Residual permutation test for high-dimensional regression coefficient testing

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Joint work with Kaiyue Wen & Tengyao Wang

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- $\varepsilon \in \mathbb{R}^{n}$: random noise vector.

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Our goal: Test whether

$$H_0 : b = 0 \quad \text{v.s.} \quad H_1 : b \neq 0$$

when $p \asymp n$.

$\hookrightarrow$ Sometimes also referred to as moderately high-dimensional regime.

Types of validity

- Asymptotic validity: *asymptotically* correct size control as $n \to \infty$.
  - Ex: OLS fit based, Freedman and Lane (1983), residual bootstrap, DiCiccio and Romano (2017), Toulis (2019), debiased lasso.

- Finite-population validity: valid size control with arbitrary n.

$\Rightarrow$ Our **target of interest**

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- **Distribution-free valid test** (Lei and Bickle, 2021): just requires ε to be exchangeable for correct size control;
  - Limitation: strong assumptions on dimension of $\boldsymbol{X}$:

$$n/p > 1/\alpha + 1$$

⇑ prespecified Type-I error

$\alpha = 0.01, n = 300 : p < 3.$

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- Minimax rate optimality: we prove that $n^{-t/(1+t)}$ matches the minimax lower bound rate for coefficient test with heavy-tailed noises.

Numerical analysis of ANOVA's validity

Simulations for general noise:

$$\boldsymbol{Y} = \boldsymbol{X}\beta + \varepsilon$$
$$\boldsymbol{Z} = \boldsymbol{X}\beta^{Z} + \boldsymbol{e}$$

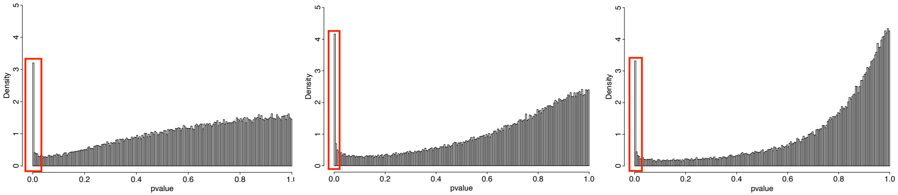
- $(n, p) = (300, 100), (600, 100), (600, 200)$;
- $\boldsymbol{X}$: Gaussian design, t_1 design;
- $\boldsymbol{e}, \varepsilon$: t_1 noise, t_2 noise, Gaussian noise.

Validity of ANOVA

n	p	X type	noise type	0.01	0.005
300	100	Gaussian	Gaussian	0.0101	0.0050
300	100	Gaussian	t_1	0.0181	0.0160
300	100	Gaussian	t_2	0.0153	0.0107
300	100	t_1	Gaussian	0.0101	0.0050
300	100	t_1	t_1	0.0243	0.0208
300	100	t_1	t_2	0.0180	0.0130
600	200	Gaussian	Gaussian	0.0101	0.0049
600	200	Gaussian	t_1	0.0141	0.0122
600	200	Gaussian	t_2	0.0150	0.0104
600	200	t_1	Gaussian	0.0101	0.0049
600	200	t_1	t_1	0.0202	0.0173
600	200	t_1	t_2	0.0170	0.0120

Table: empirical size with nominal levels $\alpha = 0.01$ and 0.005

Histogram of ANOVA's p-values



(a) (b) (c)

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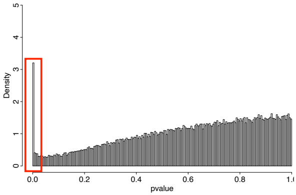
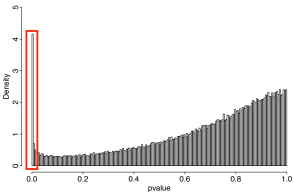


(d)



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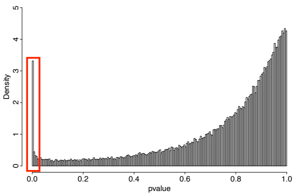

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This shows the importance of developing a distribution-free & finite-population valid test!!

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2. p-value:

$$\phi = \frac{1}{1+K}\left(1 + \sum_{k=1}^{K} \mathbb{1}\left\{\min_{1 \le k' \le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Y}\right) \le T\left(\tilde{\boldsymbol{V}}_{k}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k}^\top \boldsymbol{P}_k \boldsymbol{Y}\right)\right\}\right)$$

$\Rightarrow$ Projecting $(\boldsymbol{Y}, \boldsymbol{P}_k \boldsymbol{Y})$ onto $\mathrm{span}(\tilde{\boldsymbol{V}}_k)$ & compare.

Why residual permutation test?

- Classical regression residual:

$$\hat{\boldsymbol{R}}_Y = (\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X}^\top)\boldsymbol{Y}$$

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- $\tilde{\boldsymbol{V}}_k^\top \boldsymbol{Y}$:

$\rightsquigarrow$ a residual by regressing $\boldsymbol{Y}$ onto both $\boldsymbol{X}$ & $\boldsymbol{P}_k\boldsymbol{X}$ ...

Analysis of RPT's validity

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- We can rewrite the p-value as:

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$$\ge \frac{1}{1+K}\left(1 + \sum_{k=1}^{K} \mathbb{1}\left\{\min_{1\le k'\le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{\varepsilon}\right) \le \min_{1\le k'\le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{P}_k \boldsymbol{\varepsilon}\right)\right\}\right)$$

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\phi &= \frac{1}{1+K}\left(1+\sum_{k=1}^{K}\mathbb{1}\left\{\min_{1\le k'\le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{\varepsilon}\right) \le T\left(\tilde{\boldsymbol{V}}_{k}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k}^\top \boldsymbol{P}_k\boldsymbol{\varepsilon}\right)\right\}\right) \\
&\ge \frac{1}{1+K}\left(1+\sum_{k=1}^{K}\mathbb{1}\left\{\min_{1\le k'\le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{\varepsilon}\right) \le \min_{1\le k'\le K} T\left(\tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{Z}, \tilde{\boldsymbol{V}}_{k'}^\top \boldsymbol{P}_k\boldsymbol{\varepsilon}\right)\right\}\right) \\
&= \frac{1}{1+K}\left(1+\sum_{k=1}^{K}\mathbb{1}\left\{g(\boldsymbol{\varepsilon}) \le g(\boldsymbol{P}_k\boldsymbol{\varepsilon})\right\}\right)
\end{aligned}$$

for some $g(\cdot)$ depending only on $\boldsymbol{X}, \boldsymbol{Z}, \mathcal{P} := \{\boldsymbol{P}_0 = \boldsymbol{I}, \boldsymbol{P}_1, \cdots, \boldsymbol{P}_K\}$.

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Remaining challenge:

$$\text{Prove } \phi \geq \frac{1}{1+K}\left(1+\sum_{k=1}^{K}\mathbb{1}\left\{g(\varepsilon) \leq g(\boldsymbol{P}_k\varepsilon)\right\}\right) \text{ is a valid p-value} \quad (1)$$

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However, here $g(\cdot)$ depends on $\mathcal{P}$.

Lemma

Suppose we construct $\mathcal{P} := \{\boldsymbol{P}_0 := \boldsymbol{I}, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_K\}$ *s.t. it formalizes a group:*

$$\forall \boldsymbol{P}_i, \boldsymbol{P}_j \in \mathcal{P}, \exists \boldsymbol{P}_\ell \ s.t. \ \boldsymbol{P}_\ell := \boldsymbol{P}_i \boldsymbol{P}_j.$$

Then (1) *is a valid p-value.*

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Remark

1. Construction of $\tilde{\boldsymbol{V}}_k$ requires $p < n/2$;
2. With prespecified α, one needs to choose $K > 1/\alpha$ to have power.

Power analysis

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$$0 < \mathbb{E}[|e_1|^2] < \infty \qquad and \qquad 0 < \mathbb{E}[|\varepsilon_1|^{1+t}] < \infty$$

for $t \in [0, 1)$.

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$$\lim_{n\to\infty} \mathbb{P}\left(\phi > \frac{1}{K+1}\right) = 0.$$

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Consider a broader class of alternative where $\boldsymbol{Z}$ *is* **nonlinear** *w.r.t.* $\boldsymbol{X}$ *(i.e.,* $\boldsymbol{Z} = \boldsymbol{X}\beta^Z + \boldsymbol{h} + \boldsymbol{e}$*) and all noises are* **heteroschdastic**, *under*

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2. $\limsup \|\boldsymbol{h}\|_2/\sqrt{n} \leq r \cdot \text{noise level}$
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The signal detection rate $n^{-t/(1+t)}$ still holds.

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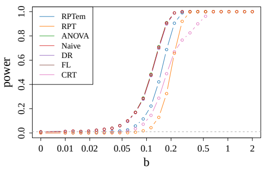
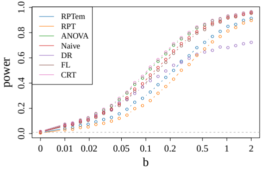
- We derive the uniform convergence rate of RPT is of $n^{-t/(1+t)+\delta}$ for any const. $\delta > 0$.

$\Rightarrow$ RPT nearly minimax rate optimal.

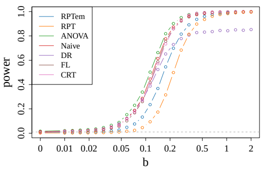
Power curves



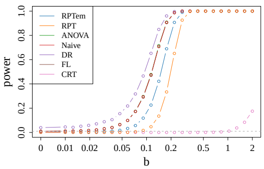
(g) Gaussian design, Gaussian noise



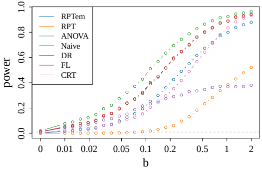
(h) Gaussian design, t_1 noise



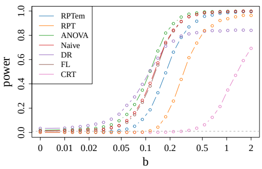
(i) Gaussian design, t_2 noise



(j) t_1 design, Gaussian noise



(k) t_1 design, t_1 noise



(l) t_1 design, t_2 noise

Figure: $n = 600, p = 100$

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- **Open question:** how to develop a distribution-free & finite-population valid test with better empirical power in small sample size.

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For theoretical details and more simulation results, please see `https://arxiv.org/abs/2211.16182`