Quasirandomness of definable subsets of definable groups in finite fields

Model theory and applications to groups and combinatorics, Luminy

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- The general idea (which has been germinating for some time) is to adapt Tao's algebraic regularity lemma for graphs (uniformly) definable in finite fields, to pairs (G, D), G a group, $D \subseteq G$, uniformly definable in finite fields.
- Tao's improvement over the conclusion of Szemeredi graph regularity consisted of so-called "power-saving" as well as the non-existence of exceptional pairs.

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- Our main point is that Tao's algebraic regularity lemma applies in the optimal manner, in particular there is a (uniformly definable, bounded index, normal) subgroup H of G such that for any two cosets V, W of H in G, the bipartitite graph $(V, W, E|(V \times W))$ is regular.

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- I did not give all the background (Szemeredi graph regularity, Tao algebraic regularity), but I will now state our results in a precise manner, so things should hopefully be clarified.

Statements of results I

- The expressions ϵ-quasirandom, ϵ-regular, ϵ-uniform for both finite bipartite graphs and subsets of finite groups, are *more or less* synonymous. We will state definitions and relationships later:

Theorem 0.1

Given M there is $C > 0$ such that for any finite field $\mathbf{F}$ and $D \subseteq G$ both definable of complexity at most M in $\mathbf{F}$ (G a group), there is a normal subgroup H of G definable in $\mathbf{F}$ with complexity at most C and index at most C such that for any two cosets V, W of H in G, the bipartite graph $(V, W, xy^{-1} \in D)$ is $C|\mathbf{F}|^{-1/2}$-quasirandom.

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- Complexity can be read as number of symbols in the language of unitary rings used in the formulas defining the data. The Theorem can also be stated in the language of uniform definability: given formula $\phi(x, y)$ there is formula $\psi(x, z)$ etc.

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- Let (V, W, E) be a finite bipartite graph, with d equal its density $|E|/|V||W|$. (V, W, E) is ϵ-quasirandom if $\sum_{v,v' \in V} |E(v, W) \cap E(v', W)|^2 \leq (d + \epsilon)|V|^2|W|^2$ (or equivalently with the same bound reversing the roles of V, W).

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- The notion has origin in work of Chung, Graham, Wilson (for unipartite graphs).

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- It is a fact that (i) ϵ-quasirandomness of (V, W, E) implies $\epsilon^{1/4}$-regularity of (V, W, E), namely for any $A \subseteq V$, $B \subseteq W$, $((|E \cap (A \times B)|/|A||B|) - d| \leq \epsilon^{1/4}|V||W|/|A||B|$,

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- Quasirandomness of subsets of finite groups is widely used in additive combinatorics on abelian groups.
- We discuss the version for possibly nonabelian groups.

Statement of results IV

- Given a finite group H let $\hat{H}$ be the set of irreducible complex (unitary) representations of H, and for $f : H \to \mathbb{C}$, the Fourier transform $\hat{f}$ of f is the map taking $\pi \in \hat{H}$ to $1/|H|(\sum_{h\in H} f(h)\pi(h^{-1}))$ an endomorphism of V_π.

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- The Fourier coefficient of f at π is $||\hat{f}(\pi)||$ (operator norm). And we will call $D \subseteq H$ ϵ-quasirandom if for all nontrivial $\pi \in \hat{H}$, $||\hat{1}_{D^{-1}}(\pi)|| \leq \epsilon$, equivalently $||\sum_{h\in D} \pi(d)|| \leq \epsilon|H|$.

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- If $(H, H, xy^{-1} \in D)$ is ϵ-quasirandom, then D is $\epsilon^{1/4}$-quasirandom.
- So the conclusion of Corollary 0.2 is that there is this definable normal subgroup H of G of complexity and index at most C such that for all $g \in G$, and nontrivial $\pi \in \hat{H}$, $||\sum_{h \in H \cap Dg} \pi(h)|| \leq C|F|^{-1/8}$.

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- One may have expected Bohr neighbourhoods of one kind or another to have shown up. The reason they do not is that if G is a group definable in a supersimple theory then $G_A^{00} = G_A^0$ (intersection of A-definable subgroups of finite index).

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- Two extreme cases of such groups are semisimple algebraic groups and commutative algebraic groups.
- Let us consider the first case. Suppose G is a connected, simply connected, semisimple algebraic group over $\mathbb{Z}$ (such as SL_2).
- So (maybe assuming some good reduction) there are no uniformly definable finite index subgroups of $G(\mathbf{F})$ as $\mathbf{F}$ ranges over all finite fields. (Explain?)

Connections II

- So by Theorem 0.1, given M, there is C such that for any finite field $\mathbf{F}$ and definable subset D of $G(\mathbf{F})$ of complexity at most M, D is $C|\mathbf{F}|^{1/8}$-quasirandom.

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- On the other hand, In the Gowers paper I referred to there is a notion of *quasirandomness* of finite groups G, with several equivalent characterizations, such as G is d-quasirandom if there are no (irreducible, unitary, nontrivial) representations of G of dimension $< d$.

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- There should be related computations of the degree of quasirandomess of arbitrary semisimple algebraic groups, in which case Theorem 0.1 may not say very much new.

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- (Green) Fix p and ϵ. Then there is $C > 0$ such that for any n if $G = (\mathbf{F}_p)^n$ and D is an arbitrary subset of G, then there is a subgroup (subspace) H of G, such that, outside a small exceptional set, for all cosets V, W of H in G, $(V, W, x - y \in D)$ is ϵ-regular (as described earlier).

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- The disadvantage is that we are only discussing uniformly definable subsets.

Connections IV

- Notice that the groups $\mathbf{F}_p^n$ as p and n vary are precisely the additive groups of the finite fields $\mathbf{F}_q$.
- So we can apply Theorem 0.1 to get that for any M there is C such that for any finite field $\mathbf{F}$ and definable subset D of complexity at most M of $\mathbf{F}$, there is a subspace H of $(\mathbf{F}, +)$ of index at most C, such that for all cosets V, W of H in $(\mathbf{F}, +)$, $(V, W, x - y \in D)$ is $C|\mathbf{F}|^{-1/8}$-regular.
- The disadvantage is that we are only discussing uniformly definable subsets.
- The advantage is that this is uniform in p, and is "power bounded" with no exceptional pairs.

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- What Daniel called the p, q, r theorem is due to Amador and me (and is a relatively straightforward extension of a result by Scanlon, Wagner and me);

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- If $p, q, r \in S_G(M)$ are types of maximal dimension (generic), and $(p/G^0_M) \times (q/G^0_M) = r/G^0_M$ in G/G^0_M, then there are pairwise independent over M realizations a, b, c of p, q, r with $a \times b = c$.

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- This plus a symmetric version and methods from Tao's paper will be enough to get the results.