Thue choice number and the counting argument

Matthieu Rosenfeld

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Many generalizations or variations were studied:

- Cubes, 4th powers, fractional powers,
- patterns, formulas (*ABABA*),
- k-abelian powers, k-binomial powers, additive powers, antipowers,
- nonrepetitive colorings of graphs (or other objects).
- ...

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Does there exists $k \in \mathbb{N}$, such that:

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The Thue choice number is the smallest such k.

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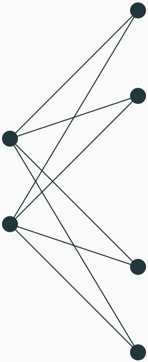

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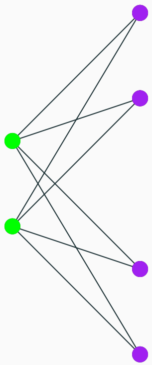

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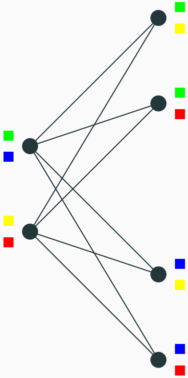

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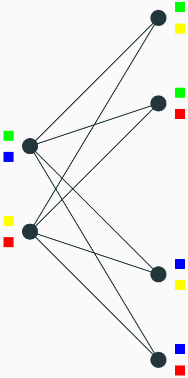

We have graphs with $\chi(G) = 2$ and $\chi_{ch}(G)$ arbitrarily large.

Avoiding squares over lists of size 4

Theorem (Grytczuk, Przybyło and Zhu (2011), ..., Rosenfeld (2020))

Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of lists such that for all i, $|A_i| \geq 4$. Then there are square-free words in $\prod_i A_i$.

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We show instead a stronger result.

Fix $(A_i)_{i\in\mathbb{N}}$. Let C_n be the set of square-free words of length n that respect $(A_i)_{i\in\mathbb{N}}$.

Lemma

For any integer n,

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For any integer n,

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$\implies |C_n| \geq 2^n \geq 1.$

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By the induction hypothesis, for all $j \in \{0, \ldots, n\}$, $|C_{n-j}| \leq \frac{|C_n|}{2^j}$.

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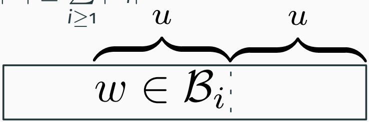

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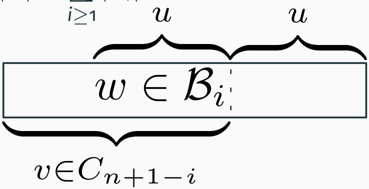

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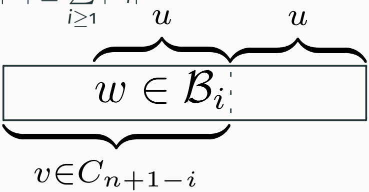

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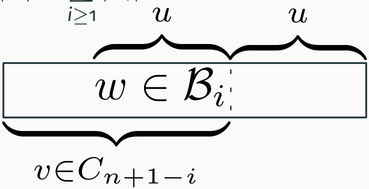

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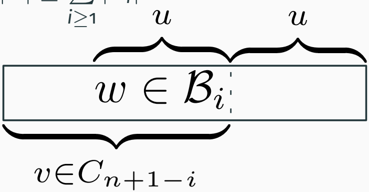

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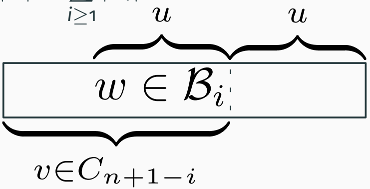

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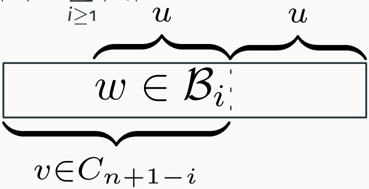

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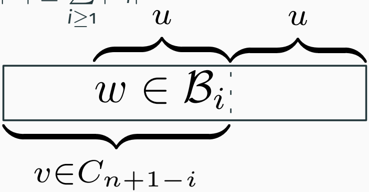

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The Thue choice number

Theorem (Grytczuk, Przybyło and Zhu (2011), ..., Rosenfeld (2020))

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Question

Is the following true ?

$$\forall i, |A_i| \geq 3 \implies \text{there exists a square-free word } w \in \prod_i A_i$$

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The main reason for which the proof is not sharp is this bound

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How to improve the bound for small i?

Compute the growth rate α_ℓ of the language of words without squares of period less than ℓ and hope for something like:

$$|C_{n+1}| \gtrsim \alpha_\ell |C_n| - \sum_{i \geq \ell} |\mathcal{B}_i|$$

Sharpening the proof sharp - II

Using elementary automata theory and linear algebra, we define a weight for each word and

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$$\widehat{C_{n+1}} \geq \alpha_\ell \widehat{C_n} - \sum_{i \geq \ell} \widehat{\mathcal{B}_i}$$

- α_ℓ is the spectral radius of the automaton,
- the weight function is given by the associated eigenvector,
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Sharpening the proof sharp - II

Using elementary automata theory and linear algebra, we define a weight for each word and

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A similar idea was used by Kolpakov, Rao, Shur to obtain bounds on the growth of power-free languages

Back to the Thue choice number

Theorem (Rosenfeld, 2023)

Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of lists such that

- *for all i,* $|A_i| \geq 3$
- *and for all i,* $A_i \subseteq \{0, 1, 2, 3\}$.

Then there are square-free words in $\prod A_i$.

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A rough estimation seems to indicate that 10^7GB or memory should be enough to remove the second condition on A_i (under the hypothesis that the growth rate is the same)

Other application of this proof technique

A generic sufficient condition for avoidability

Theorem

Let $\mathcal{A}$ be an alphabet and $\mathcal{F}$ be a set of forbiden factors over $\mathcal{A}$. Suppose that there exists a positive real x such that

$$|\mathcal{A}| - \sum_{f \in \mathcal{F}} x^{1-|f|} \geq x \,,$$

then there exists arbitrarily long words over $\mathcal{A}$ that avoid $\mathcal{F}$.

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Fix $\mathcal{A}$ and $\mathcal{F}$.

For any n, let $\mathcal{C}_n$ be the set of words of length n over $\mathcal{A}$ that avoid $\mathcal{F}$.

Lemma

Under the Theorem hypothesis, for all $n \in \mathbb{N}$,

$$|\mathcal{C}_{n+1}| \geq x \cdot |\mathcal{C}_n| \,.$$

$\implies |\mathcal{C}_n| \geq x^n$

The proof by induction that $|\mathcal{C}_{n+1}| \geq x \cdot |\mathcal{C}_n|$

By the induction hypothesis, for all $j \in \{0, \ldots, n\}$, $|\mathcal{C}_{n-j}| \leq \frac{|\mathcal{C}_n|}{x^j}$.

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Back to the result

Theorem (Ochem, 2016 and Rosenfeld, 2022)

Let $\mathcal{A}$ be an alphabet and $\mathcal{F}$ be a set of forbiden factors over $\mathcal{A}$. Suppose that there exists a positive real x such that

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Corollary

Let $\mathcal{F}$ be a set of forbidden factors that contain at most one factor of each size in $\{5, 6, 7, \ldots\}$ and no shorter factor. Then the number of words of size n avoiding $\mathcal{F}$ over $\{0, 1\}$ is at least

$$\alpha_1^n \geq 1.755^n \,,$$

where α_1 is the largest root of $x^3 - 2x^2 + x - 1$.

Fractional repetitions

Let $\alpha(k, x)$ be the growth of the language of x-free words over the alphabet $\{1, 2, \ldots, k\}$.

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Conjecture (Shur, 2009)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) = k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

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Thue game is between Ann and Ben

- at their turn Ann and Ben add a letter at the end of the word,
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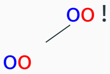

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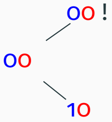

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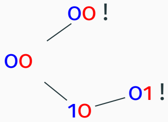

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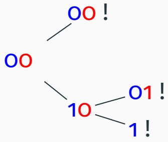

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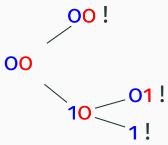

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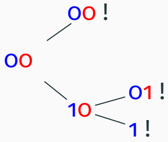

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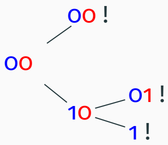

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Theorem (Rosenfeld, 2022)

Over 4 letters Ann has a winning stategy.

The counting argument applied to other objects

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Theorem (Rosenfeld, 2022)

Let G be a countable amenable group and $\mathcal{F}$ a set of forbidden patterns over the alphabet $\mathcal{A}$. If there exists a positive real β such that,

$$\beta + \sum_{f \in \mathcal{F}} |f| \beta^{1-|f|} \leq |\mathcal{A}|$$

then

$$\alpha(\mathcal{X}_{\mathcal{F}}) \geq \beta\,.$$

Thanks !