A Borovik-Cherlin bound for primitive pseudo-finite permutation groups

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Permutation Groups

- A *permutation group* (G, X) is a group G together with a faithful action of G on the set X.
- The permutation group (G, X) is called *transitive* if it has only one orbit—that is, given any $x, y \in X$, there is $g \in G$ with $g \cdot x = y$.
- Similarly, (G, X) is called 2-*transitive* if the induced action on pairs of distinct elements from X has only one orbit—that is, if $x \neq x'$ and $y \neq y'$ are from X, there is $g \in G$ with $g \cdot (x, x') = (y, y')$.
- Finally, (G, X) is called *primitive* if there is no nontrivial equivalence relation on X preserved by G (trivial equivalence relations are $=$ or X^2). We have

$$\text{2-transitive} \implies \text{primitive} \implies \text{transitive}$$

Primitivity

- Permutation groups are everywhere and the *primitive* permutation groups are the building blocks.
- If (G, X) is a transitive permutation group, then (G, X) is primitive if and only if each/some point stabilizer G_x is a maximal subgroup of G.
- (Higman) If (G, X) is a permutation group, then there is an induced action of $G \curvearrowright \binom{X}{2}$ and an orbit Ω of this action is called an *orbital*. The group G acts as automorphisms of the graph (X, Ω) on X with edge relation Ω and (G, X) is primitive if and only if each orbital graph is connected.

The O'Nan-Scott Theorem

- Finite primitive permutation groups (G, X) are roughly classified by the O'Nan-Scott Theorem into classes based on the *socle* $\mathrm{Soc}(G)$, which is the subgroup generated by all minimal normal nontrivial subgroups. The socle $\mathrm{Soc}(G)$ is always a product of isomorphic finite simple groups. The following is roughly the breakdown:

1. Abelian socle: then X can be identified with an n-dimensional $\mathbb{F}_p$-space V and $G = V \rtimes H$ for some irreducible $H = G_0 \leq GL_n(p)$.
2. Non-abelian socle: $\mathrm{Soc}(G) = T^k$ for non-abelian simple T and (G, X) is one of
   2.1 Almost simple: $k = 1$ and $T \leq G \leq \mathrm{Aut}(T)$.
   2.2 Diagonal action: $k \geq 2$ and X is identified with a diagonal copy of T in T^k and G acts on the cosets.
   2.3 Product action: $k \geq 2$ and there is a primitive (H, Y) in the almost simple case, transitive $P \leq S_k$, and $G = H \wr K = H^k \rtimes P$ acting primitively on Y^k.

Definable primitivity

- Suppose (G, X) is a permutation group definable in some structure. We say (G, X) is *definably primitive* if there is no non-trivial *definable* G-invariant equivalence relation on X.
- The transitive definable permutation group (G, X) is definably primitive if and only if each point stabilizer G_x is *definably maximal*—that is, there is no proper definable subgroup $H \lneq G$ with $G_x \lneq H$.
- We have primitivity $\implies$ definable primitivity.

The Borovik-Cherlin Bound

Theorem (Borovik-Cherlin)

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if (G, X) is a definably primitive permutation group of finite Morley rank, then

$$\mathrm{RM}(G) \leq f(\mathrm{RM}(X)),$$

in other words, the rank of the group can be bounded by the rank of the set on which it acts.

- The proof relies heavily on the finite Morley rank analogue of the O'Nan-Scott Theorem due to Macpherson and Pillay, as well as results of Wagner on fields of finite Morley rank.

Question

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Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that if (G, X) is a definably primitive pseudo-finite permutation group of finite SU-rank, then

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Theorem (Karhumäki-R.)

There is such a bound.

Precursors

- Elwes-Jaligot-Macpherson-Ryten studied primitive pseudo-finite permutation groups (G, X) in a finite SU-rank theory eliminating $\exists^\infty$. They proved that if $SU(X) = 1$, (G, X) is either
  1. an elementary abelian p-group or torsion free divisible abelian group acting regularly on itself ($\mathrm{SU}(G) = 1$)
  2. there is an interpretable pseudo-finite field F and $(G, X) = (\mathrm{AGL}_1(F), F)$ ($\mathrm{SU}(G) = 2$) or $X = \mathrm{PG}_1(F)$ and $\mathrm{PGL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$ ($\mathrm{SU}(G) = 3$).

  This was later generalized to the setting of finite dimensional groups satisfying a chain condition on centralizers and stabilizers by Zou.
- Elwes-Jaligot-Macpherson-Ryten proved in their setting that if $(G, X) = \prod_{i \in I}(G_i, X_i)/\mathcal{D}$, then there is some c such that $|G_i| \leq |X_i|^c$ for a $\mathcal{D}$-large set of i.

Supersimple groups

- SU-rank has the following useful properties in supersimple theories of finite rank:
  - If $H \leq G$ is a definable subgroup, then $[G : H]$ is finite if and only if $\mathrm{SU}(H) = \mathrm{SU}(G)$.
  - Let $H \leq G$ be a definable subgroup. Then $\mathrm{SU}(H) + \mathrm{SU}(G/H) = \mathrm{SU}(G)$.
- If G is a non-abelian definably simple group, then G is a simple group.

Definable Primitivity $\neq$ Primitivity

Definable primitivity for pseudo-finite permutation groups is, in general, weaker than primitivity (following Macpherson-Pillay):

- Let G be a fixed finite group and let I denote an infinite set of primes such that for some d, for each $p \in I$, there is an irreducible representation V_p of G where V_p is a d-dimensional vector space over $\mathbb{F}_p$.
- Irreducibility entails that each permutation group $(V_p \rtimes G, V_p)$ is primitive.
- Let $(G, V) = \prod_{i \in I}(V_p \rtimes G, V_p)/\mathcal{D}$ be a nonprincipal ultraproduct and let $\mathbb{F} = \prod_{i \in I} \mathbb{F}_p/\mathcal{U}$ be the corresponding pseudo-finite field.
- The permutation group $(V \rtimes G, V)$ is interpretable in $\mathbb{F}$ so of finite SU-rank and, as an ultraproduct of primitive permutation groups it is definably primitive.
- $|V| = 2^{\aleph_0}$ and G is finite so there is some countable G-invariant subgroup $U \leq V$. Then

$$\mathrm{Stab}(0) = G \lneq U \rtimes G \lneq V \rtimes G$$

so $(V \rtimes G, V)$ is not primitive.

Definable Primitivity = Primitivity (Almost)

In the previous example, the permutation group which was definably primitive but not primitive had finite point-stabilizer. This turns out to be the only obstruction:

Theorem

Suppose (G, X) is a definably primitive permutation group of finite SU-rank. Then (G, X) is primitive if and only if G_x is infinite for some/all $x \in X$.

- Elwes-Jaligot-Macpherson-Ryten earlier proved that if $SU(G) > SU(X)$ and the ambient theory T eliminates $\exists^\infty$ in T^{eq} then (G, X) is primitive. Elwes-Ryten also proved this for (G, X) definable in a measurable theory.
- Our proof is straight out of Frank's playbook: to establish primitivity, we reduce to showing the almost normalizer $\tilde{N}_G(G_x) = G_x$ by indecomposability and then prove this using Schlichting's theorem.
- To show that if G_x is finite, (G, X) is not primitive, we use the following theorem of Smith: if (G, X) is primitive, X is infinite, and G_x is finite, then G is finitely generated.

Reduction to primitivity

- If (G, X) is definably primitive pseudo-finite and of finite SU-rank, then either (G, X) is outright primitive or G_x is finite.
- But if G_x is finite, then since X may be definably identified with G/G_x, we have

$$SU(G) = SU(G/G_x) = SU(X),$$

so we have a bound on $SU(G)$ in terms of $SU(X)$.
- Therefore, it is enough to prove the Borovik-Cherlin bound when the action is outright primitive.

LMT Classification

- By Higman's criterion, a family $\mathcal{C}$ of finite primitive permutation groups will have primitive ultraproducts if and only if the diameter of the orbital graphs are bounded.
- A classification of primitive ultraproducts of finite primitive permutation groups was given by Liebeck-Macpherson-Tent.

LMT Classification

Suppose $(G, X) = \prod_{i \in I}(G_i, X_i)/\mathcal{D}$ is an infinite primitive ultraproduct of finite permutation groups. Then (G, X) is one of the following:

1. A primitive non-principal ultraproduct of affine primitive permutation groups of the form $(V_d(q), V_d(q) \rtimes H)$, for fixed d, increasing q, and $H \leq \mathrm{GL}_d(q)$ is irreducible.
2. A non-principal ultraproduct of almost simple groups $S_i \leq G_i \leq \mathrm{Aut}(S_i)$ for S_i simple of fixed Lie type and G_i acting in standard t-action.
3. An ultraproduct of diagonal type with $\mathrm{Soc}(G) = T^k$ with T an infinite non-abelian simple pseudo-finite group.
4. An ultraproduct of permutation groups product type for fixed k will embed in a group of form $H \wr S_k$ in product action on a set Y^k. The associated permutation group (H, Y) be one of the above types.
5. Situations involving limits of groups of unbounded Lie rank, e.g. a non-principal ultraproduct of affine groups $(V_n(q), V_n(q)\mathrm{Cl}_n(q_0))$, where $\mathrm{Cl}_n(q_0)$ denotes a classical group of Lie rank n over the field $\mathbb{F}_{q_0}$.

Finding fields

- The bound then comes from an analysis of the cases in the LMT classification. Finite SU-rank rules out the situations involving ultraproducts of A_n for $n \to \infty$ or simple groups of unbounded Lie rank or situations with infinite field automorphisms (by Zou).
- The basic idea is to find an interpretable pseudo-finite field F (not necessarily pure) such that both $SU(X)$ and $SU(G)$ can be calculated in terms of $SU(F)$. These come from two places:
- **Field interpretation**: Let $S = A \rtimes G$ be a pseudofinite supersimple group of finite SU-rank, where G is definable, $A = C_S(A)$ is an infinite abelian subgroup of G with no definable G-invariant subgroups. Assume that G has an infinite definable abelian normal subgroup H. Then there is an interpretable pseudofinite field F and finite index subgroup G_0 of G so that A is a d-dimensional F-vector space and $G_0 \leqslant \mathrm{GL}_d(F)$ via its action on A.
- **Wilson's Theorem**: An infinite non-abelian simple pseudo-finite group is a (possibly twisted) Chevalley group over a pseudo-finite field and **Ryten's thesis**: The class of Chevalley groups of fixed Lie type over finite fields are uniformly bi-interpretable with finite fields (or certain finite difference fields).

Happy birthday, Frank!