QUANTUM GALILEI GROUP AS QUANTUM REFERENCE FRAME TRANSFORMATIONS

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Applications of NonCommutative Geometry to Gauge Theories, Field Theories, and Quantum Space-Time

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Two notions of "quantum spacetime"

Nontrivial commutators between spacetime coordinates formalise a fundamental uncertainty in the localisability of spacetime points in analogy to the quantisation of phase space in quantum mechanics

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(\hat{x}) \quad \longrightarrow \quad \Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \frac{1}{2} |\langle \Theta^{\mu\nu} \rangle|$$

Most work on **noncommutative spacetime** explored models that can be generically described by

$$[\hat{x}^\mu, \hat{x}^\nu] = \ell^2 \theta^{\mu\nu} + \ell \, \gamma^{\mu\nu}_\beta \, \hat{x}^\beta$$

The quantisation parameter ℓ is in principle independent from $\hbar$

Quantisation of spacetime is independent from quantisation of phase space [(1)] — in fact, extension to a full noncommutative phase space can be nontrivial

[(1)] *G. Gubitosi, F. Lizzi, J. Relancio, P. Vitale Phys.Rev.D 105 (2022) 12, 126013*

Two notions of "quantum spacetime"

The **quantum reference frames** proposal in quantum mechanics is inherently linked to the presence of quantum matter, which serves as the quantum reference frame — quantisation of the phase space induces 'quantisation of spacetime'

One separates the subsystem playing the role of reference frame (e.g. a quantum particle) and considers separately the phase spaces of the reference frame and of the system (e.g. another quantum particle)

$$[\hat{x}_A, \hat{p}_A] = i\kappa \qquad\qquad [\hat{x}_B, \hat{p}_B] = i\hbar$$

Associating a reference frame to a quantum system is in some sense assigning quantum properties to spacetime itself, since the reference frame fuzzy properties (system A) will affect the description of the other particles (system B).

In this case the 'quantisation of spacetime' is governed by the constant governing the non commutativity of phase space

How are these two concepts of 'quantum spacetime' linked? Could the QRF construction be understood as some limiting case of spacetime quantisation? Or are they different limits of a more general picture? Is spacetime a good tool when both matter and reference frames are quantised?

A symmetry-based approach to "quantum spacetimes"

To clarify the link between noncommutative spacetime and quantum reference frames, we compare the properties of the symmetry transformations defined in each framework.

Two-steps discussion, based on:

- *"The group structure of dynamical transformations between quantum reference frames"*
  A. Ballesteros, F. Giacomini, G. Gubitosi
  Quantum 5 (2021) 470 [arXiv: 2012.15769 [quant-ph]]

We identify the canonical transformations on the phase space of the quantum systems comprising the quantum reference frames, and show that these transformations close a **group structure defined by a 7-dimensional Lie algebra** $\mathscr{D}(7)$, which is different from the usual Galilei algebra of quantum mechanics, but reduces to it in the classical $\hbar \to 0$ limit.

- *"Quantum Galilei group as quantum reference frame transformations"*
  A. Ballesteros, D. Fernandez-Silvestre, F. Giacomini, G. Gubitosi
  arXiv: 2504.00569 [quant-ph]

We establish a **correspondence between quantum reference frame transformations** described by the algebra $\mathscr{D}(7)$ **and transformations generated by a quantum deformation of the Galilei group** with commutative time, taken at the first order in the quantum deformation parameter.

Preliminaries: the 1+1 extended Galilei group

Centrally-extended Galilei algebra in 1+1D:

$$[\hat{K}, \hat{P}_0] = -i\hbar \hat{P}_1 \,, \qquad [\hat{K}, \hat{P}_1] = -i\hbar \hat{M} \,, \qquad [\hat{P}_0, \hat{P}_1] = 0 \,, \qquad [\hat{M}, \cdot\,] = 0$$

The group element is parameterised by the (commutative) group coordinates (θ, b, a, v)

$$G = e^{\frac{i}{\hbar}\theta \hat{M}} \, e^{\frac{i}{\hbar} b \hat{P}_0} \, e^{\frac{i}{\hbar} a \hat{P}_1} \, e^{\frac{i}{\hbar} v \hat{K}}$$

Group law:

$$G(\theta', b', a', v') \cdot G(\theta, b, a, v) = G(\theta'', b'', a'', v'')$$

$$\longrightarrow \qquad (\theta'', b'', a'', v'') = (\theta' + \theta + v' a + \tfrac{1}{2} v'^2 b, b' + b, a' + a + v' b, v' + v)$$

We are writing the generators in 'physical dimensions', so that they can be represented on the phase space of a quantum particle $[\hat{x}, \hat{\pi}] = i\hbar$ with mass m

$$\hat{K} = t\hat{\pi} - m\hat{x} \,, \qquad \hat{P}_1 = \hat{\pi} \,, \qquad \hat{P}_0 = \frac{\hat{\pi}^2}{2m} \,, \qquad \hat{M} = m$$

Galilean quantum reference frame transformations

Given three quantum particles, A, B, C one can define relational phase space operators for particles A and B taking C as reference frame

$$[\hat{x}_A^{(C)}, \hat{\pi}_A^{(C)}] = i\kappa \,, \qquad [\hat{x}_B^{(C)}, \hat{\pi}_B^{(C)}] = i\hbar$$

(we use two different quantum phase space parameters to distinguish the quantum properties of each particle)

A QRF transformation maps the quantum state in the Hilbert space of A and B relative to C, $\mathscr{H}_A^{(C)} \otimes \mathscr{H}_B^{(C)}$, to the quantum state in the Hilbert space of B and C relative to A, i.e., $\mathscr{H}_B^{(A)} \otimes \mathscr{H}_C^{(A)}$

$$\hat{S}^{(C\to A)} |\psi\rangle_{AB}^{(C)} = |\psi\rangle_{CB}^{(A)}$$

When A is in a quantum state that is delocalised in position (resp. momentum), one cannot apply standard translations (resp. boosts) on particle B to obtain its description in the reference frame of A. **The parameters of transformations acting on B are promoted to quantum operators** on $\mathscr{H}_A$, $\hat{x}_A$ for translations and $\hat{p}_A/m_A$ for boosts

$$\hat{U}_P = e^{\frac{i}{\hbar}\hat{x}_A \otimes \hat{P}_B} \,, \qquad \hat{U}_K = e^{\frac{i}{\hbar}\frac{\hat{\pi}_A}{m_A}\otimes \hat{K}_B} \,.$$

F. Giacomini, E. Castro-Ruiz, and C. Brukner, Nat. Commun. 10 494 (2019)

Group structure of Galilean quantum reference frame transformations

Because of the noncommutativity of the transformation parameters, the operators $\hat{U}_P$ and $\hat{U}_K$ act on the phase space of both particles A and B

$$\hat{U}_P\, \hat{x}_A\, \hat{U}_P^{-1} = \hat{x}_A,$$
$$\hat{U}_P\, \hat{p}_A\, \hat{U}_P^{-1} = \hat{p}_A - \frac{\kappa}{\hbar}\, \hat{p}_B,$$
$$\hat{U}_P\, \hat{x}_B\, \hat{U}_P^{-1} = \hat{x}_B + \hat{x}_A,$$
$$\hat{U}_P\, \hat{p}_B\, \hat{U}_P^{-1} = \hat{p}_B,$$

$$\hat{U}_K\, \hat{x}_A\, \hat{U}_K^{-1} = \hat{x}_A + \frac{\kappa}{\hbar}\, \frac{1}{m_A}\, \left(\hat{p}_B t - m_B \hat{x}_B\right),$$
$$\hat{U}_K\, \hat{p}_A\, \hat{U}_K^{-1} = \hat{p}_A,$$
$$\hat{U}_K\, \hat{x}_B m_A\, \hat{U}_K^{-1} = \hat{x}_B + t\,\frac{\hat{p}_A}{m_A},$$
$$\hat{U}_K\, \hat{p}_B\, \hat{U}_K^{-1} = \hat{p}_B + \frac{m_B}{m_A}\, \hat{p}_A\,.$$

The **group structure defined by these operators** can be investigated by searching for the corresponding algebra for the operators $\hat{P}_{AB} = \hat{x}_A \otimes \hat{p}_B$ and $\hat{K}_{AB} = \dfrac{\hat{p}_A}{m_A} \otimes \hat{K}_B$. One finds that these operators belong to a **7-dimensional Lie algebra**, generated by

$$\hat{P}_{AB} = \hat{x}_A \otimes \hat{\pi}_B\ , \qquad \hat{K}_{AB} = \frac{\hat{\pi}_A}{m_A} \otimes \hat{K}_B$$

$$\hat{Q}_A = \frac{\hat{\pi}_A^2}{2m_A} \otimes \mathbf{I}_B\ , \qquad \hat{Q}_B = \mathbf{I}_A \otimes \frac{\hat{\pi}_B^2}{2m_B}\ ,$$

$$\hat{D}_A = \frac{1}{2}(\hat{x}_A\hat{\pi}_A + \hat{\pi}_A\hat{x}_A) \otimes \mathbf{I}_B\ , \qquad \hat{D}_B = \mathbf{I}_A \otimes \frac{1}{2}(\hat{x}_B\hat{\pi}_B + \hat{\pi}_B\hat{x}_B)\ , \qquad \hat{T} = \hat{\pi}_A \otimes \hat{\pi}_B$$

Group structure of Galilean quantum reference frame transformations

The action on phase space variables defines canonical transformations:

	$\hat{x}_A$	$\hat{p}_A$	$\hat{x}_B$	$\hat{p}_B$
$U_P = \mathrm{e}^{\frac{i}{\hbar}\hat{P}_{AB}}$	$\hat{x}_A$	$\hat{p}_A - \frac{\kappa}{\hbar}\hat{p}_B$	$\hat{x}_B + \hat{x}_A$	$\hat{p}_B$
$U_G = \mathrm{e}^{\frac{i}{\hbar}\hat{K}_{AB}}$	$\hat{x}_A + \frac{\kappa}{\hbar}\frac{1}{m_A}(\hat{p}_B t - m_B\hat{x}_B)$	$\hat{p}_A$	$\hat{x}_B + t\,\frac{\hat{p}_A}{m_A}$	$\hat{p}_B + \frac{m_B}{m_A}\hat{p}_A$
$\mathrm{e}^{\frac{i}{\kappa}\alpha\hat{D}_A}$	$\mathrm{e}^{\alpha}\hat{x}_A$	$\mathrm{e}^{-\alpha}\hat{p}_A$	$\hat{x}_B$	$\hat{p}_B$
$\mathrm{e}^{\frac{i}{\hbar}\alpha\hat{D}_B}$	$\hat{x}_A$	$\hat{p}_A$	$\mathrm{e}^{\beta}\hat{x}_B$	$\mathrm{e}^{-\beta}\hat{p}_B$
$\mathrm{e}^{\frac{i}{\kappa}\alpha\hat{Q}_A}$	$\hat{x}_A + \frac{\alpha}{m_A}\hat{p}_A$	$\hat{p}_A$	$\hat{x}_B$	$\hat{p}_B$
$\mathrm{e}^{\frac{i}{\hbar}\alpha\hat{Q}_B}$	$\hat{x}_A$	$\hat{p}_A$	$\hat{x}_B + \frac{\alpha}{m_B}\hat{p}_B$	$\hat{p}_B$
$\mathrm{e}^{\frac{i}{\hbar}\alpha\hat{T}}$	$\hat{x}_A + \alpha\,\frac{\kappa}{\hbar}\hat{p}_B$	$\hat{p}_A$	$\hat{x}_B + \alpha\,\hat{p}_A$	$\hat{p}_B$

The operators $\hat{D}_A, \hat{D}_B$ act as dilations on phase space, $\hat{Q}_A, \hat{Q}_B$ generate the free motion of particles A and B, respectively. $\hat{T}$ is an evolution of spatial coordinates, without rescaling of momenta.

Interestingly, these additional operators appear naturally when requiring that QRF transformations are extended symmetries of the Hamiltonian of the quantum particles.

Quantum Galilei groups

Quantum Galilei groups are a fully noncommutative version of the exponential map for the Galilei group, where the **group coordinates** (θ, b, a, v) **become noncommutative operators** $(\hat{\theta}, \hat{b}, \hat{a}, \hat{v})$

$$\hat{G}_\alpha = e^{\frac{i}{\hbar}\hat{\theta}\otimes\hat{M}} e^{\frac{i}{\hbar}\hat{b}\otimes\hat{P}_0} e^{\frac{i}{\hbar}\hat{a}\otimes\hat{P}_1} e^{\frac{i}{\hbar}\hat{v}\otimes\hat{K}}$$

The commutators between quantum coordinates $(\hat{\theta}, \hat{b}, \hat{a}, \hat{v})$ are ruled by a **quantum deformation parameter** α**,** which also governs possible deformations of the group law

$$(\theta'', b'', a'', v'') = (\theta' + \theta + v'a + \frac{1}{2}v'^2 b, b' + b, a' + a + v'b, v' + v) + \mathcal{O}(\alpha)$$

as well as deformations of the algebra of the generators $(\hat{M}, \hat{P}_0, \hat{P}_1, \hat{K})$.

The Galilei group in (1+1) dimensions admits 26 inequivalent quantum group structures[1].

Comparison with the with the algebra of QRF transformation parameters leads us to **single out a specific quantum Galilei group** based on two criteria:

- the quantum transformation parameters $\hat{a}$ and $\hat{v}$ have a nonvanishing commutator, similarly to the QRF transformation parameters $\hat{x}_A$ and $\hat{p}_A/m_A$;
- $\hat{b}$ is a central element of the algebra of quantum group coordinates, since in the QRF approach time is a commutative parameter

[1] *A. Opanowicz, J. Phys. A 31 (1998), A. Ballesteros, E. Celeghini, and F. J. Herranz, J. Phys. A 33 (2000), A. Opanowicz,J. Phys. A 33 (2000)*

A quantum Galilei group with commutative time

To construct the relevant quantum Galilei group we start from its semi-classical counterpart, the Poisson-Lie group

$$G_\alpha = e^{\frac{i}{\hbar}\theta \hat{M}}\, e^{\frac{i}{\hbar} b \hat{P}_0}\, e^{\frac{i}{\hbar} a \hat{P}_1}\, e^{\frac{i}{\hbar} v \hat{K}}$$

such that the parameters (θ, b, a, v) define the fundamental brackets of a Poisson algebra structure which is ruled by the parameter α.

The two conditions we impose allow us to select the Poisson-Lie Galilei group with structure

$$\{a, v\} = \alpha v\ , \qquad \{a, \theta\} = \alpha\theta\ , \qquad \{v, \theta\} = -\frac{1}{2}\alpha v^2\ , \qquad \{b, \cdot\,\} = 0$$

Quantisation, with quantisation parameter κ, leads to

$$[\hat{a}, \hat{v}] = i\kappa\alpha\hat{v}\ , \qquad [\hat{a}, \hat{\theta}] = i\kappa\alpha\hat{\theta}\ , \qquad [\hat{v}, \hat{\theta}] = -\frac{i}{2}\kappa\alpha\hat{v}^2\ , \qquad [\hat{b}, \cdot\,] = 0\ .$$

These commutators are compatible with an undeformed group law, but require a deformation of the algebra of the generators

$$[\hat{P}_0, \hat{P}_1] = 0 \qquad [\hat{K}, \hat{M}] = \frac{i}{2}\kappa\,\alpha\, e^{\frac{\kappa}{\hbar}\alpha\hat{P}_1}\hat{M}^2 \qquad [\hat{K}, \hat{P}_0] = i\hbar\frac{1 - e^{\frac{\kappa}{\hbar}\alpha\hat{P}_1}}{\frac{\kappa}{\hbar}\alpha} \qquad [\hat{K}, \hat{P}_1] = -\,i\hbar e^{\frac{\kappa}{\hbar}\alpha\hat{P}_1}\hat{M}$$

Phase-space realisation of the quantum Galilei group

The connection with the QRF approach can be found upon **representing the algebra of generators on the phase space** $[\hat{q}_B, \hat{p}_B] = i\hbar$**, and the algebra of quantum group coordinates on the phase space** $[\hat{q}_A, \hat{p}_A] = i\kappa$

Phase space realisation of the coordinates:

$$\hat{a} = \hat{q}_A + \gamma \,, \qquad \hat{v} = \phi e^{\alpha \hat{p}_A} \,, \qquad \hat{\theta} = \frac{\phi}{4}(e^{\alpha \hat{p}_A}\hat{q}_A + \hat{q}_A e^{\alpha \hat{p}_A}) \,, \qquad \hat{b} = \delta$$

with ϕ a free parameter with dimensions of velocity. The parameters (γ, δ) label the representation.

Representation of the generators:

$$\hat{M} = m_B e^{-\frac{\kappa}{2\hbar}\alpha \hat{p}_B} \,, \qquad \hat{P}_0 = \frac{1}{m_B(\frac{\kappa}{2\hbar}\alpha)^2}\left(\cosh\left(\frac{\kappa}{2\hbar}\alpha \hat{p}_B\right) - 1\right) + \xi \,,$$

$$\hat{P}_1 = \hat{p}_B \,, \qquad \hat{K} = -\frac{m_B}{2}\left(e^{\frac{\kappa}{2\hbar}\alpha \hat{p}_B}\hat{q}_B + \hat{q}_B e^{\frac{\kappa}{2\hbar}\alpha \hat{p}_B}\right) + t\hat{p}_B \,,$$

where ξ is an arbitrary constant whose physical interpretation is that of internal energy and can be set to zero.

Phase-space realisation of the quantum Galilei group

Putting together the representation of the quantum group coordinates and of the Galilei generators one can define **operators acting on the two sets of phase space coordinates** $[\hat{q}_A, \hat{p}_A] = i\kappa$ and $[\hat{q}_B, \hat{p}_B] = i\hbar$

$$\hat{M}^{AB}_{\alpha} \equiv \hat{\theta} \otimes \hat{M} = \frac{\phi}{4}(e^{\alpha \hat{p}_A}\hat{q}_A + \hat{q}_A e^{\alpha \hat{p}_A}) \otimes m_B e^{-\frac{\kappa}{2\hbar}\alpha \hat{p}_B} ,$$

$$\hat{P}^{AB}_{0,\alpha} \equiv \hat{b} \otimes \hat{P}_0 = \delta \otimes \left[\frac{1}{m_B(\frac{\kappa}{2\hbar}\alpha)^2} \left(\cosh\left(\frac{\kappa}{2\hbar}\alpha \hat{p}_B \right) - 1 \right) + \xi \right] ,$$

$$\hat{P}^{AB}_{1,\alpha} \equiv \hat{a} \otimes \hat{P}_1 = (\hat{q}_A + \gamma) \otimes \hat{p}_B ,$$

$$\hat{K}^{AB}_{\alpha} \equiv \hat{v} \otimes \hat{K} = -\phi e^{\alpha \hat{p}_A} \otimes \left(\frac{m_B}{2}(e^{\frac{\kappa}{2\hbar}\alpha \hat{p}_B}\hat{q}_B + \hat{q}_B e^{\frac{\kappa}{2\hbar}\alpha \hat{p}_B}) - t\hat{p}_B \right) .$$

This allows us to define the quantum Galilei transformation operators in a similar fashion as the QRF operators

$$\hat{U}^{\alpha}_{M} \equiv e^{\frac{i}{\hbar}\hat{M}^{AB}_{\alpha}} = e^{\frac{i}{\hbar}\hat{\theta}\otimes\hat{M}} , \qquad \hat{U}^{\alpha}_{P_0} \equiv e^{\frac{i}{\hbar}\hat{P}^{AB}_{0,\alpha}} = e^{\frac{i}{\hbar}\hat{b}\otimes\hat{P}_0} ,$$

$$\hat{U}^{\alpha}_{P_1} \equiv e^{\frac{i}{\hbar}\hat{P}^{AB}_{1,\alpha}} = e^{\frac{i}{\hbar}\hat{a}\otimes\hat{P}_1} , \qquad \hat{U}^{\alpha}_{K} \equiv e^{\frac{i}{\hbar}\hat{K}^{AB}_{\alpha}} = e^{\frac{i}{\hbar}\hat{v}\otimes\hat{K}} .$$

Connection between the quantum Galilei group and QRF Galilei transformations

We **identify the two phase spaces** on which we represented the quantum Galilei operators **as the phase spaces of quantum particles**. Moreover, we take the **first order in the quantum deformation parameter α.**

The transformation parameters read

$$\hat{a} = \hat{q}_A + \gamma \,, \quad \hat{v} = \phi(1 + \alpha \hat{p}_A) + \mathcal{O}(\alpha^2) \,, \quad \hat{\theta} = \frac{\phi}{2}(\hat{q}_A + \frac{\alpha}{2}(\hat{p}_A \hat{q}_A + \hat{q}_A \hat{p}_A)) + \mathcal{O}(\alpha^2) \,, \quad \hat{b} = \delta$$

In order to recognise the parameter $\hat{v}$ as the velocity of a quantum particle, we define the '**renormalised momentum**' $\hat{p}'_A = m_A \hat{v} = m_A \phi \, (1 + \alpha \hat{p}_A)$

If we also take the **ansatz** $\alpha = \dfrac{1}{m_A \phi}$, we can write the renormalised momentum and the velocity as

$$\hat{p}'_A = m_A \phi + \hat{p}_A \qquad\qquad \hat{v} = \phi + \hat{v}_A$$

The transformation generators read

$$\hat{M} = m_B \left(1 - \frac{\kappa}{2\hbar} \alpha \hat{p}_B\right) + \mathcal{O}(\alpha^2) \,, \qquad \hat{P}_0 = \frac{\hat{p}_B^2}{2m_B} + \xi + \mathcal{O}(\alpha^2) \,,$$

$$\hat{P}_1 = \hat{p}_B \,, \qquad \hat{K} = -\, m_B \left(\hat{q}_B + \frac{\kappa}{4\hbar} \alpha (\hat{p}_B \hat{q}_B + \hat{q}_B \hat{p}_B)\right) + t \hat{p}_B + \mathcal{O}(\alpha^2) \,,$$

Connection between the quantum Galilei group and QRF Galilei transformations

Putting everything together, the representation of the operators defining quantum Galilei transformations takes the form

$$\hat{M}^{AB}_{\alpha} = -\frac{1}{4}\frac{\kappa}{\hbar}\frac{m_B}{m_A}(\hat{q}_A \otimes \hat{p}_B) + \frac{1}{4}\frac{m_B}{m_A}(\hat{p}'_A\hat{q}_A + \hat{q}_A\hat{p}'_A) \otimes \mathbf{1}_B ,$$

$$\hat{P}^{AB}_{0,\alpha} = \delta\mathbf{1}_A \otimes \left(\frac{\hat{p}_B^2}{2m_B} + \xi\mathbf{1}_B\right) ,$$

$$\hat{P}^{AB}_{1,\alpha} = (\hat{q}_A + \gamma\mathbf{1}_A) \otimes \hat{p}_B ,$$

$$\hat{K}^{AB}_{\alpha} = \frac{\hat{p}'_A}{m_A} \otimes (t\hat{p}_B - m_B\hat{q}_B) - \frac{1}{4}\frac{\kappa}{\hbar}\frac{m_B}{m_A}\mathbf{1}_A \otimes (\hat{p}_B\hat{q}_B + \hat{q}_B\hat{p}_B) .$$

which can be compared to the QRF operators defined before

$$\hat{P}_{AB} = \hat{x}_A \otimes \hat{\pi}_B , \qquad \hat{K}_{AB} = \frac{\hat{\pi}_A}{m_A} \otimes \hat{K}_B$$

$$\hat{Q}_A = \frac{\hat{\pi}_A^2}{2m_A} \otimes \mathbf{I}_B , \qquad \hat{Q}_B = \mathbf{I}_A \otimes \frac{\hat{\pi}_B^2}{2m_B} ,$$

$$\hat{D}_A = \frac{1}{2}(\hat{x}_A\hat{\pi}_A + \hat{\pi}_A\hat{x}_A) \otimes \mathbf{I}_B , \qquad \hat{D}_B = \mathbf{I}_A \otimes \frac{1}{2}(\hat{x}_B\hat{\pi}_B + \hat{\pi}_B\hat{x}_B) , \qquad \hat{T} = \hat{\pi}_A \otimes \hat{\pi}_B$$

The quantum Galilei group operators are linear combinations of the QRF operators.

Connection between the quantum Galilei group and QRF Galilei transformations

Therefore, the quantum Galilei group element can be written in terms of the QRF operators

$$\hat{G}_\alpha = e^{\frac{i}{\hbar}\hat{\theta}\otimes\hat{M}}\, e^{\frac{i}{\hbar}\hat{b}\otimes\hat{P}_0}\, e^{\frac{i}{\hbar}\hat{a}\otimes\hat{P}_1}\, e^{\frac{i}{\hbar}\hat{v}\otimes\hat{K}} = e^{\frac{i}{\hbar}\hat{M}^{AB}_\alpha}\; e^{\frac{i}{\hbar}\hat{P}^{AB}_{0,\alpha}}\; e^{\frac{i}{\hbar}\hat{P}^{AB}_{1,\alpha}}\; e^{\frac{i}{\hbar}\hat{K}^{AB}_\alpha}$$
$$= e^{\frac{i}{\hbar}(-\frac{\kappa}{4\hbar}\frac{m_B}{m_A}\hat{P}_{AB}+\frac{1}{2}\frac{m_B}{m_A}\hat{D}_A)}\, e^{\frac{i}{\hbar}\delta\hat{Q}_B}\, e^{\frac{i}{\hbar}\hat{P}_{AB}}\, e^{\frac{i}{\hbar}(\hat{K}_{AB}-\frac{\kappa}{2\hbar}\frac{m_B}{m_A}\hat{D}_B)}$$

The four operators $\{\hat{M}^{AB}_\alpha, \hat{P}^{AB}_{0,\alpha}, \hat{P}^{AB}_{1,\alpha}, \hat{K}^{AB}_\alpha\}$ generate through commutations the full 7d dynamical Lie algebra $\mathscr{D}(7)$: there is a **complete algebraic equivalence between the Galilei quantum group and the dynamical Lie group of QRFs**.

The fact that the quantum Galilei group operators close a 7-dimensional algebra when taken to the first order in α is a non-trivial observation, since in general quantum group operators would generate an infinite-dimensional algebra. This is possible because at the first order in α the representation of the quantum group operators are realised as quadratic functions of phase space coordinates.

Semiclassical limit

Taking the semiclassical limit from the quantum group to the Poisson-Lie group corresponds to taking a limit in which the quantum particle serving the role of reference frame is classical, but dynamical

$$\{x_A, p_A\} = \lim_{\kappa \to 0} \frac{[\hat{x}_A, \hat{p}_A]}{i\kappa} = 1$$

In this limit, the exponents of the quantum Galilei group transformations become a product of classical phase space functions for particle A and quantum phase space operators for particle B

$$\begin{aligned}
\hat{M}_c^{AB} &= \frac{\phi}{2} q_A (1 + \alpha p_A) \otimes m_B \mathbf{1}_B = q_A \frac{p'_A}{2m_A} \otimes m_B \mathbf{1}_B \ , \\
\hat{P}_{0,c}^{AB} &= \delta \otimes \frac{\hat{p}_B^2}{2m_B} \ , \\
\hat{P}_{1,c}^{AB} &= q_A \otimes \hat{p}_B \ , \\
\hat{K}_c^{AB} &= \phi(1 + \alpha p_A) \otimes (t\hat{p}_B - m_B \hat{q}_B) = \frac{p'_A}{m_A} \otimes (t\hat{p}_B - m_B \hat{q}_B) \ ,
\end{aligned}$$

where we used again the ansatz $\alpha = (m_A \phi)^{-1}$ and we set $\gamma = \xi = 0$

QRF transformations as transformations on superposition on semiclassical states

The quantum Galilei group acts on the Hilbert space $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, where states can be written as combinations of elements $|\phi\rangle_A |\psi\rangle_B$. The correspondence with QRFs tells us that these can be interpreted as states of the quantum particles A and B.

The semiclassical limit corresponds to considering semiclassical states on $\mathscr{H}_A$, such that

$$[\hat{q}_A, \hat{p}'_A] \, |\chi(t)\rangle_A \approx 0$$

Therefore, in the semiclassical limit, QRF transformations act on states of the form $|\chi(t)\rangle_A |\psi\rangle_B$: the reference frame A is described by a semiclassical (dynamical) state, while the other particle B is fully quantum

Taking for particle A a free particle, $|\chi(t)\rangle_A | = e^{-\frac{i}{\kappa}\hat{H}_A t} |\chi(0)\rangle_A$, the action of the quantum Galilei group transformations is

$$e^{\frac{i}{\hbar}\hat{M}^{AB}_\alpha} |\chi(t)\rangle_A |\psi\rangle_B \approx e^{\frac{i}{\hbar}\frac{q_A(t) p'_A(t)}{2m_A} \otimes \hat{M}} |\chi(t)\rangle_A |\psi\rangle_B$$

$$e^{\frac{i}{\hbar}\hat{P}^{AB}_{0,\alpha}} |\chi(t)\rangle_A |\psi\rangle_B \approx e^{\frac{i}{\hbar}\delta \otimes \hat{P}_0} |\chi(t)\rangle_A |\psi\rangle_B$$

$$e^{\frac{i}{\hbar}\hat{P}^{AB}_{1,\alpha}} |\chi(t)\rangle_A |\psi\rangle_B \approx e^{\frac{i}{\hbar}q_A(t) \otimes \hat{P}_1} |\chi(t)\rangle_A |\psi\rangle_B$$

$$e^{\frac{i}{\hbar}\hat{K}^{AB}_\alpha} |\chi(t)\rangle_A |\psi\rangle_B \approx e^{\frac{i}{\hbar}\frac{p'_A(t)}{m_A} \otimes \hat{K}} |\chi(t)\rangle_A |\psi\rangle_B$$

which is exactly what one would write using the Galilei Poisson-Lie group

QRF transformations as transformations on superposition on semiclassical states

How do QRF transformations emerge when going beyond semiclassical states? Which states of the reference frame lead to this kind of transformations?

Let us consider a quantum superposition of two semiclassical states for particle A

$$\frac{1}{\sqrt{2}}\left(|\chi_1(t)\rangle_A + |\chi_2(t)\rangle_A\right)|\psi\rangle_B$$

The action of a quantum Galilean boost on such a state reduces to the action of a QRF boost

$$e^{\frac{i}{\hbar}\hat{K}_\alpha^{AB}}\frac{1}{\sqrt{2}}(|\chi_1(t)\rangle_A + |\chi_2(t)\rangle_A)|\psi(t)\rangle_B \approx \frac{1}{\sqrt{2}}\left(|\chi_1(t)\rangle_A e^{\frac{i}{\hbar}\frac{p'_{A,1}(t)}{m_A}\otimes\hat{K}} + |\chi_2(t)\rangle_A e^{\frac{i}{\hbar}\frac{p'_{A,2}(t)}{m_A}\otimes\hat{K}}\right)|\psi(t)\rangle_B$$

$$\approx e^{\frac{i}{\hbar}\frac{\hat{\pi}_A}{m_A}\otimes\hat{K}}\frac{1}{\sqrt{2}}\left(|\chi_1(t)\rangle_A + |\chi_2(t)\rangle_A\right)|\psi(t)\rangle_B$$

This is only possible for superpositions of semiclassical states, for generic quantum states of the reference frame A we have to resort to the full quantum Galilei boost operator $\hat{K}_\alpha^{AB}$

Summary

QRF transformations generalise Galilean transformations, introducing transformation parameters as noncommutative operators to describe transformations between physical reference frames with quantum properties.

We found that these transformations close a finite, 7-dimensional algebra.

This algebra is recovered in the quantum Galilei group setting, when one considers only the first order in the quantum deformation parameter α, and upon linking this parameter to physical properties of the quantum reference frame system, $\alpha = (m_A \phi)^{-1}$.

The correspondence allows us to interpret the quantum Galilei group as transformations acting on the phase space of quantum particles.

Using the quantum Galilei group formalism applied to quantum states, one finds that QRF transformations are recovered when the state of the reference frame is a quantum superposition of semiclassical states.

We conjecture that the all-orders quantum Galilei group describes the situation where the state of the reference frame is a generic quantum state.