Finite-dimensional pseudofinite groups of small dimension, without CFSG

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Finite-dimensional groups

Definition 1.1. (Wagner) A group G is *finite-dimensional* if there is a dimension function $\dim$ from the collection of all interpretable sets in models of $\mathrm{Th}(G)$ to $\mathbb{N} \cup \{-\infty\}$ such that, for any formula $\phi(x, y)$ and interpretable sets X and Y, the following hold:

1. Invariance: If $a \equiv a'$ then $\dim(\phi(x, a)) = \dim(\phi(x, a'))$.
2. Algebraicity: If $X \neq \emptyset$ is finite then $\dim(X) = 0$, and $\dim(\emptyset) = -\infty$.
3. Union: $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
4. Fibration: If $f : X \to Y$ is an interpretable map such that $\dim(f^{-1}(y)) \geqslant d$ for all $y \in Y$ then $\dim(X) \geqslant \dim(Y) + d$.

In this talk, 'a finite-dimensinal group' always means a finite-dimensional group with *fine* and *additive* dimension: $\dim$ is

- additive if $\dim(a, b/C) = \dim(a/b, C) + \dim(b/C)$ holds for any tuples a and b and for any set C; and
- fine if $\dim(X) = 0$ implies that X is finite.

Additivity ensures that Lascar equality holds: If G is a finite-dimensional group and $H \leqslant G$ is definable, then $\dim(G) = \dim(H) + \dim(G/H)$.

Example 1.2. Any supersimple finite SU-rank group is finite-dimensional; thus $\mathrm{PSL}_2(F)$ where F is pseudofinite field, is finite-dimensional.

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icc^0, commensurability and almost operators

Let G be a group, $H, K \leqslant G$. Then H is *almost contained* in K, written $H \lesssim K$, if $[H : (H \cap K)] < \infty$; and $H, K \leqslant G$ are *commensurable*, written $H \simeq K$, if both $H \lesssim K$ and $K \lesssim H$. A family $\mathcal{H}$ of subgroups of G is *uniformly commensurable* if there is $n \in \mathbb{N}$ so that $|H_1 : H_1 \cap H_2| < n$ for all $H_1, H_2 \in \mathcal{H}$.

Definition 1.3. For subgroups H, K of a group G, define:

1. $\widetilde{N}_K(H) = \{k \in K : H \simeq H^k\}$, the *almost normaliser* of H in K.
2. $\widetilde{C}_K(H) = \{k \in K : H \lesssim C_H(k)\}$, the *almost centraliser* of H in K.
3. $\widetilde{Z}(G) = \widetilde{C}_G(G)$, the *almost centre* of G.

The commensurability is *uniform* in $\widetilde{N}_K(H)$ (resp. in $\widetilde{C}_K(H)$) if there is some $m \in \mathbb{N}$ so that if $H \simeq H^k$ then $|H : H \cap H^k| < m$ (resp. if $H \lesssim C_H(k)$ then $|H : C_H(k)| < m$).

It is easy to see that a finite-dimensional group G satisfies the chain condition on interscetions of uniformly definable subgroups, icc^0:

Given a family $\mathcal{H}$ of uniformly definable subgroups of G, there is $m < \omega$ so that there is no sequence $\{H_i : i \leqslant m\} \subset \mathcal{H}$ with $|\bigcap_{i<j} H_i : \bigcap_{i \leqslant j} H_i| \geqslant m$ for all $j \leqslant m$.

$\Rightarrow$: it follows that if $H, K \leqslant G$ are definable then the commensurability is uniform in $\widetilde{N}_K(H)$ and in $\widetilde{C}_K(H)$; hence $\widetilde{N}_K(H)$ and $\widetilde{C}_K(H)$ are definable subgroups.

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Pseudofinite groups

Definition 1.4. An infinite group G is *pseudofinite* if for every L_{gr}-sentence σ s.t. $G \models \sigma$ there is a finite group G_0 s.t. $G_0 \models \sigma$.

Łoś: A group G is pseudofinite $\Leftrightarrow$ $G \equiv \prod_{i \in I} G_i/U$, where U is a non-principal ultrafilter on I and for U-many i, the groups G_i's are finite of increasing orders.

Example 1.5.

- $(\mathbb{Z}, +)$ is not pseudofinite: Let σ say 'If the map $x \mapsto x + x$ is injective then it is surjective'. Then σ is true in all finite groups and false in $(\mathbb{Z}, +)$.
- $(\mathbb{R}, +) \equiv (\mathbb{Q}, +) \equiv \prod_{p_i \in P} C_{p_i}/U$, where P is the set of primes, are pseudofinite.
- The group $G = \prod_{n_i \in \mathbb{N}} Alt_{n_i}/U$, is pseudofinite and definably simple but not simple.

We denote a group of (twisted) Lie type over a field F by $X(F)$.

Theorem 1.6 (Wilson 1995, uses CFSG). A simple group G is pseudofinite $\Leftrightarrow$ $G \equiv X(F)$, where F is a pseudofinite field.

- Ryten: '$\equiv$' generalises to '$\cong$' without *further* use of CFSG.
- Theorem 1.6 implies that a) a simple pseudofinite group is supersimple of finite SU-rank, thus finite-dimensional b) the only simple pseudofinite group of dimension $\leqslant 3$ is $\mathrm{PSL}_2(F)$ with $\dim(\mathrm{PSL}_2(F)) = 3$.

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Structural results for pseudofinite groups

If the *soluble radical* (i.e. a maximal soluble normal subgroup) of a group G exists, then it is denoted by $\mathrm{Rad}(G)$.

Theorem 1.7 (Wilson 2009, uses CFSG). Let G be a finite group. Then there is an L_{gr}-formula $\phi_R(x)$ s.t. $\mathrm{Rad}(G) = \phi_R(G)$.

$\Rightarrow$: If G is pseudofinite then $\overline{G} := G/\phi_R(G)$ is *semi-simple* (i.e. no proper non-trivial abelian normal subgroup), and $\mathrm{Rad}(G)$ exists $\Leftrightarrow \mathrm{Rad}(G) = \phi_R(G) \Leftrightarrow \phi_R(G)$ is soluble.

- Khukhro 2009: If the *centraliser dimension* $\mathrm{cd}(G)$ of G is $k \in \mathbb{N}$ then $\mathrm{Rad}(G)$ exists.
- Milliet 2013: If G is supersimple then $\mathrm{Rad}(G)$ exists.
- K.-Wagner 2024: If G is finite-dimensional then $\mathrm{Rad}(G)$ exists.

1. Macpherson-Tent 2012: If G is NIP then $\overline{G}$ is finite; if G is non-stable then G is not soluble-by-finite in general.
2. Milliet 2013: If G is supersimple then $\overline{G}$ has a definable normal subgroup $\overline{L}$ which is a finite direct product of simple (pseudo)finite groups and $\overline{G}/\overline{L}$ is finite-by-abelian.
3. K. 2022: If $\mathrm{cd}(G) = k$ then G is as in 2.
4. K.-Uğurlu Kowalski 2024: If $\mathrm{cd}(G) = k$ and for any $N \trianglelefteq \trianglelefteq G$ we have $C_G(N) = 1$, then there is a simple $\emptyset$-definable subgroup $X(F) \trianglelefteq G$ with $G/X(F)$ finite-by-abelian.

Theorem 1.17 and 1-4 all use CFSG!!

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Our results

Question 1.8 (Elwes-Jaligot-Macpherson-Ryten; Macpherson; Palacìn,...).
What can be proven without CFSG? In particular, can one prove without CFSG that a supersimple of finite SU-rank, simple, pseudofinite group with $SU(G) = 3$ is isomorphic to $\mathrm{PSL}_2(F)$?

Let G be a pseudofinite finite-dimensional group.

- Wagner 2020 (in similar settings EJMR, Hempel-Palacín,...): If $\dim(G) = 1$ then G is finite-by-abelian-by-finite (FAF).
- Wagner 2020: If $\dim(G) = 2$ then G is soluble-by-finite (follows from Wilson's classification of simple pseudofinite groups).

Theorem 1.9 (Without CFSG, Wagner/K.-Wagner). Let G be a pseudofinite finite-dimensional group. If $\dim(G) = 2$ then G is soluble-by-finite. Moreover, if G is not FAF, then there is a definable subgroup $N \trianglelefteq G$ with $\dim(N) = 1$.

Theorem 1.10 (Without CFSG, K.-Wagner). Let G be a pseudofinite finite-dimensional group. If $\dim(G) = 3$, then either G is soluble-by-finite, or $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ has a definable normal subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ where F is a pseudofinite field.

Question 1.11. Can one prove without CFSG that a simple pseudofinite group is finite-dimensional?

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Theorem 1.10 (Without CFSG, K.-Wagner). Let G be a pseudofinite finite-dimensional group. If $\dim(G) = 3$, then either G is soluble-by-finite, or $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ has a definable normal subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ where F is a pseudofinite field.

Question 1.11. Can one prove without CFSG that a simple pseudofinite group is finite-dimensional?

Tools from finite group theory

We use:

Theorem 1.12 (Feit-Thompson 1963). *A finite group with no involutions is soluble.*

Theorem 1.13. *Let G be a finite simple non-abelian and 'large' group. If at least one of the following holds then $G \cong X(\mathbb{F}_q)$, where $X \in \{\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3, \mathrm{Sz}\}$.*

- *(Gorenstein et al.) $m_2(G) \leqslant 2$, i.e., if $j, k, \ell \in I(G)$ are distinct and pairwise commuting then $jk = \ell$.*
- *(Bender) G has a strongly embedded subgroup, i.e., there is $1 \neq C < G$ so that $I(C) \neq \emptyset$ and, whenever $g \in G \setminus C$, then $I(C \cap C^g) = \emptyset$.*

$\Rightarrow$: If $G \equiv \prod_{i \in I} G_i/U$ is a semi-simple pseudofinite finite-dimensional group with $\dim(G) \leqslant 3$. Then $I(G) \neq \emptyset$. Further, if either

- $m_2(G) \leqslant 2$, or
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We also use the following generalisation of the famous Brauer-Fowler Theorem (1955):

Theorem 1.14 (Hempel-Palacín). *Let G be a pseudofinite group with $\widetilde{Z}(G) = 1$. Then $C_G(i)$ is infinite for any $i \in I(G)$.*

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Schlichting's Theorem

Theorem 1.15 (Schlichting/Wagner). *Let G be a group and $\mathcal{H}$ a uniformly commensurable family of subgroups of G. Then there is a subgroup $H \leqslant G$ which is uniformly commensurable to all members of $\mathcal{H}$ and is invariant under all automorphisms of G which fix $\mathcal{H}$ setwise.*

An infinite group is called *almost simple* if it is not abelian-by-finite and has no definable normal subgroup of infinite index. It is not hard to verify the following things:

1. Let G be an almost simple icc^0-group.
   a. If H is an infinite definable subgroup of infinite index then $\widetilde{N}_G(H) < G$.
   b. If H is an infinite definable solvable-by-finite subgroup, then $\widetilde{N}_G(H)$ has infinite index in G (note that almost simple implies semi-simple).
2. Let G be a pseudofinite finite-dimensional group with $\dim(G) \leqslant 3$ and $H < G$ a definable subgroup of G s.t. $\dim(H) = 2$ and $\widetilde{N}_G(H) \neq G$. Then either $\widetilde{Z}(H)$ is finite or $\dim(\widetilde{Z}(G)) \geqslant 1$.

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2-dimensional subgroups

Lemma 1.16. *Let G be an almost simple pseudofinite finite-dimensional group with $\dim(G) = 3$. Assume that all definable subgroups of dimension 2 are soluble-by-finite. If L is a definable subgroup of dimension 2, then:*

1. *$B = \widetilde{N}_G(L)$ is a maximal definable subgroup of dimension 2. Moreover, $B = \widetilde{N}_G(B)$; if $Z = \widetilde{Z}(B)$ then Z is finite, and B/Z is a Frobenius group with Frobenius kernel U/Z and Frobenius complement T/Z, where $U = \widetilde{C}_G(U)$ and $T = \widetilde{C}_B(T)$ have dimension 1 and are finite-by-abelian.*
2. *There is a pseudofinite field F such that $U/Z \cong F^+$ and T/Z embeds into $F^\times$ as a subgroup of finite index. In particular T/Z is abelian, and T has only finitely many elements of any given order.*
3. *If $g \in G \setminus B$ then $U \cap U^g = 1$. If B contains involutions, then either B is strongly embedded, or there is an involution in $T \setminus Z$ and no involution in $U \setminus Z$.*
4. *Suppose $Z \neq 1$. If $g \in G \setminus B$ then $B \cap B^g$ is a finite index subgroup of T^u for some $u \in U$. Moreover, if $x \in B$ with $\dim(C_G(x)) = 2$ then $x \in Z$.*

Identifying $\mathrm{PSL}_2(F)$

Let G be an almost simple pseudofinite finite-dimensinal group with $\dim(G) \leqslant 3$; if $\dim(G) = 3$, let 2-dimensional subgroups be soluble-by-finite. Then G is a finite extension of $\mathrm{PSL}_2(F)$:

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- Claim 3: Any two $x, y \in I(C_i)$ with $\dim(C_G(x)) = \dim(C_G(y)) = 1$ commute. Further, if $x, y, z \in I(C_i)$ are distinct with $\dim(C_G(x)) = \dim(C_G(y)) = \dim(C_G(z)) = 1$ then $xyz = 1$.
- Claim 4: $\dim(C_G(j)) = 1$ for all $j \in I(G)$.

$\Rightarrow$: By $m_2(G) > 2$ there are pairwise commuting $x, y, z \in I(G)$ with $xyz \neq 1$. Claim 4 and 3: either $C_x = C_y \neq C_z$ or $C_x \neq C_y \neq C_z$; in the former case $z \in N_G(C_x) \setminus C_x$ centralises $x, y \in C_x$ contradicts Claim 2 and in the latter case $yz \in C_x$ is centralised by $y \in N_G(C_x) \setminus C_x$, thus Claim 2 implies $yz = x$ contradicting $xyz \neq 1$.

Concluding our results

- Let G be finite-dimensional pseudofinite group with $\dim(G) = 2$. Then G is soluble-by-finite:

Proof: If $\exists N \trianglelefteq G$ definable with $\dim(N) = 1$ then, setting $\overline{G} = G/C_N((\widetilde{Z}(N))')$, $\overline{S} = C_{\overline{G}}((\widetilde{Z}(\overline{G}))')$ is 2-nilpotent of finite index in $\overline{G}$; its preimage is soluble of finite index in G. Otherwise, every finite normal subgroup of G is contained in $\widetilde{Z}(G)$; if G is not finite-by-abelian-by-finite then $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ is almost simple of dimension 2. By the last slide, $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$ so $\dim(G) = 3$; a contradiction.

- Let G be finite-dimensional pseudofinite group with $\dim(G) = 3$. If G is not soluble-by-finite then $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$.

Proof: If $\exists N \trianglelefteq G$ definable with $1 \leqslant \dim(N) \leqslant 2$, then, by the above, N and G/N are soluble-by-finite and $C_G(N/\mathrm{Rad}(N))$ is a soluble-by-finite finite index subgroup of G. If such N does not exists and G is not finite-by-abelian-by-finite then $\widetilde{Z}(G)$ is finite and we get that $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$.

Concluding our results

- Let G be finite-dimensional pseudofinite group with $\dim(G) = 2$. Then G is soluble-by-finite:

Proof: If $\exists N \trianglelefteq G$ definable with $\dim(N) = 1$ then, setting $\overline{G} = G/C_N((\widetilde{Z}(N))')$, $\overline{S} = C_{\overline{G}}((\widetilde{Z}(\overline{G}))')$ is 2-nilpotent of finite index in $\overline{G}$; its preimage is soluble of finite index in G. Otherwise, every finite normal subgroup of G is contained in $\widetilde{Z}(G)$; if G is not finite-by-abelian-by-finite then $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ is almost simple of dimension 2. By the last slide, $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$ so $\dim(G) = 3$; a contradiction.

- Let G be finite-dimensional pseudofinite group with $\dim(G) = 3$. If G is not soluble-by-finite then $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$.

Proof: If $\exists N \trianglelefteq G$ definable with $1 \leqslant \dim(N) \leqslant 2$, then, by the above, N and G/N are soluble-by-finite and $C_G(N/\mathrm{Rad}(N))$ is a soluble-by-finite finite index subgroup of G. If such N does not exists and G is not finite-by-abelian-by-finite then $\widetilde{Z}(G)$ is finite and we get that $G/\widetilde{Z}(G)$ is a finite extension of $\mathrm{PSL}_2(F)$.

Thank you!