\hat{x}^1, \hat{x}^3] = i \left[s \left(\hat{x}^0 - \hat{x}^3 \right) - \hat{x}^1 \right] \zeta_1^{(9)} + i \, \hat{x}^2 \zeta_2^{(9)} , \quad [\hat{x}^2, \hat{x}^3] = i \theta^{(9)} - i \, s \left(\hat{x}^0 - \hat{x}^3 \right) \zeta_2^{(9)} .$$

List of T-Minkowski spacetimes

Case 10

$$[\hat{x}^0, \hat{x}^1] = i\left(\hat{x}^3 - \hat{x}^0 - \hat{x}^2 - \theta_1{}^{(10)}\right), \quad [\hat{x}^0, \hat{x}^2] = -i\theta_2{}^{(10)},$$

$$[\hat{x}^1, \hat{x}^2] = i\left(\hat{x}^0 - \hat{x}^3\right), \quad [\hat{x}^0, \hat{x}^3] = 0,$$

$$[\hat{x}^1, \hat{x}^3] = i\left(\hat{x}^0 + \hat{x}^2 - \hat{x}^3 - \theta_1{}^{(10)}\right) \quad [\hat{x}^2, \hat{x}^3] = i\theta_2{}^{(10)}.$$

Case 11

$$[\hat{x}^0, \hat{x}^1] = -i\theta_1{}^{(11)}, \quad [\hat{x}^0, \hat{x}^2] = i\left(\hat{x}^1 - \theta_2{}^{(11)}\right), \quad [\hat{x}^1, \hat{x}^2] = i\left(\hat{x}^0 - \hat{x}^3\right),$$

$$[\hat{x}^0, \hat{x}^3] = 0 \quad [\hat{x}^1, \hat{x}^3] = i\theta_1{}^{(11)} \quad [\hat{x}^2, \hat{x}^3] = -i\left(\theta_2{}^{(11)} + \hat{x}^1\right).$$

Case 12 (mistaken in Zakrzewski, corrected by Tolstoy)

$$[\hat{x}^0, \hat{x}^1] = -i\left(\theta_2{}^{(12)} + \hat{x}^0 - \hat{x}^3\right), \quad [\hat{x}^0, \hat{x}^2] = -i\theta_3{}^{(12)}, \quad [\hat{x}^1, \hat{x}^2] = 0,$$

$$[\hat{x}^0, \hat{x}^3] = -2i\theta_1{}^{(12)} \quad [\hat{x}^1, \hat{x}^3] = i\left(-\theta_2{}^{(12)} + \hat{x}^0 - \hat{x}^3\right) \quad [\hat{x}^2, \hat{x}^3] = i\theta_3{}^{(12)}.$$

List of T-Minkowski spacetimes

Cases 13, 14 & 15

ζ-Minkowski (ρ-Minkowski when $\zeta_3 = 0$, λ-Minkowski when $\zeta_0 = 0$).

$$[\hat{x}^0, \hat{x}^1] = i\hat{x}^2 {\zeta_0}^{(\varrho)}\,, \quad [\hat{x}^0, \hat{x}^2] = -i\hat{x}^1 {\zeta_0}^{(\varrho)}\,, \quad [\hat{x}^1, \hat{x}^2] = -i{\theta_2}^{(\varrho)}\,,$$

$$[\hat{x}^0, \hat{x}^3] = -i{\theta_1}^{(\varrho)}\,, \quad [\hat{x}^1, \hat{x}^3] = i\hat{x}^2 {\zeta_3}^{(\varrho)}\,, \quad [\hat{x}^2, \hat{x}^3] = -i\hat{x}^1 {\zeta_3}^{(\varrho)}\,.$$

Case 16

$$[\hat{x}^0, \hat{x}^1] = i\hat{x}^3\,, \quad [\hat{x}^0, \hat{x}^2] = 0\,, \quad [\hat{x}^1, \hat{x}^2] = -i{\theta_2}^{(16)}\,,$$

$$[\hat{x}^0, \hat{x}^3] = -i{\theta_1}^{(16)}\,, \quad [\hat{x}^1, \hat{x}^3] = -i\hat{x}^0\,, \quad [\hat{x}^2, \hat{x}^3] = 0\,.$$

Case 17 (non unimodular)

$$[\hat{x}^0, \hat{x}^1] = -i{\theta_2}^{(17)}\,, \quad [\hat{x}^0, \hat{x}^2] = 0\,, \quad [\hat{x}^1, \hat{x}^2] = -i{\theta_1}^{(17)}\,,$$

$$[\hat{x}^0, \hat{x}^3] = i\left(\hat{x}^3 - \hat{x}^0\right)\,, \quad [\hat{x}^1, \hat{x}^3] = i{\theta_2}^{(17)}\,, \quad [\hat{x}^2, \hat{x}^3] = 0\,.$$

List of T-Minkowski spacetimes

Case 18 (non unimodular)

$$[\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^1] = -i\hat{\boldsymbol{x}}^2 \zeta^{(18)}\,, \quad [\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^2] = i\hat{\boldsymbol{x}}^1 \zeta^{(18)}\,, \quad [\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^2] = -i\theta^{(18)}\,,$$

$$[\hat{\boldsymbol{x}}^0, \hat{\boldsymbol{x}}^3] = i\left(\hat{\boldsymbol{x}}^3 - \hat{\boldsymbol{x}}^0\right)\,, \quad [\hat{\boldsymbol{x}}^1, \hat{\boldsymbol{x}}^3] = i\hat{\boldsymbol{x}}^2 \zeta^{(18)}\,, \quad [\hat{\boldsymbol{x}}^2, \hat{\boldsymbol{x}}^3] = -i\hat{\boldsymbol{x}}^1 \zeta^{(18)}\,.$$

Cases 19, 20 & 21

θ-Minkowski/Moyal
(3 automorphism classes of θ-Poincaré quantum group)

$$[\hat{\boldsymbol{x}}^\mu, \hat{\boldsymbol{x}}^\nu] = i\,\theta^{\mu\nu}\,.$$