Symbolic dynamics and representations of matrices

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Plan

Conjugacy

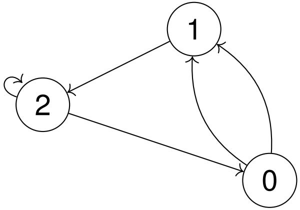
Main objective: decide conjugacy of shifts of finite type

Upto conjugacy, every subshift of finite type is effectively an edge shift, so this can be discussed using matrices.

Edge shift

An edge shift is the set of all biinfinite walks (on edges) in a finite graph.

Edge shifts are conjugate if they're isomorphic via local transformations (cellular automata).



$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

SSE

This can be defined directly on adjacency matrices:

Definition

Two matrices M and N are 1-step equivalent if $M = RS$ and $N = SR$ for (nonnecessarily square) nonnegative integral matrices R, S

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Example:

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

SSE

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Two matrices M and N are 1-step equivalent if $M = RS$ and $N = SR$ for (nonnecessarily square) nonnegative integral matrices R, S

Definition

Strong shift equivalence (SSE) is the transitive closure of 1-step equivalence.

Graphs G_1 and G_2 represent conjugate edge shifts iff their adjacency matrices are SSE.

History

Main open problem of symbolic dynamics: decide conjugacy/SSE

- Open since the 70s
- Decidable for matrices in $\mathbb{Z}$ (Krieger, 1980)
  - (almost the) same as conjugacy in $GL_n(\mathbb{Z})$
- Decidable for one-sided edge-shifts (Williams, 1973)
  - The rewriting system on graphs is confluent.

In this talk:

- Represent matrices with generators and relations
- Use it to obtain invariants.

Plan

Fact

Every matrix over $\mathbb{Z}_+$ can be obtained from the following 6 matrices:

$$\begin{pmatrix}1\end{pmatrix} \qquad \begin{pmatrix}1 & 1\end{pmatrix} \qquad \begin{pmatrix}1\\1\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad (\quad) \qquad \left(\ \right)$$

with two composition laws:

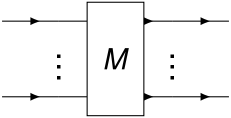
- Multiplication: $A \times B$
- Sum: $A \oplus B = \begin{pmatrix}A & 0\\0 & B\end{pmatrix}$

$(\quad)$ is the matrix with two columns and 0 rows, equivalently the unique function : $\mathbb{Z}_+^2 \to \{0\}$

Example

$$\begin{pmatrix}1 & 3\\ 2 & 0\end{pmatrix}$$
$$=$$
$$\left(\begin{pmatrix}1 & 1\end{pmatrix} \oplus 1\right)\left(\begin{pmatrix}1 & 1\end{pmatrix} \oplus \begin{pmatrix}1 & 1\end{pmatrix} \oplus \begin{pmatrix}1 & 1\end{pmatrix}\right) \times$$
$$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix} \times$$
$$\left(\begin{pmatrix}1\\ 1\end{pmatrix} \oplus 1 \oplus \begin{pmatrix}1\\ 1\end{pmatrix} \oplus 1\right)\left(\begin{pmatrix}1\\ 1\end{pmatrix} \oplus \begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

To understand that, we represent matrix graphically



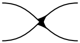
identifying the matrix M of m rows and n columns with the linear function $\mathbb{Z}_+^n \to \mathbb{Z}_+^m$ it represents.

Dictionary

$$\begin{pmatrix} 1 \end{pmatrix}$$

Dictionary

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

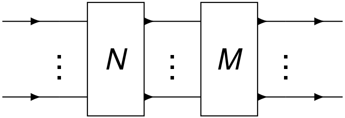


Dictionary

$$MN$$

Dictionary

MN



associativity is for free in the graphical representation

Dictionary

$$M \oplus N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

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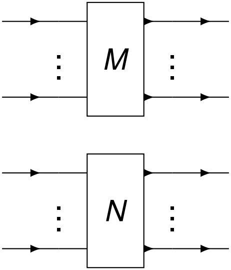
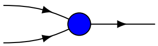


Dictionary

$\begin{pmatrix}1 & 1\end{pmatrix}$ is essentially the addition.



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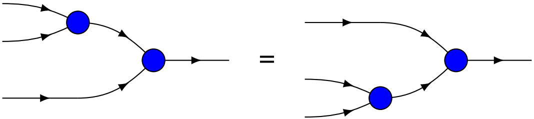

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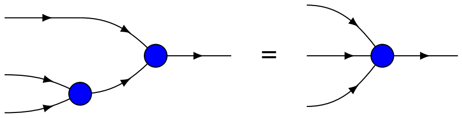

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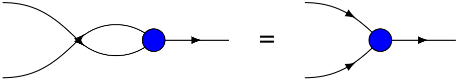
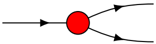


Dictionary

$\begin{pmatrix}1\\1\end{pmatrix}$ is the copy $x \mapsto (x, x)$.



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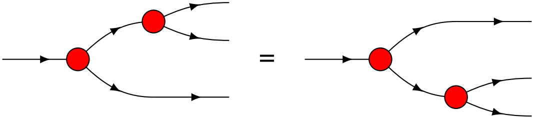

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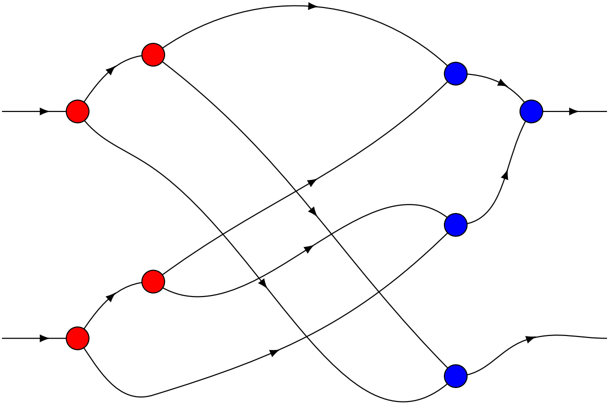

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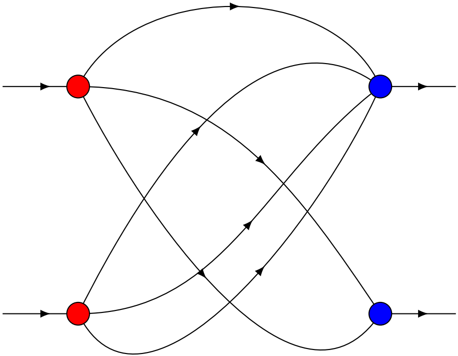


props

Instead of dealing with matrices, we can deal with pictures

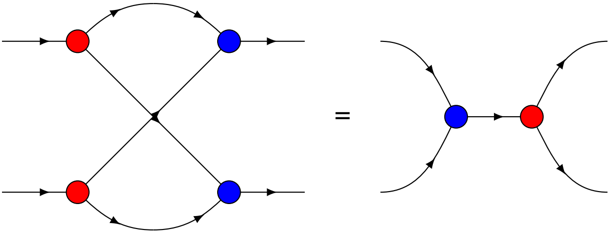
This gives a presentation of matrices in terms of generators and relations.

Presentation as a *prop* which is something from category theory with two composition rules.

Axioms

Two composition laws: sequential, parallel.
Two generators:

Relations: (co)associativity, (co)commutativity, and bialgebra:



(folklore, see also Pirashvili 2002, and Zanasi 2015)

Plan

Goal

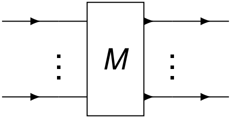
Can we do the same for matrices quotiented by SSE ?

(to decide conjugacy)

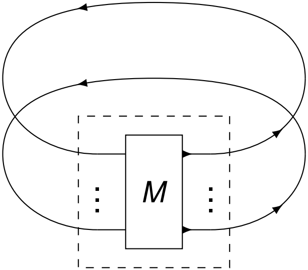
How to say that $MN \equiv NM$?

(Note: We cannot directly say $MN = NM$: a 2×2 matrix $\neq 3 \times 3$ matrix)

Idea: replace

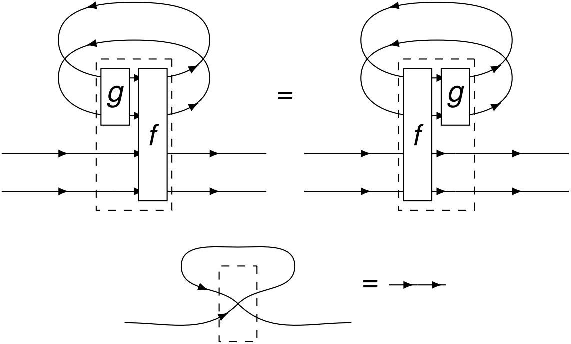


by



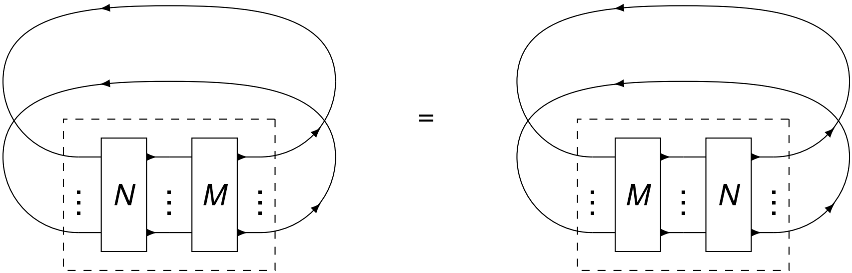
Trace

The trace (feedback loop) is an operator that takes a diagram and loops one wire s.t.:



Trace

In particular

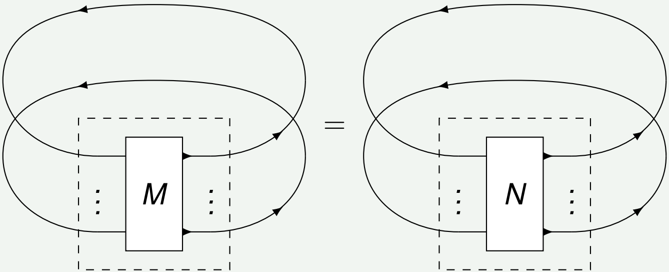


Trace

- Consider the prop given by the previous generators and relations
- Add a trace (feedback loop)

Theorem

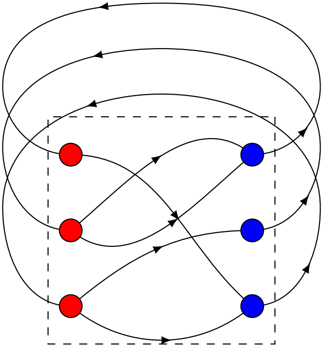
M and N are SSE iff

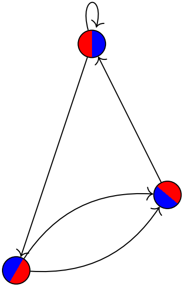


using the equations

Trace

$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$





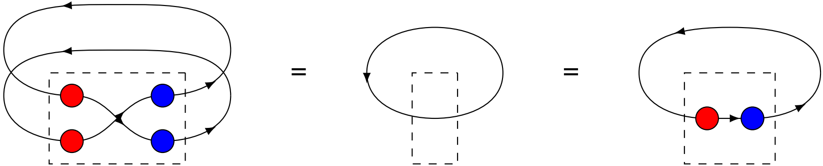
Problem

Theorem

The previous theorem is not a theorem

Flow equivalence

In this context: $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N = (1)$ are equivalent, and they shouldn't be.



Flow equivalence

Theorem

The graphical representation captures flow equivalence, not SSE equivalence.

(First observed by David Hillman, 1995)

We have lost a notion of time, that we need to recover

Solution

We add a notion of time: something that commutes with everything, and just needs to be there.

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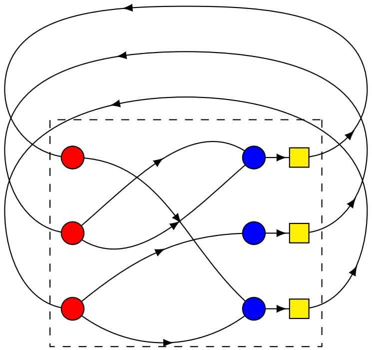
$$M = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



Theorem

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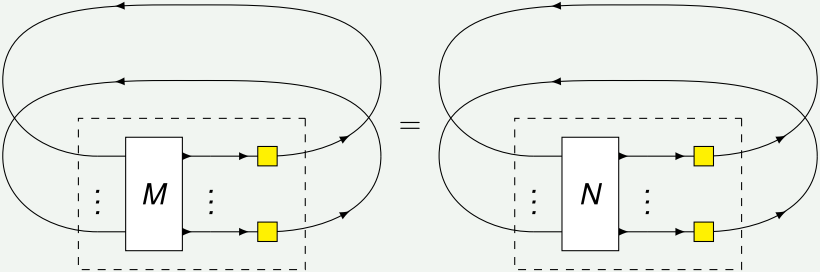


using the equations.

Idea of the proof: Now we are representing matrices in $\mathbb{Z}_+[t]$, use positive K-theory from Boyle-Wagoner.

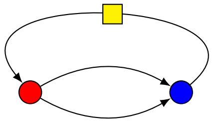
Example

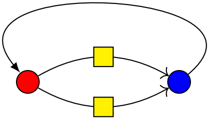
$$\begin{pmatrix}2\end{pmatrix}$$

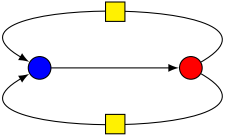
is SSE to

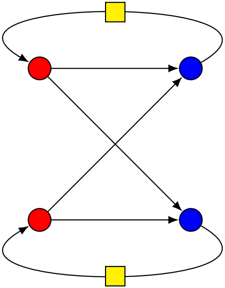
$$\begin{pmatrix}1 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$$

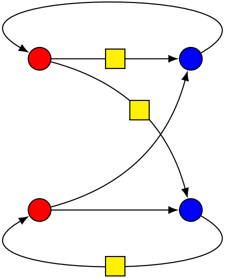
(2)

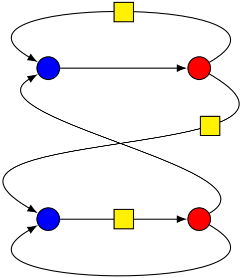


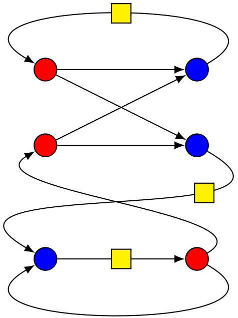


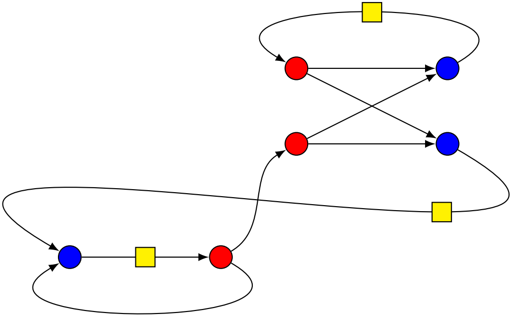


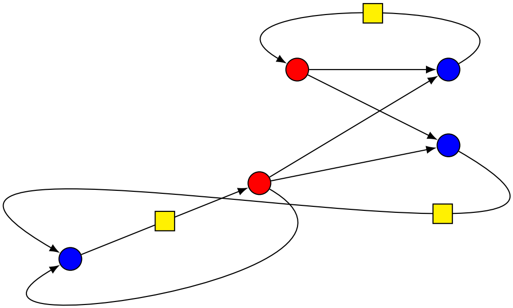


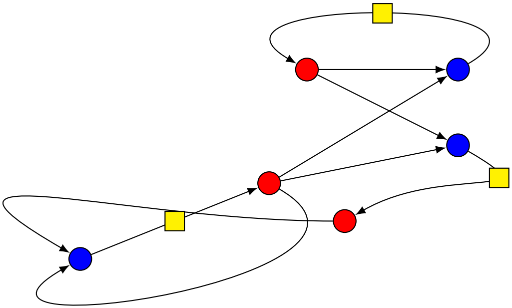


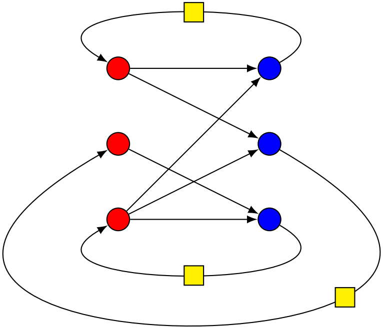












$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Plan

Applications

What do we gain from it ?

- If one knows a concrete representation of this prop, one can decide conjugacy
- One can give a definition of SSE equivalence that is compositional
- One can find invariants !

Methodology

- Forget about matrices
- Find structures in the wild that have “objects” that satisfy the axioms

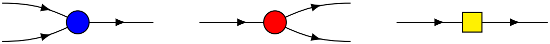
Then

- Structure A (matrices in $\mathbb{Z}_+[t]$ (with loops)) satisfies all the axioms, and no additional ones
- Structure B also satisfies the axiom
- There is a morphism from A to B.

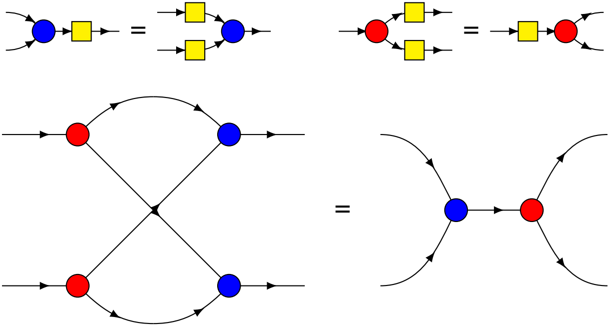
We obtain an invariant of conjugacy: $M \equiv N$ implies $\phi(M) = \phi(N)$.

Axioms

Two composition laws: sequential, parallel. One trace (loop)
Three generators:



Relations: (co)associativity, (co)commutativity, and:



Good news/Bad news

Good news: There are a lot of such structures in the wild

Bad news:

- They either have less structure, or more structure
- Some of them are Hopf algebras
  - Hopf algebras represent matrices with coefficients in $\mathbb{Z}$, not in $\mathbb{Z}_+$
- Some of them do not have loops
  - We need to tweak them to have loops.

But it still works !

Example: Monoids

- Let $\mathcal{M}$ be a commutative monoid.
- Product: monoid addition
- Coproduct: copy
- Trace: fixed points

Proposition

Let $\mathcal{M}$ be a commutative monoid and h a morphism. The number $\psi(A)$ of solutions of the system

$$\begin{array}{rcl} h(a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n) & = & x_1 \\ h(a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n) & = & x_2 \\ & \dots & \\ h(a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n) & = & x_n \end{array}$$

is an invariant of conjugacy.

$\mathcal{M} = \mathbb{C}$, $h(x) = \lambda x$: The nonzero eigenvalues are invariant.

Example: Polynomials

- Maps $n \to m$ are linear maps from $\mathbb{K}[X_1 \dots X_n]$ to $\mathbb{K}[Y_1, Y_m]$
- Product $(2 \to 1)$: $\mathbb{K}[X_1, X_2] \to \mathbb{K}[Y]$ identifies X_1 and X_2.
- Coproduct $(1 \to 2)$ is defined by $\Delta(X^k) = (Y_1 + Y_2)^k$
- Trace: If p is a polynomial $\mathbb{K}[X_1 \dots X_{n-1}]$ to $\mathbb{K}[Y_1, Y_{m-1}]$, then

$$(tr\ f)(p) = \sum_k [Z^k] f(p, Z^k)$$

where $[Z^k]q$ is the coefficient of degree Z^k of q

Problem: Does not work: infinite sum

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Solution: Replace $\mathbb{K}[X]$ by formal series in a complete semiring (like $\mathbb{R}_+ \cup \{+\infty\}$).

If we take the morphism $h(X) = tX$ we get:

Theorem

The quantity $f_M(t) = \frac{1}{det(I-tM)}$ is an invariant for SSE

This is the well-known Zeta function of a subshift.

Example: Groups

- Maps $n \to m$ are abelian groups with $n + m$ distinguished points
- Composition = identifying points (quotienting)
- Product: $2 \to 1$ the group $\langle x_1, x_2, y | y = x_1 + x_2 \rangle$ with points x_1, x_2, y.
- Coproduct: $1 \to 2$ the group $\langle x, y_1, y_2 | y_1 = y_2 = x \rangle$ with points x, y_1, y_2
- Trace: identifying points

We obtain the *Bowen-Franks* group (1977)

Example: $\mathbb{Z}[t]$-modules

- Maps $n \to m$ are $\mathbb{Z}[t]$ modules with $n + m$ distinguished points
- Product: $2 \to 1$ the module $\langle x_1, x_2, y | y = x_1 + x_2 \rangle$
- Coproduct: $1 \to 2$ the module $\langle x, y_1, y_2 | y_1 = y_2 = x \rangle$
- Trace: identifying points
- Morphism: h is the $\mathbb{Z}[t]$ module $\langle x, y | y = tx \rangle$

We obtain the dimension group of Krieger (1977)

Plan

Where to go from here ?

A systematic way to obtain invariants for symbolic dynamics by looking at algebraic structures.
We recover the classical invariants, which proves the method works:

- The Zeta function
- The Bowen-Franks group
- The Dimension group

Now: find new invariants!