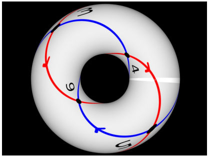

Embedding theorems, absolute retracts and the map extension property for multidimensional subshifts

Complexity of Simple Dynamical Systems, 12 – 16 February, 2024, CIRM

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- Introduce a new class of multidimensional subshifts of finite type:Subshifts satisfying the map extension property, or absolute retracts.
- Note: Closely related notions have recently been presented by Leo Poirier and Ville Salo in "Contractible subshifts", arXiv:2401.16774.

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Γ-subshift $Y \subseteq A^\Gamma$ is call a subshift of finite type (SFT) if there exists a finite set $W \Subset \Gamma$ and a finte set of forbidden patterns $\mathcal{F} \subset A^W$ such that

$$Y = \{y \in A^\Gamma \ : \ \sigma_v(y)_W \notin \mathcal{F}\}.$$

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Definition (Admissible patterns)

Given $F \Subset \Gamma$, we denote

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- If X factors onto Y then $h(X) \geq h(Y)$.

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Here, $P_n(X)$ is the number of points in X that have least period n.

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- Krieger's theorem implies: A $\mathbb{Z}$-subshift X properly topologically embeds in a topologically mixing $\mathbb{Z}$-SFT Y if and only if X non-densely Borel embeds in Y.

Lightwood's embedding theorem (aperiodic subsystems)

Theorem (Lightwood, 2003+2004)

Let X be an aperiodic $\mathbb{Z}^2$ subshift and let Y be a square-filling mixing $\mathbb{Z}^2$-SFT. Then X embeds in Y if and only if $h(X) < h(Y)$.

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Given a Γ-subshift X and a subgroup $\Gamma_0 \leq \Gamma$ let

$$X_{[\Gamma_0]} = \{x \in X \ : \ \Gamma_0 \subseteq stab(x)\}.$$

and

$$X_{\Gamma_0} = \begin{cases} X_{[\Gamma_0]} & \text{if } [\Gamma : \Gamma_0] = \infty \\ \{x \in X \ : \ stab(x) = \Gamma_0\} & \text{if } [\Gamma : \Gamma_0] < \infty. \end{cases}$$

Quotients of stabilizers of points in subshifts

Definition

Given a Γ-subshift $X \subseteq A^\Gamma$ and a subgroup $\Gamma_0 < \Gamma$ let $\overline{X}_{[\Gamma_0]}, \overline{X}_{\Gamma_0} \subseteq \overline{A}^{\Gamma/\Gamma_0}$ denote the natural images of $X_{[\Gamma_0]}$ and X_{Γ_0} respectively. Namely,

$$\overline{X}_{[\Gamma_0]} = \left\{ \overline{x} \in A^{\Gamma/\Gamma_0} \ : \ \exists x \in X_{[\Gamma_0]} \text{ s.t. } x_v = \overline{x}_{v+\Gamma_0} \ \forall v \in \Gamma \right\}$$

and

$$\overline{X}_{\Gamma_0} = \left\{ \overline{x} \in A^{\Gamma/\Gamma_0} \ : \ \exists x \in X_{\Gamma_0} \text{ s.t. } x_v = \overline{x}_{v+\Gamma_0} \ \forall v \in \Gamma \right\}.$$

Both $\overline{X}_{[\Gamma_0]}$ and $\overline{X}_{\Gamma_0}$ are closed (Γ/Γ_0)-invariant sets, and that there are continuous bijections $\overline{X}_{\Gamma_0} \leftrightarrow X_{\Gamma_0}$ and $\overline{X}_{[\Gamma_0]} \leftrightarrow X_{[\Gamma_0]}$. Thus, $\overline{X}_{\Gamma_0}$ and $\overline{X}_{[\Gamma_0]}$ are (Γ/Γ_0)-subshifts (whenever they are non-empty).

Subsystems of the full-shift in $\mathbb{Z}^2$

Theorem (M. 2023+)

A $\mathbb{Z}^2$-subshift X embeds in $Y = A^{\mathbb{Z}^2}$ if and only if either X is topologically conjugate to $Y = A^{\mathbb{Z}^2}$ or $h(X) < h(Y) = \log|A|$ and the following conditions hold:

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2. *For every pair of linearly independent vectors $v, w \in \mathbb{Z}^2$, $|X_{\langle v,w \rangle}| \leq |Y_{\langle v,w \rangle}|$.*

The multidimensional embedding theorem

Theorem (M. 2023+)

Let Γ be a countable abelian group and let X, Y be Γ-subshifts. Suppose that Y has the map extension property. Then $X \hookrightarrow Y$ if and only if for every subgroup $\Gamma_0 \leq \Gamma$ one of the conditions holds:

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Where:

$$Ker(X) = \{v \in \Gamma \ : \ \sigma_v(x) = x \text{ for every } x \in X\}.$$

Equivalently,

$$Ker(X) = \bigcap_{x \in X} stab(x).$$

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Remark: The precise definition involves another condition. At least for finitely generated abelian groups the additional condition is a consequence of the above definition.

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- Definition: Let $\mathcal{C}$ be a class of subshifts, which is closed under isomorphism and passing to subsystems. A subshift Y is said to be an absolute retract for the class $\mathcal{C}$ if $Y \in \mathcal{C}$ and whenever Y embeds in $X \in \mathcal{C}$ then Y is a retract of X.

Absolute retracts and the map extension propery

- Proposition: A Γ-subshift Y has the map extension property if and only if it there exists a finite set $\mathcal{G} \Subset \mathrm{Sub}(\Gamma)$ such that Y is an absolute retract for the class of $\mathcal{G}$-free subshifts.

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- In particular a $\mathbb{Z}$-subshift Y is a topologically mixing SFT if and only if Y has the map extension property, if and only if there exists $F \Subset \mathbb{N}$ such that Y is an absolute retract for the class of subshifts having no k-periodic points for all $k \in \mathbb{N}$.

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- The map extension property passes to $Y_{[\Gamma_0]}$: If Y has the map extension property the $Y_{[\Gamma_0]}$ also has the map extension property.
- If Y has the map extension property then Y has "many periodic points". Eg.

$$\lim_{\Gamma_0 \to \{0\}} h(\overline{Y}_{[\Gamma_0]}) = h(Y).$$

and if Γ is residually finite then the points with finite orbit are dense in Y.

Some examples of subshifts with the map extension property

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- k-colored chessboards (in $\mathbb{Z}^2$) with $k \geq 5$.

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- k-colored chessboards (in $\mathbb{Z}^2$) with$k \geq 5$.
- More generally proper $(|S|+1)$-colorings of a symmetric Cayley graph of Γ with respect to a finite symmetric generating set $S \Subset \Gamma \setminus \{0\}$.
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The relative multidimensional embedding theorem

Theorem (M. 2023+)

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- Reduction to the case of finitely generated abelian groups (a soft argument, using that the target is a subshift of finite type).
- Proceeds via induction on the "size" of the finitely generated abelian group (rank, size of torsion).
- Apply a a version of Krieger's marker lemma and Voronoi diagrams using convex geometry to form equivariant partial tiling by almost-invariant subsets.
- At an intermediate step, embed into a slightly bigger subshift $\hat{Y}$ that contains Y, then retract into Y.

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Let Γ be a countable group.

Question

If a Γ-subshift X Borel embeds in $Y = A^\Gamma$ so that the image of X_{Γ_0} is not dense in Y_{Γ_0} for any $\Gamma_0 < \Gamma$, does X topologically embeds in $Y = A^\Gamma$?

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Theorem (Boyle, 1983)

Let X, Y be irreducible $\mathbb{Z}$-subshifts of finite type with $h(X) > h(Y)$. Then there exists a factor map from X to Y if and only if $X \rightsquigarrow Y$.

The Briceno-Mcgoff-Pavlov lower entropy factor theorem:

Theorem (Briceno-Mcgoff-Pavlov , 2018)

Let X be a block gluing $\mathbb{Z}^d$-subshift and let Y be a $\mathbb{Z}^d$-subshift of finite type with a fixed point and the finite extension property such that $h(X) > h(Y)$. Then there exists a factor map from X to Y.

A multidimensional lower entropy factor theorem

Theorem (M. 2023+)

Let X be a block gluing $\mathbb{Z}^d$-subshift and let Y be a $\mathbb{Z}^d$-subshift with the map extension property such that $h(X) > h(Y)$. Then there exists a factor map from X to Y if and only if $X \rightsquigarrow Y$.