Learning with a linear loss function

Applications to sign clustering and robustness in sparse PCA.

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Oracle:

$$Z^* \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle \mathbb{E}A, Z \rangle$$

$A \in \mathbb{R}^{d \times d}$ is a random matrix and $\mathcal{C} \subset \mathbb{R}^{d \times d}$.

Estimator:

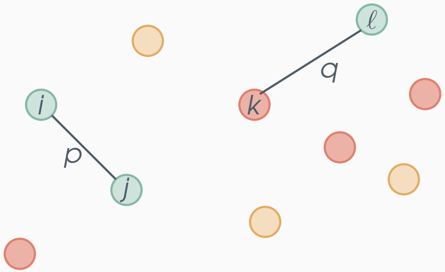
$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle A, Z \rangle$$

We look at $\hat{Z}$ as an ERM = empirical risk minimizer for a **linear loss function** $\ell_Z(A) = -\langle A, Z \rangle$.

Semi-definite program (SDP): optimization problem of the form

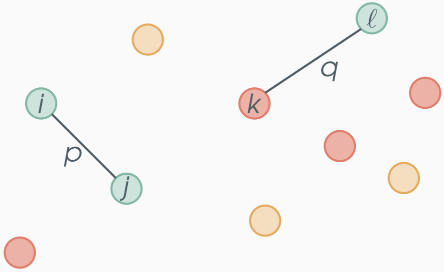
$$\max_{\substack{Z \in \mathbb{R}^{d \times d} \\ Z \succeq 0, \\ \langle B_j, Z \rangle = b_j \text{ for } j=1,\dots,m}} \langle A, Z \rangle.$$

where $A, B_1, \ldots B_m \in \mathbb{R}^{d \times d}$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given.



$0 < q < p < 1$

- $G = (V, E)$, graph with d nodes
- V can be partitioned into K communities $\mathcal{C}_1, \ldots, \mathcal{C}_K$
- We observe $A \in \mathbb{R}^{d \times d}$, adjacency matrix:
$$A_{ij} \sim \begin{cases} \text{Bern(p) if } i \sim j \\ \text{Bern(q) if } i \nsim j. \end{cases}$$
- We do not observe $\mathcal{C}_1, \ldots, \mathcal{C}_K$
- **Aim:** estimate the membership matrix $\bar{Z} = (1_{\{i \sim j\}})_{1 \le i,j \le d}$



$$0 < q < p < 1$$

- $\lambda := \sum_{k=1}^K |\mathcal{C}_k|^2$
- $\mathcal{C} := \left\{Z \succeq 0 : Z \geq 0, \mathrm{diag}(Z) \preceq I_d, \sum_{i,j=1}^d Z_{ij} \leq \lambda\right\}$,
- Oracle:

$$\bar{Z} = Z^* \in \underset{Z \in \mathcal{C}}{\mathrm{argmax}} \langle \mathbb{E}A, Z\rangle.$$

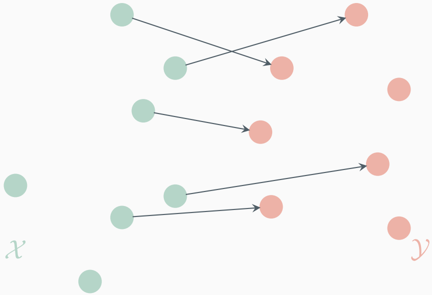
But we only observe A

- Natural estimator:

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\mathrm{argmax}} \langle A, Z\rangle$$

(An ERM w.r.t. a linear loss function).

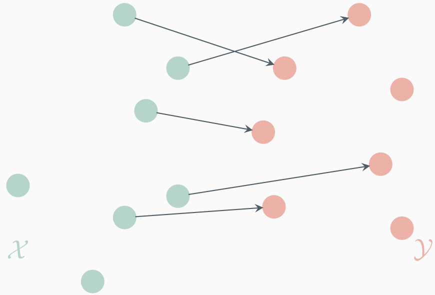
- Data: two clouds of points, $\mathcal{X} = (x_1, \dots, x_d)$ and $\mathcal{Y} = (y_1, \dots, y_d)$.



- Data: two clouds of points, $\mathcal{X} = (x_1, \dots, x_d)$ and $\mathcal{Y} = (y_1, \dots, y_d)$.
- Aim: match each point of $\mathcal{X}$ with a point of $\mathcal{Y}$ reaching the Wasserstein distance

$$\tau^* \in \min_{\tau \in \mathfrak{S}_d} \sum_{i=1}^{d} \|x_i - y_{\tau(i)}\|^2 := W_2^2(\mathcal{X}, \mathcal{Y})$$

(quadratic assignment problem)



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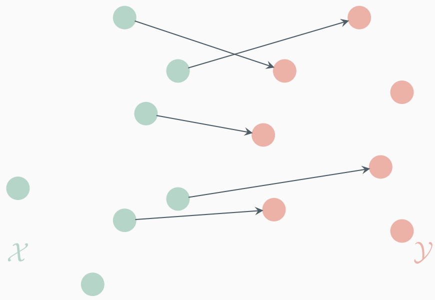
(quadratic assignment problem)

- Equivalent matrix formulation:

Define $Z^* := \left(1_{\{j=\tau^*(i)\}}\right)_{1 \le i,j \le d}$

$$\to \quad Z^* \in \operatorname*{argmin}_{Z \in \mathcal{C}} \sum_{i,j} \|x_i - y_j\|_2^2 Z_{ij},$$

with $\mathcal{C} := \{d \times d \text{ bistochastic matrices}\}$.



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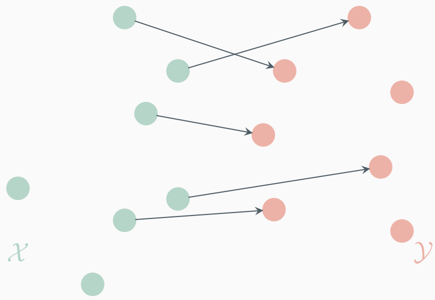
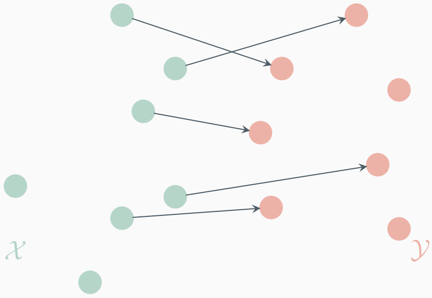


- Oracle:

$$Z^* \in \operatorname*{argmin}_{Z \in \mathcal{C}} \sum_{i,j} \|x_i - y_j\|_2^2 Z_{ij},$$

- We observe: $X_i = x_i + G_i$ and $Y_i = y_i + G_i'$, G_i, G_i': i.i.d. centered noises.



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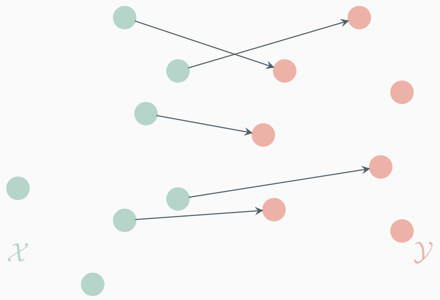
- Quadratic assigment problem:

$$\hat{Z} \in \operatorname*{argmin}_{Z\in\mathcal{C}} \langle A, Z\rangle \text{ where } A = \left(\|X_i - Y_j\|_2^2\right)_{1\leq i,j\leq N}$$



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We show that in the noiseless case:

$$Z^* \in \operatorname*{argmin}_{Z\in\mathcal{C}} \langle \mathbb{E}A, Z\rangle$$

We look at all the previous estimators as ERMs for a linear loss function.

Other problems

Angular synchronization	$A = \left(e^{\iota \delta_{ij}}\right)_{1\leq i,j,\leq d}$	Bandeira, Boumal, Singer,'16
Variable clustering	$A = (1/N)\sum_{i=1}^N X_i X_i^\top$	Bunea, Giraud, Luo, Royer, Verzelen'18
Max-Cut	$A = \left(s_{ij}A^*_{ij}\right)_{1\leq i,j,\leq d}$	Hong, Lee, Wei,'21
Phase recovery	$A = (XX^\top) \circ (I_N - BB^+)$	Waldspurger, D'Aspremont, Mallat'13
Single index model	$A = (1/N)\sum_{i=1}^N Y_i T(X_i)$	Yang, Balasubramanian, Liu'18
Distance metric learning	$A = \sum_{i,j=1}^N (X_i - X_j)(X_i - X_j)^\top$	Xing, Jordan, Russels'02

2 | General excess risk and estimation bounds with applications in signed clustering and robust sparse PCA.

- H: Hilbert space
- X: random vector in H, distributed according to P
- $Pg := \mathbb{E}[g(X)]$ and $\|g\|_{L_p} = (P|g|^p)^{\frac{1}{p}}$,
- $\mathcal{C}$: subset of H
- $\ell_Z : X \in H \to -\langle X, Z\rangle$, the linear loss function of $Z \in \mathcal{C}$.

Definition | Star-shaped

We say that a set $\mathcal{C}$ is star-shaped in Z^* when for all $Z \in \mathcal{C}$, the segment $[Z, Z^*]$ is in $\mathcal{C}$.

In practice, $\mathcal{C}$ will always be star-shaped in Z^*, since convex sets are star-shaped in any of their elements.

- We are interested in the oracle

$$Z^* \in \operatorname*{argmin}_{Z \in \mathcal{C}} P\ell_Z.$$

- We observe $\mathcal{D} := \{X_1, \ldots, X_N\}$, a sample distributed according to P.

Learning with a linear loss function.

From the observations $\mathcal{D}$, we construct several estimators $\hat{Z}$ for the oracle Z^* and provide statistical guarantees on:

- the excess risk $P\mathcal{L}_{\hat{Z}} = P(\ell_{\hat{Z}} - \ell_{Z^*})$
- the error rate $\|\hat{Z} - Z^*\|_{L_2} := \sqrt{\mathbb{E}\left[\langle X, \hat{Z} - Z^* \rangle^2\right]}$

Here, we consider the following natural estimator for the oracle $Z^* \in \operatorname{argmin}_{Z \in \mathcal{C}} P\ell_Z$:

Definition | ERM estimator

$$\hat{Z}^{\mathrm{ERM}} \in \operatorname*{argmin}_{Z \in \mathcal{C}} P_N \ell_Z$$

where $P_N \ell_Z := \frac{1}{N}\sum_{i=1}^N \ell_Z(X_i) = \frac{1}{N}\sum_{i=1}^N -\langle X_i, Z\rangle = \langle -\bar{X}_N, Z\rangle$.

Definition | Local complexity fixed point

Consider $G : H \to \mathbb{R}$ and $0 < \delta < 1$. The fixed point complexity parameter with respect to the local curvature function G at deviation $1 - \delta$ is

$$r_G^*(\delta) = \inf\left(r > 0 : \mathbb{P}\left[\sup_{Z \in \mathcal{C} : G(Z - Z^*) \leq r} (P_N - P)\mathcal{L}_Z \leq \frac{r}{2}\right] \geq 1 - \delta\right)$$

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local curvature assumption

For all $Z \in \mathcal{C}$, if $P\mathcal{L}_Z \leq r_G^*(\delta)$ then $P\mathcal{L}_Z \geq G(Z^* - Z)$. For example, $G(Z) = P\mathcal{L}_Z$, $G(Z) = \|Z\|_{L_2}^2$ or $G(Z) = \|Z\|_1$.

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Theorem | Excess risk for the ERM estimator with G-localization

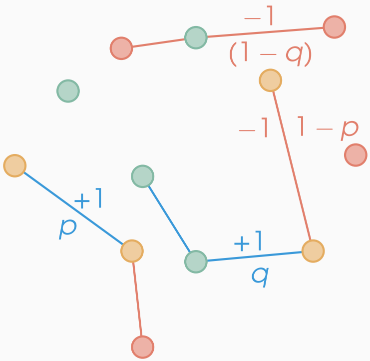
We assume that $\mathcal{C}$ is star-shaped in Z^* and that the local curvature Assumption holds for some $0 < \delta < 1$. Then, with probability at least $1 - \delta$, $G(Z^* - \hat{Z}^{ERM}) \leq P\mathcal{L}_{\hat{Z}^{ERM}} \leq r_G^*(\delta)$.

Application | The ERM estimator for signed clustering

Signed stochastic block model (SSBM)

- $G = (V, E)$, graph with d nodes
- V can be partitioned into K communities $\mathcal{C}_1, \dots, \mathcal{C}_K$

[1] This SSBM is the one introduced by Cucuringu et al.'19 and adapts the one of Fei and Chen'19.



$0 \leq q < 1/2 < p \leq 1$

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- V can be partitioned into K communities $\mathcal{C}_1, \dots, \mathcal{C}_K$
- We observe $A \in \mathcal{S}_d(\mathbb{R})$ s.t.: $A_{ij} = 1$,

$$2\mathrm{B}_{ij} - 1 \text{ , with } \mathrm{B}_{ij} \sim \begin{cases} \mathrm{Bern}(p) \text{ if } i \sim j \\ \mathrm{Bern}(q) \text{ if } i \not\sim j \end{cases}$$

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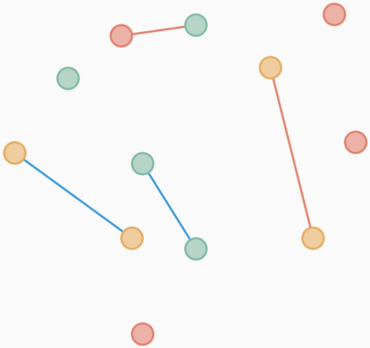

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and $s_{ij} \sim \mathrm{Bern}(\delta)$

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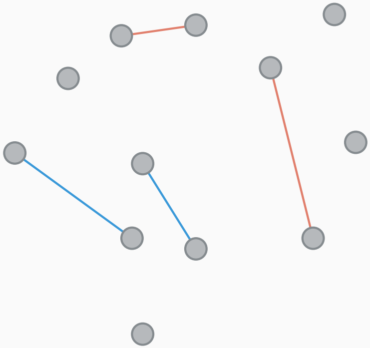

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  and $s_{ij} \sim \mathrm{Bern}(\delta)$
- $\{\mathrm{B}_{ij}, s_{ij},\ 1 \leq i < j \leq d\}$ are independent.

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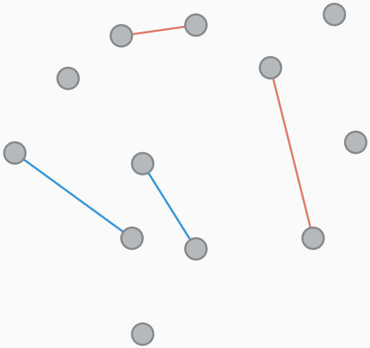

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  and $s_{ij} \sim \mathrm{Bern}(\delta)$
- $\{\mathrm{B}_{ij}, s_{ij},\ 1 \leq i < j \leq d\}$ are independent.

Aim: recover $\bar{Z} := \left(1_{\{i \sim j\}}\right)_{1 \leq i,j \leq d}$ the membership matrix.

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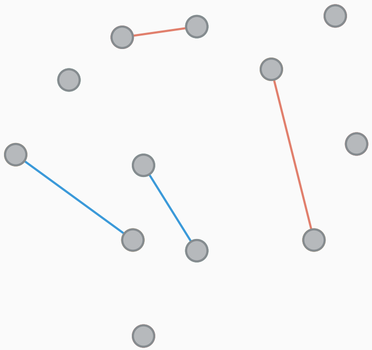


Property

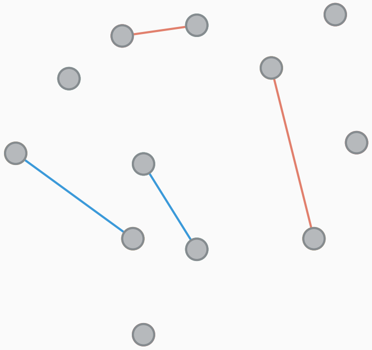
We show that $\bar{Z} = Z^*$, where Z^* is the oracle

$$Z^* \in \operatorname*{argmax}_{Z \in \mathcal{C}} \langle \mathbb{E}A - \alpha J, Z \rangle$$

$\alpha = \delta(p + q - 1)$, $J = (1)_{d \times d}$, $\mathcal{C} = \{Z \succeq 0 : Z_{ij} \in [0, 1], Z_{ii} = 1\}$.

ERM estimator for signed clustering

$$\hat{Z} \in \operatorname*{argmax}_{Z \in \mathcal{C}} \langle A - \alpha J, Z \rangle$$

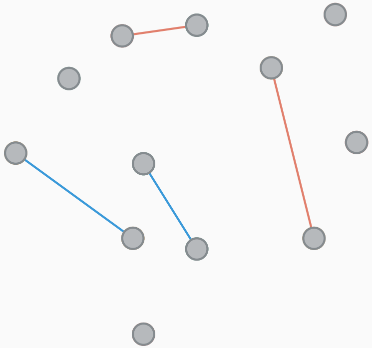


Curvature of the excess risk

For $\theta = \delta(p - q)$, we have for all $Z \in \mathcal{C}$,

$$\langle \mathbb{E}A - \alpha J, Z^* - Z \rangle = \theta \|Z^* - Z\|_1.$$

$\rightarrow$ We compute a complexity fixed point for the localization function $G : Z \rightarrow \|Z\|_1$.



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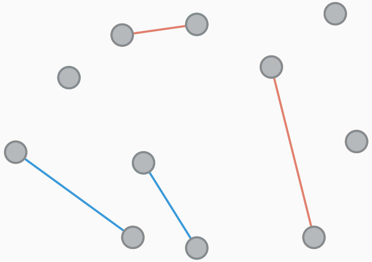
Complexity fixed point parameter

$$r_G^*(\Delta) = \inf\left(r > 0 : \mathbb{P}\left[\sup_{Z \in \mathcal{C} : \|Z - Z^*\|_1 \leq r} \langle A - \mathbb{E}[A], Z - Z^* \rangle \leq \frac{r}{2} \right] \geq 1 - \Delta \right)$$



Assumptions

(i) $|\mathcal{C}_k| \sim d/k$ for any $k \in [K]$

(ii) $d\nu\delta \geq \log d$

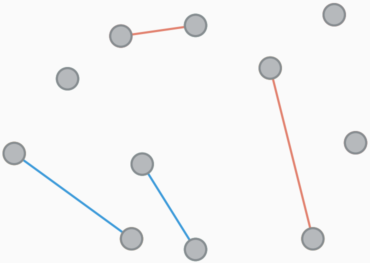
(iii) $sd \gtrsim K^2\nu$

(iv) $\frac{K}{d}\log(2eKd) \leq \max\left(\frac{\theta^2}{\rho}, \frac{9\rho}{32}\right)$

Theorem | Exact recovery in signed clustering

Let $\Delta := \exp(-\delta\nu d) + 3/(2eKd)$. Then, $r_G^*(\Delta) = 0$, that is, $\hat{Z} = Z^*$ with probability at least $1 - \Delta$.

for $\nu = \max(2p-1, 1-2q)$, $\rho = \delta\max(1-\delta(2p-1)^2, 1-\delta(2q-1)^2)$, $s = \delta(p-q)^2$ and $\theta = \delta(p-q)$



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Theorem | Exponential rate in signed clustering

Let $\Delta := \exp(-\delta\nu d) + 3/(2eKd)$. Then, with probability at least $1-\Delta$,

$$\|Z^* - \hat{Z}\|_1 \leq \frac{c_0 d^2}{\theta}\exp\left(-\frac{c_1 sd}{K}\right).$$

Similar to Fei and Chen 17':

- exact recovery in the regime $K^2 + K\log(d) \lesssim d$
- exponential rate with exponent $\simeq -d/K$ otherwise

When Z^* has some structure (e.g. sparsity), we consider the following type of estimator:

Definition | regularized ERM estimator

$$\hat{Z}^{\mathrm{RERM}} \in \underset{Z \in \mathcal{C}}{\mathrm{argmin}} \left(P_N \ell_Z + \lambda \|Z\| \right)$$

$\lambda > 0$ the regularization parameter and $\|\cdot\|$ a norm.

Definition | Complexity fixed point parameter

For $A > 0$, $\rho > 0$ and $\delta \in (0,1)$, the complexity fixed point for the structural learning problem is

$$r^*(\rho) = \inf\left(r > 0 : \mathbb{P}\left(\sup_{Z \in \mathcal{C}: \|Z - Z^*\| \leq \rho, \|Z - Z^*\|_2^2 \leq r} |(P - P_N)\mathcal{L}_Z| \leq \frac{r}{3A}\right) \geq 1 - \delta\right)$$

when the local excess risk curvature is well described by $G : Z \to \|Z\|_2^2$.

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Assumption | Local curvature assumption

There exists $A > 0$, $\rho^* > 0$ and $\delta \in (0, 1)$ such that, for any $Z \in \mathcal{C}$, if $\|Z - Z^*\|_2 = r^*(\rho^*)$ and $\|Z - Z^*\| \leq \rho^*$, then $AP\mathcal{L}_Z \geq \|Z - Z^*\|_2^2$.

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ρ^* is chosen to take advantage of the structure inducing property of the regularization norm:
$\rightarrow$ sparsity equation.

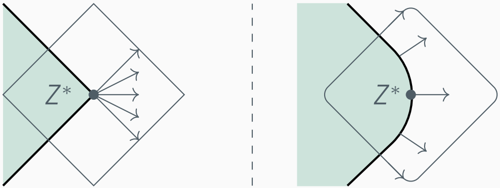
Definition - Subdifferential of $\|.\|$ at a point Z:

$$\partial\|.\|(Z) := \{\Phi \in H : \|Z+h\| - \|Z\| \geq \langle \Phi, h\rangle \text{ for all } h \in H\}.$$

Property - For S^* and B^* the dual sphere and dual ball, we have

$$\partial\|\cdot\|(Z) = \begin{cases} \{\Phi \in S^* : \langle \Phi, Z\rangle = \|Z\|\} & \text{if } Z \neq 0 \\ B^* & \text{if } Z = 0 \end{cases}$$

Subdifferential are large sets at points where $\|.\|$ is not differentiable



$$\partial\|\cdot\|_1 \left(1,0\ldots 0\right)^\top = \left\{\left(1,x_2\ldots x_d\right) : |x_i| \leq 1\right\} \quad \Big| \quad \partial\|\cdot\|_1 \left(1\ldots 1\right)^\top = \left\{\left(1\ldots 1\right)^\top\right\}$$

For $\rho > 0$, we consider

$$\Gamma_{Z^*}(\rho) = \bigcup_{\|Z - Z^*\| \leq \rho/20} (\partial \|.\|)_Z,$$

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We want $\Gamma_{Z^*}(\rho)$ to be a large subset of S^* (or B^*) when Z^* is structured or close to a structured element in H.

Definition | Sparsity equation

For $\rho > 0$ we define:

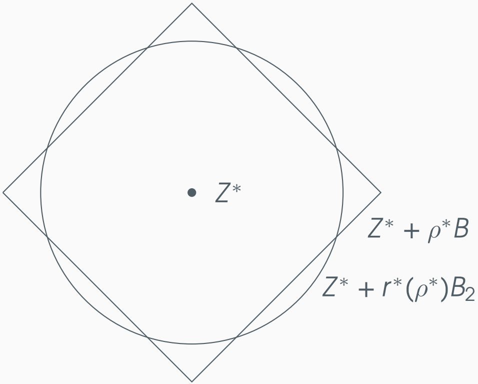
$$H_\rho := \left\{Z \in \mathcal{C} : \|Z - Z^*\| = \rho \;\; and \;\; \|Z - Z^*\|_2^2 \leq r^*(\rho)\right\}$$

and

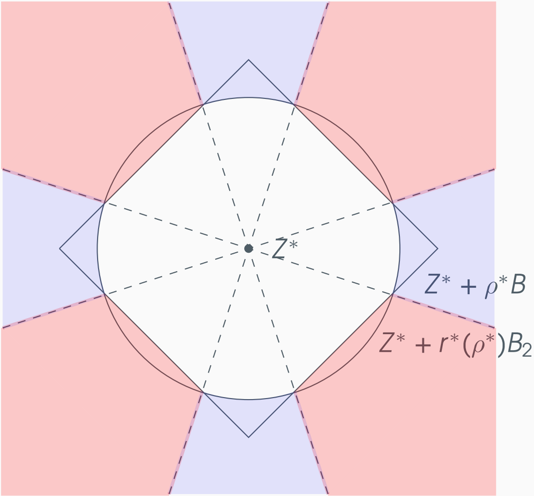
$$\Delta(\rho) := \inf_{Z\in H_\rho} \sup_{\Phi\in\Gamma_{Z^*}(\rho)} \langle \Phi, Z - Z^*\rangle.$$

We say that $\rho > 0$ satisfies the **sparsity equation** when $\Delta(\rho) \geq (4/5)\rho$.

Sparsity equation and curvature



Sparsity equation and curvature



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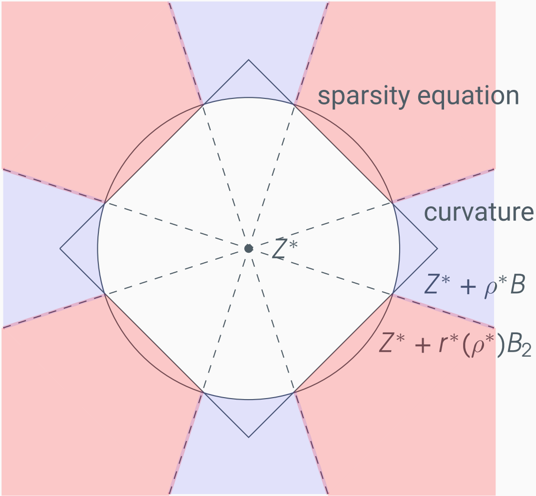

Assumptions

(i) $\mathcal{C}$ is star-shaped in Z^*

(ii) $\rho^* > 0$ s.t. the local curvature assumption holds and ρ^* satisfies the sparsity equation

(iii) $\lambda \simeq r^*(\rho^*)/\rho^*$

Theorem | Excess risk and estimation bounds for the RERM with ℓ_2-localization

With probability at least $1 - \delta$,

$$\|\hat{Z}^{\text{RERM}} - Z^*\| \leq \rho^* \quad , \quad \|\hat{Z}^{\text{RERM}} - Z^*\|_2^2 \leq r^*(\rho^*) \text{ and } P\mathcal{L}_{\hat{Z}^{\text{RERM}}} \leq \frac{r^*(\rho^*)}{A}.$$

- We observe $X_1, \ldots, X_N \in \mathbb{R}^d$, $i.i.d.$ centered and $\mathrm{cov}(X_1 X_1^\top) := \Sigma$,
- Aim[2]: find a 1st k-sparse principal component of Σ:

$$v^* \in \underset{\substack{v \in \mathbb{R}^d \\ \|v\|_2 = 1 \\ \|v\|_0 \leq k}}{\mathrm{argmax}} \langle \Sigma, vv^\top \rangle$$

[2] Johnstone and Lu, 09'; Berthet and Rigollet, 13'; Wang, Berthet and Samworth, 16'

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Lifting
$$Z \in \underset{\substack{Z \succeq 0 \\ \mathrm{rank}(Z)=1 \text{ and } \|Z\|_0 \leq k^2 \\ \mathrm{Tr}(Z)=1}}{\mathrm{argmax}} \langle \Sigma, Z \rangle$$

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SDP Relaxation $$Z^* \in \underset{\substack{Z \succeq 0 \\ \mathrm{Tr}(Z)=1}}{\mathrm{argmax}} \langle \Sigma, Z \rangle$$

Estimator for Z^*: $$\hat{Z}_\lambda^{RERM} \in \underset{\substack{Z \succeq 0 \\ \mathrm{Tr}(Z)=1}}{\mathrm{argmax}} \left(\langle \hat{\Sigma}_N, Z \rangle + \lambda \|Z\|_1 \right) \text{ for } \hat{\Sigma}_N = \frac{1}{N} \sum_{i=1}^N X_i X_i^\top$$

Estimator for v^*: $\hat{v}$ a top unit-norm eigenvector of $\hat{Z}$

[2] Johnstone and Lu, 09'; Berthet and Rigollet, 13'; Wang, Berthet and Samworth, 16'

Spiked covariance model: $\Sigma = I_d + \theta\beta^*(\beta^*)^\top$ with $\beta^* \in \mathcal{S}_2^d$, k-sparse, $\theta > 0$

Exactness in the spiked covariance model

In the spiked covariance model, we have $Z^* = (\beta^*)(\beta^*)^\top$.

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Curvature of the excess risk

In the spiked covariance model: for all $Z \in \mathcal{C} = \{Z \succeq 0 : \mathrm{Tr}(Z) = 1\}$,

$$P\mathcal{L}_Z = \langle \Sigma, Z^* - Z \rangle \geq (\theta/2)\|Z^* - Z\|_2^2.$$

$\rightarrow$ The local curvature assumption holds globally for $G : Z \rightarrow \|Z\|_2^2$.

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$\longrightarrow$ The local curvature assumption holds globally for $G : Z \to \|Z\|_2^2$.

Sparsity equation

ρ satisfies the sparsity equation when $\rho \geq 10k\sqrt{r^*(\rho)}$.

Assumptions:

i. For all $p, q \in [d]$ and all $2 \leq r \leq 3\log(ed/k)$, we have
$$\|X_{1p}X_{1q} - \mathbb{E}(X_{1p}X_{1q})\|_{L_r} \leq c_0 r$$

ii. $N \gtrsim \log(ed/k)$

iii. $\lambda \simeq \sqrt{(1/N)\log(d/k)}$

Theorem | Excess risk and estimation bounds for the RERM in sparse PCA

In the Spiked covariance model: $\Sigma = I_d + \theta\beta^*(\beta^*)^\top$, with probability at least $1 - 10k/ed$:

$$\|\hat{Z}_\lambda^{\mathrm{RERM}} - Z^*\|_1 \lesssim k^2\sqrt{\frac{1}{N\theta^2}\log\left(\frac{ed}{k}\right)}, \quad \|\hat{Z}_\lambda^{\mathrm{RERM}} - Z^*\|_2 \lesssim \sqrt{\frac{k^2}{N\theta^2}\log\left(\frac{ed}{k}\right)}$$

and

$$P\mathcal{L}_{\hat{Z}_\lambda^{\mathrm{RERM}}} \lesssim \frac{k^2}{N\theta}\log\left(\frac{ed}{k}\right).$$

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iii. $\lambda \simeq \sqrt{(1/N)\log(d/k)}$

Corollary | Estimation bounds for β^*

Let $\hat{\beta} \in \mathbb{R}^d$ be a leading unit length eigenvector of $\hat{Z}_\lambda^{\mathrm{RERM}}$. Applying the sin-$\theta$ theorem: with probability at least $1 - 10k/ed$:

$$\|\hat{\beta}\hat{\beta}^\top - \beta^*(\beta^*)^\top\|_2 \lesssim \sqrt{\frac{k^2}{N\theta^2}\log\left(\frac{ed}{k}\right)}.$$

This improves the result from Wang, Berthet and Samworth 16':

- $\log(d/k)$ rate instead of $\log(d)$, thanks to localization.
- we only need $\log(d/k)$ moments.

Adversarial contamination setup

good data $\widetilde{X}_1, \quad \widetilde{X}_2 \quad \dots \quad \widetilde{X}_p \quad \dots \quad \widetilde{X}_N \quad \in H, \quad i.i.d$

adversary $\widetilde{X}_1, \quad X_2 \quad \dots \quad X_p \quad \dots \quad \widetilde{X}_N \quad \in H$

$$\mathcal{I} := \left\{ i \in [N] : \widetilde{X}_i = X_i \right\} \qquad \mathcal{O} := [N] \backslash \mathcal{I}$$

$(X_i)_{i \in \mathcal{I}}$: inliers | $(X_i)_{i \in \mathcal{O}}$: outliers

$\mathcal{O} \cup \mathcal{I} = [N]$ is unknown to the statistician.

In what follows: $P\mathcal{L}_Z := \mathbb{E}[-\langle \widetilde{X}, Z \rangle], \quad Z^* \in \mathrm{argmin}_{Z \in \mathcal{C}} P\mathcal{L}_Z.$

MOM principle

- $[N] = B_1 \cup \ldots \cup B_K$, $|B_k| = N/K$

[3] Lecué and Lerasle '20

MOM principle

- $[N] = B_1 \cup \ldots \cup B_K$, $|B_k| = N/K$
- $P_{B_k}(g) = (K/N)\sum_{i\in B_k} g(X_i)$

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- $P_{B_k}(g) = (K/N) \sum_{i \in B_k} g(X_i)$
- $MOM_K(g) := Med\left(P_{B_1}(g), \ldots, P_{B_K}(g)\right)$

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Definition | Regularized minmax MOM estimator[3]

$$\hat{Z}^{\mathrm{RMOM}}_{K,\lambda} \in \underset{Z\in\mathcal{C}}{\mathrm{argmin}} \sup_{Z'\in\mathcal{C}} \left(\mathrm{MOM}_K(\ell_Z - \ell_{Z'}) + \lambda(\|Z\| - \|Z'\|)\right)$$

Idea: Use this procedure for the linear loss function.

[3] Lecué and Lerasle '20

We consider the case where the excess risk $P\mathcal{L}_Z$ is well described by $G : Z \to \|Z\|_2^2$.

Definition | Complexity fixed point parameter

Let $\sigma_1, \dots, \sigma_N$ be independent Rademacher variables independent of the $\tilde{X}_i$'s. The complexity fixed point for the robust structural learning problem is

$$r^*_{\mathrm{RMOM}}(\rho) := \inf \left\{ r > 0 \; : \; \max\left(E(r,\rho), V_K(r,\rho) \right) \leq r^2 \right\}$$

where, for all $r, \rho > 0$,

$$E(r,\rho) := \mathbb{E}\left[\sup_{Z \in \mathcal{C}_{\rho,r}} \left| \frac{1}{N} \sum_{i=1}^N \sigma_i \mathcal{L}_Z(\tilde{X}_i) \right| \right] \text{ and } V_K(r,\rho) := \sqrt{\frac{K}{N}} \sup_{Z \in \mathcal{C}_{\rho,r}} \sqrt{\mathrm{Var}(\mathcal{L}_Z(\tilde{X}))},$$

with $\mathcal{C}_{\rho,r} = \{Z \in \mathcal{C} : \|Z - Z^*\| \leq \rho, \|Z - Z^*\|_2^2 \leq r^2\}$.

Definition | Sparsity equation

Let

$$\bar{H}_\rho := \left\{Z \in \mathcal{C} : \|Z - Z^*\| = \rho \text{ and } \|Z - Z^*\|_2^2 \leq r^*_{\mathrm{RMOM}}(\rho)^2\right\}$$

and

$$\bar{\Delta}(\rho) := \inf_{Z \in \bar{H}_\rho} \sup_{\Phi \in \Gamma_{Z^*}(\rho)} \langle \Phi, Z - Z^* \rangle.$$

We say that ρ satisfies the **sparsity equation** if $\bar{\Delta}(\rho) \geq 4\rho/5$.

Assumption | Local curvature assumption

Let ρ^* satisfies the sparsity equation. We assume that for all $Z \in \mathcal{C}$, if $\|Z - Z^*\|_2^2 = r^*_{\mathrm{RMOM}}(\rho^*)^2$ and $\|Z - Z^*\| \leq \rho^*$, then $AP\mathcal{L}_Z \geq \|Z - Z^*\|_2^2$.

Assumptions

i. Adversarial contamination framework with $|\mathcal{O}|$ outliers
ii. $\mathcal{C}$ is star-shaped in Z^*
iii. $K \gtrsim |\mathcal{O}|$
iv. the local curvature assumption holds and ρ^* satisfies the sparsity equation
v. $\lambda \simeq r^*_{\text{RMOM}}(\rho^*)/\rho^*$

Theorem | Excess risk and estimation bounds for the RMOM with ℓ_2-localization

With probability at least $1 - 2\exp(-72K/625)$,

$$\|\hat{Z}_{K,\lambda}^{\text{RMOM}} - Z^*\| \leq \rho^* \quad , \quad \|\hat{Z}_{K,\lambda}^{\text{RMOM}} - Z^*\|_2^2 \leq r^*_{\text{RMOM}}(\rho^*)^2 \;\; \text{and} \;\; P\mathcal{L}_{\hat{Z}_{K,\lambda}^{\text{RMOM}}} \leq r^*_{\text{RMOM}}(\rho^*)^2.$$

Assumptions:
- Spiked covariance model: $\Sigma = I_d + \theta\beta^*(\beta^*)^\top$ with $\beta^* \in \mathcal{S}_2^d$, k-sparse, $\theta > 0$
- Adversarial contamination with $|\mathcal{O}|$ outliers
- For all $p, q \in [d]$ and all $2 \leq r \leq 2\log(ed/k)$, $\|\tilde{X}_{1p}\tilde{X}_{1q} - \mathbb{E}(\tilde{X}_{1p}\tilde{X}_{1q})\|_{L_r} \leq c_0 r$
- $N \gtrsim \log(d/k)$, $\theta \leq k$ and $K \gtrsim |\mathcal{O}|$

Theorem | Excess risk and estimation bounds for the RMOM in sparse PCA

Let $\lambda \simeq r^*_{\mathrm{RMOM}}(\rho^*)/\rho^*$. With probability at least $1 - \exp(-72K/625)$:

$$\|\hat{Z}_{K,\lambda}^{\mathrm{RMOM}} - Z^*\|_1 \lesssim \frac{k}{\sqrt{N\theta^2}} \max\left(k\sqrt{\log\left(\frac{ed}{k}\right)}; \sqrt{K}\right),$$

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$$P\mathcal{L}_{\hat{Z}_{K,\lambda}^{\mathrm{RMOM}}} \lesssim \frac{1}{N\theta} \max\left(k^2\log\left(\frac{ed}{k}\right); K\right).$$

Thanks !

Example for the minmax MOM version of the LASSO:

$$\hat{u} \in \operatorname*{argmin}_{u \in \mathbb{R}^d} \sup_{u' \in \mathbb{R}^d} MOM_K(\ell_u - \ell_{u'}) + \lambda_K \left(\|u\|_1 - \|u'\|_1 \right)$$

where $\ell_u(x, y) = (y - \langle x, u \rangle)^2$ and $\lambda_K \sim \sigma \sqrt{(1/N) \log \left(\sigma^2 d / K \right)}$.

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At iteration $(u_t, u_{t'})$ we do:

u1 random partition: $\{1, \ldots, N\} = B_1 \cup \cdots \cup B_K$

u2 median block: $P_{B_k}(\ell_{u_t} - \ell_{u'_t}) = MOM_K(\ell_{u_t} - \ell_{u'_t})$

u3 descent direction: $\nabla_t := \nabla(u \to P_{B_k} \ell_u)_{|u=u_t}$

u4 $u_{t+1} = prox_{\lambda_K \|\cdot\|_1}(u_t - \eta_t \nabla_t)$

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u4 $u_{t+1} = prox_{\lambda_K \|\cdot\|_1}(u_t - \eta_t \nabla_t)$
$u'1$ random partition: $\{1, \ldots, N\} = B_1 \cup \cdots \cup B_K$
$u'2$ median block: $P_{B_{k'}}(\ell_{u_{t+1}} - \ell_{u'_t}) = MOM_K(\ell_{u_{t+1}} - \ell_{u'_t})$
$u'3$ ascent direction: $\nabla'_t := -\nabla(u \to P_{B_{k'}}\ell_u)_{|u=u'_t}$
$u'4$ $u'_{t+1} = prox_{\lambda_K \|\cdot\|_1}\left(u'_t + \eta_t \nabla'_t\right)$

Problem: Find a top unit-eigenvector of $A \in \mathcal{S}_d(\mathbb{R})$.

$$x^* \in \underset{x \in \mathbb{R}^d, \|x\|_2 = 1}{\operatorname{argmax}} \langle A, xx^\top \rangle \tag{1}$$

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Fact: $\{xx^\top : x \in \mathbb{R}^d, \|x\|_2 = 1\} = \{X \in \mathbb{R}^{d \times d} : X \succeq 0, \mathbf{Tr}(X) = 1, \mathbf{rank}(X) = 1\}$

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Lifting: quadratic functions of x are linear functions of $xx^\top$

$$(1) \Leftrightarrow x^*(x^*)^\top \in \operatorname*{argmax}_{\substack{X \in \mathbb{R}^{d \times d}, \\ X \succeq 0, \\ \mathbf{Tr}(X) = 1, \\ \mathbf{rank}(X) = 1}} \langle A, X \rangle$$

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Lifting + SDP relaxation: a classical strategy for many problems (Goemans and Williamson, 95') and a source of many **ERMs w.r.t. a linear loss function** for us.

Problem: estimate d unknown angles $\theta_1, \cdots, \theta_d$.

Data: a noisy pairwise measurements of their offsets:

$$\delta_{ij} = (\theta_i - \theta_j + \sigma g_{ij})[2\pi]$$

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Equivalent problem: estimate $x^* \in \mathbb{R}^d$ with coordinates $x_j^* := e^{\iota\theta_j}$

$\rightarrow$ we define $A := \left(e^{\iota\delta_{ij}}\right)_{1\leq i,j,\leq d} = S \circ [x^*(\overline{x^*})^\top]$, with $S = (S_{ij})_{d\times d}, S_{ij} = \begin{cases} e^{\iota\sigma g_{ij}} & \text{if } i < j \\ 1 & \text{if } i = j \\ e^{-\iota\sigma g_{ij}} & \text{if } i > j \end{cases}$

We have

$$x^* \in \underset{x\in\mathbb{C}^d:|x_i|=1}{\text{argmax}} \left\{\langle \mathbb{E}A, x\bar{x}^\top \rangle\right\} = \left\{(e^{\iota(\theta_i+\theta_0)})_{i=1}^d : \theta_0 \in [0, 2\pi)\right\}.$$

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Lifting

$$\left\{ x\bar{x}^\top : \; x \in \mathbb{C}^d : |x_i| = 1 \right\} = \{ Z \in \mathbb{H}_d : Z \succeq 0, \operatorname{diag}(Z) = \mathbf{1}_d, \operatorname{rank}(Z) = 1 \}$$

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Relaxation

$$Z^* \in \underset{\substack{Z\in\mathbb{H}_d,\\ Z\succeq 0,\\ \operatorname{diag}(Z)=\mathbf{1}_d}}{\operatorname{argmax}} \langle \mathbb{E}A, Z\rangle$$

x^* can be obtain as a first eigen-vector of Z^* up to a global rotational shift $e^{\iota\theta_0}$ of the coordinates.

We have

$$x^* \in \operatorname*{argmax}_{x\in\mathbb{C}^d : |x_i|=1} \left\{\langle \mathbb{E}A, x\bar{x}^\top\rangle\right\} = \left\{(e^{\iota(\theta_i+\theta_0)})_{i=1}^d : \theta_0 \in [0, 2\pi)\right\}.$$

Lifting

$$\left\{x\bar{x}^\top : x \in \mathbb{C}^d : |x_i| = 1\right\} = \{Z \in \mathbb{H}_d : Z \succeq 0, \operatorname{diag}(Z) = \mathbf{1}_d, \operatorname{rank}(Z) = 1\}$$

Estimator

$$\hat{Z} \in \operatorname*{argmax}_{\substack{Z\in\mathbb{H}_d,\\ Z\succeq 0,\\ \operatorname{diag}(Z)=\mathbf{1}_d}} \langle A, Z\rangle$$

(An ERM w.r.t. a linear loss function).

- We observe $X = |Bx| \in \mathbb{R}^N$, with $B \in \mathbb{C}^{N\times d}$, random matrix.
- Aim: recover x from the observation of X.
- Equivalent problem: (d'Aspremont et al.)

$$z^* \in \underset{\substack{z\in\mathbb{C}^N \\ |z_i|=1,\forall i\in[N]}}{\operatorname{argmin}} \langle \mathbb{E}[A], z\bar{z}^\top \rangle,$$

where $A := \left(XX^\top\right) \circ \left(I_N - BB^+\right)$, B^+ pseudo-inverse of B.

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- Lifting

$$\min_{\substack{Z\succeq 0 \\ Z_{ii}=1 \forall i \\ rank(Z)=1}} \langle \mathbb{E}[A], Z \rangle$$

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where $A := \left(XX^\top\right) \circ \left(I_N - BB^+\right)$, B^+ pseudo-inverse of B.

- Relaxation:

$$Z^* \in \underset{\substack{Z\succeq 0 \\ Z_{ii}=1\forall i}}{\operatorname{argmin}} \langle \mathbb{E}[A], Z\rangle \quad \Big| \quad z^* : 1^{st} \text{ eigen vector of } Z^*$$

- Estimator

$$\hat{Z} \in \underset{\substack{Z\succeq 0 \\ Z_{ii}=1\forall i}}{\operatorname{argmin}} \langle A, Z\rangle \quad \Big| \quad \hat{z} : 1^{st} \text{ eigen vector of } \hat{Z}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_j \\ \vdots \\ X_d \end{bmatrix}$$

- We observe $X_1, \cdots, X_N$, independant copies of $X \in \mathbb{R}^{d\times d}$
  $\Sigma := cov(X)$

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- Aim: find a partition $G = \{G_1, \dots, G_K\}$ of $\{1, \dots, d\}$ that separates the components of X.

Membership matrix:

$$Q := \left(1_{\{a\in G_k\}}\right)_{\substack{1\le a\le d\\ 1\le k\le K}}$$

Partnership matrix:

$$Z := \left(|G_k|^{-1} 1_{\{i\in G_k, j\in G_k\}}\right)_{1\le i,j\le d}$$

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- exact G-block covariance model: $\Sigma = QCQ^\top + \Gamma$, with $C \in \mathcal{S}_K(\mathbb{R})$ and Γ diagonal.

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_j \\ \vdots \\ X_d \end{bmatrix}$$

- Estimator (Verzelen, Giraud et al.)

$$\hat{Z} \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle A, Z \rangle, \qquad \mathcal{C} := \left\{ Z \succeq 0, Z \geq 0, \sum_j Z_{ij} = 1 \forall i, Tr\left(Z\right) = K \right\}$$

- Noiseless case:

$$Z^* \in \underset{Z \in \mathcal{C}}{\operatorname{argmax}} \langle \mathbb{E} A, Z \rangle$$

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_j \\ \vdots \\ X_d \end{bmatrix}$$

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- Noiseless case:

$$Z^* \in \underset{Z \in \mathcal{C}}{\mathrm{argmax}} \langle \mathbb{E}A, Z \rangle$$

3 | Key technical tools.

Reminders

Complexity fixed point parameter

$$r_G^*(\delta) = \inf\left(r > 0 : \mathbb{P}\left[\sup_{Z\in\mathcal{C}:G(Z-Z^*)\leq r}(P_N - P)\mathcal{L}_Z \leq \frac{r}{2}\right] \geq 1-\delta\right)$$

Local curvature assumption

For all $Z \in \mathcal{C}$, if $P\mathcal{L}_Z \leq r_G^*(\delta)$ then $P\mathcal{L}_Z \geq G(Z^* - Z)$.

Theorem | Excess risk for the ERM estimator with G-localization

We assume that $\mathcal{C}$ is star-shaped in Z^* and that the local curvature Assumption holds for some $0 < \delta < 1$. Then, with probability at least $1-\delta$, it holds true that

$$G(Z^* - \hat{Z}) \leq P\mathcal{L}_{\hat{Z}} \leq r_G^*(\delta).$$

Assume $r^* > 0$

- Define $\Omega^* := \{\forall Z \in \mathcal{C} \text{ s.t } \langle \mathbb{E}[A], Z^* - Z\rangle \leq r^*, \langle A - \mathbb{E}[A], Z - Z^*\rangle \leq r^*/2\}$
  By definition of r^*, Ω^* holds with probability at least $1 - \delta$.

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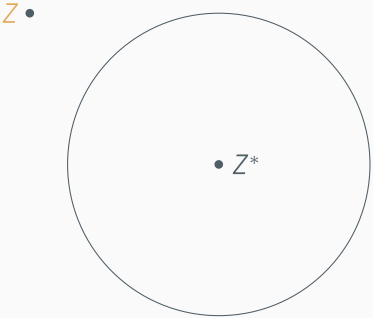
  By definition of r^*, Ω^* holds with probability at least $1 - \delta$.
- Consider $Z \in \mathcal{C}$ such that $\langle \mathbb{E}[A], Z^* - Z\rangle > r^*$.



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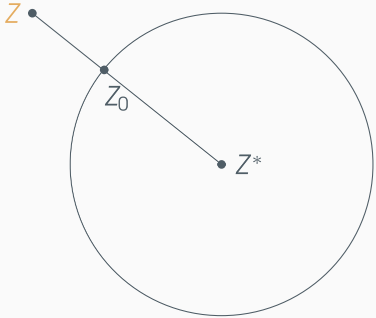

Star-shaped property of $\mathcal{C}$:

$\exists\, 0 < \lambda < 1$, $Z_0 \in \mathcal{C}$ s.t $Z_0 - Z^* = \lambda(Z - Z^*)$ and $\langle \mathbb{E}[A], Z^* - Z_0\rangle = r^*$.

Assume $r^* > 0$

- Define $\Omega^* := \{\forall Z \in \mathcal{C} \text{ s.t } \langle \mathbb{E}[A], Z^* - Z\rangle \leq r^*, \langle A - \mathbb{E}[A], Z - Z^*\rangle \leq r^*/2\}$
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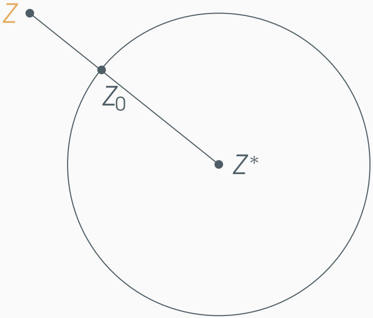

Star-shaped property of $\mathcal{C}$:

$\exists\, 0 < \lambda < 1,\ Z_0 \in \mathcal{C}$ s.t $Z_0 - Z^* = \lambda(Z - Z^*)$ and $\langle \mathbb{E}[A], Z^* - Z_0\rangle = r^*$.

Homogeneity of the loss function:

On $\Omega^* \Rightarrow \langle A - \mathbb{E}[A], Z_0 - Z^*\rangle \leq r^*/2$
and then $\langle A, Z - Z^*\rangle \leq -r^*/2 < 0$

Assume $r^* > 0$

- Define $\Omega^* := \{\forall Z \in \mathcal{C} \text{ s.t } \langle \mathbb{E}[A], Z^* - Z\rangle \leq r^*, \langle A - \mathbb{E}[A], Z - Z^*\rangle \leq r^*/2\}$
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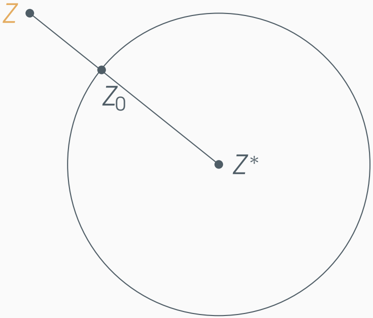
- Consider $Z \in \mathcal{C}$ such that $\langle \mathbb{E}[A], Z^* - Z\rangle > r^*$.



Star-shaped property of $\mathcal{C}$:

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Homogeneity of the loss function:
On $\Omega^* \Rightarrow \langle A - \mathbb{E}[A], Z_0 - Z^*\rangle \leq r^*/2$
and then $\langle A, Z - Z^*\rangle \leq -r^*/2 < 0$
$\rightarrow Z$ cannot be a maximizer of $Z \rightarrow P_N \ell_Z$ on $\mathcal{C}$

- Necessarily, $\langle \mathbb{E}[A], Z^* - \hat{Z} \rangle \leq r^*$.

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- From the curvature equation, we get that $G(Z^* - \hat{Z}) \leq P\mathcal{L}_{\hat{Z}}$.

- Necessarily, $\langle \mathbb{E}[A], Z^* - \hat{Z} \rangle \leq r^*$.
- From the curvature equation, we get that $G(Z^* - \hat{Z}) \leq P\mathcal{L}_{\hat{Z}}$.

Finally :

$$G(Z^* - \hat{Z}) \leq P\mathcal{L}_{\hat{Z}} \leq r_G^*(\delta)$$

.

4 | Appendix.

Reminders

Assumption | Local curvature assumption

There exists $A > 0$, $\gamma > 0$ and $\rho^* > 0$ such that ρ^* satisfies the spartsity equation and for both $b \in \{1, 2\}$ and all $Z \in \mathcal{C}$, if $\|Z - Z^*\|_2^2 = r^*_{\mathrm{RMOM},\mathrm{G}}(\gamma^*, b\rho^*)^2$ and $\|Z - Z^*\| \leq b\rho^*$, then $AP\mathcal{L}_Z \geq \|Z - Z^*\|_2^2$.

Theorem | Excess risk and estimation bounds for the RMOM with ℓ_2-localization

Grant Assumptions (i) to (v). Then, with probability at least $1 - 2\exp(-72K/625)$, the following bounds hold for the RMOM estimator with ℓ_2-localization:

$$\|\hat{Z}^{\mathrm{RMOM}}_{K,\lambda} - Z^*\| \leq 2\rho^* \quad , \quad \|\hat{Z}^{\mathrm{RMOM}}_{K,\lambda} - Z^*\|_2^2 \leq r^*_{\mathrm{RMOM},\mathrm{G}}(\gamma, 2\rho^*)^2 \ \ and \ \ P\mathcal{L}_{\hat{Z}^{\mathrm{RMOM}}_{K,\lambda}} \leq \frac{93}{100} r^*_{\mathrm{RMOM},\mathrm{G}}(\gamma, 2\rho^*)^2.$$

Structure of the proof

- For $b \in \{1, 2\}$, we define $r_b^* = r^*_{\mathrm{RMOM},G}(\gamma, b\rho^*)$, and $\mathcal{B}_b := \{Z \in \mathcal{C} : G(Z - Z^*) \leq (r_b^*)^2 \text{ and } \|Z - Z^*\| \leq b\rho^*\}$.
- We consider the event

$$\Omega_{K,G} = \left\{ \forall b \in \{1,2\}, \forall Z \in \mathcal{B}_b,\ \sum_{k=1}^{K} 1\left(|(P_{B_k} - P)\mathcal{L}_Z| \leq \frac{1}{20}(r_b^*)^2 \right) > \frac{K}{2} \right\}.$$

- Two steps

1| $\hat{Z}$ has the desired properties on $\Omega_{K,G}$.

2| $\Omega_{K,G}$ holds with high probability.

Step 1 |

Lemma 1.

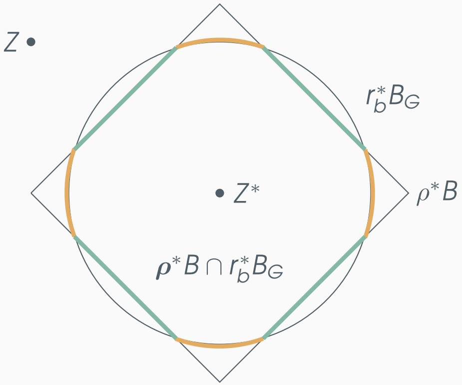
If there exists $\eta > 0$ such that

$$\sup_{Z \in \mathcal{C} \setminus \mathcal{B}_2} \mathrm{MOM}_K(\ell_{Z^*} - \ell_Z) + \lambda(\|Z^*\| - \|Z\|) < -\eta \tag{2}$$

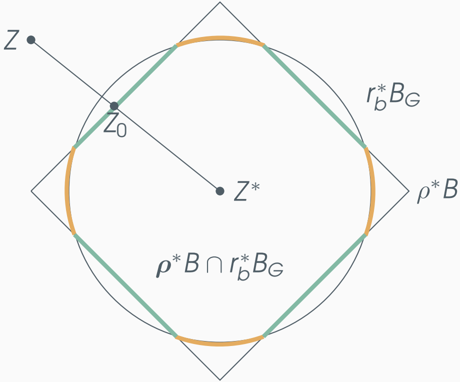
and

$$\sup_{Z \in \mathcal{C}} \mathrm{MOM}_K(\ell_{Z^*} - \ell_Z) + \lambda(\|Z^*\| - \|Z\|) \leq \eta \tag{3}$$

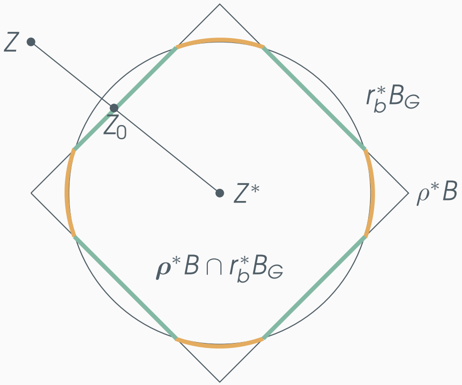
then $\|\hat{Z} - Z^*\| \leq 2\rho^*$ and $G(\hat{Z} - Z^*) \leq r^*_{\mathrm{RMOM},G}(\gamma, 2\rho^*)^2$.



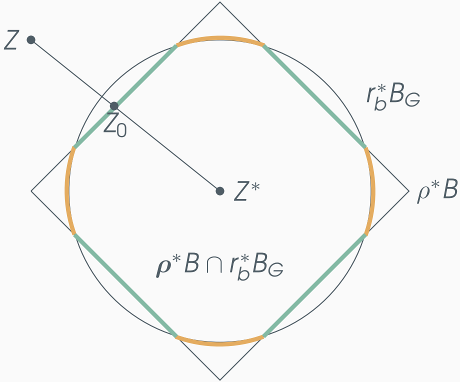
- Consider $Z \in \mathcal{C} \backslash \mathcal{B}_b$.



- Consider $Z \in \mathcal{C} \backslash \mathcal{B}_b$.
- Regularity of G and star-shaped property of $\mathcal{C} \to \exists Z_0 \in \partial \mathcal{B}_b$ and $\alpha > 1$ such that
$$Z = Z^* + \alpha(Z_0 - Z^*).$$



- Consider $Z \in \mathcal{C} \backslash \mathcal{B}_b$.
- Regularity of G and star-shaped property of $\mathcal{C} \rightarrow$ $\exists Z_0 \in \partial \mathcal{B}_b$ and $\alpha > 1$ such that
$$Z = Z^* + \alpha(Z_0 - Z^*).$$
- Linearity of ℓ_Z and convexity of $\|\cdot\|$:
$$\forall k \in [K], P_{B_k} \mathcal{L}_Z^\lambda \geq \alpha P_{B_k} \mathcal{L}_{Z_0}^\lambda. \tag{4}$$



- Consider $Z \in \mathcal{C} \backslash \mathcal{B}_b$.
- Regularity of G and star-shaped property of $\mathcal{C} \to \exists Z_0 \in \partial \mathcal{B}_b$ and $\alpha > 1$ such that
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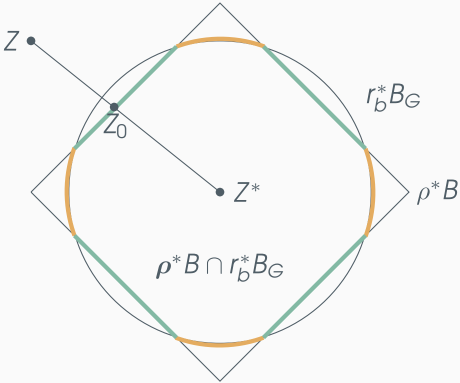
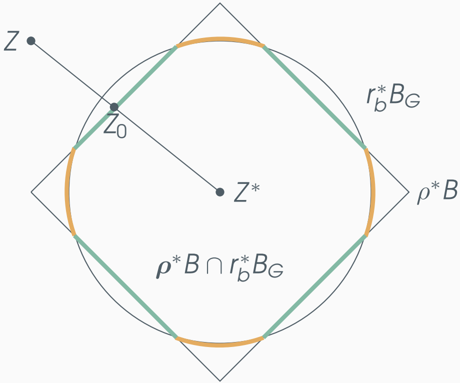
- Linearity of ℓ_Z and convexity of $\|\cdot\|$:
$$\forall k \in [K], P_{B_k} \mathcal{L}_Z^\lambda \geq \alpha P_{B_k} \mathcal{L}_{Z_0}^\lambda. \tag{4}$$
- Either a $\Big|\, G(Z_0 - Z^*) = (r_b^*)^2$ and $\|Z_0 - Z^*\| < b\rho^*$
  or b $\Big|\, G(Z_0 - Z^*) < (r_b^*)^2$ and $\|Z_0 - Z^*\| = b\rho^*$.



a | On $\Omega_{K,G}$, using the local curvature assumption, we find at least $K/2$ blocks on which

$$P_{B_k} \mathcal{L}_{Z_0}^{\lambda} \geq \begin{cases} (r_2^*)^2/5 & \text{for } b = 1 \\ 2(r_2^*)^2/5 & \text{for } b = 2 \end{cases} \tag{5}$$



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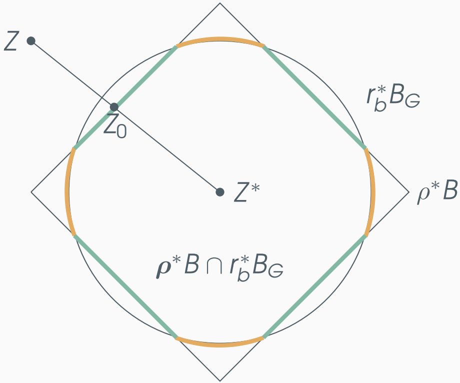
b | Here, $Z_0 \in \bar{H}_{b\rho^*}$. The sparsity equation also holds for $\rho = b\rho^*$. Then, on $\Omega_{K,G}$, we find at least $K/2$ blocks on which

$$P_{B_k}\mathcal{L}_{Z_0}^{\lambda} \geq \begin{cases} 57(r_2^*)^2/400 & \text{for } b = 1 \\ 134(r_2^*)^2/400 & \text{for } b = 2 \end{cases} \tag{6}$$



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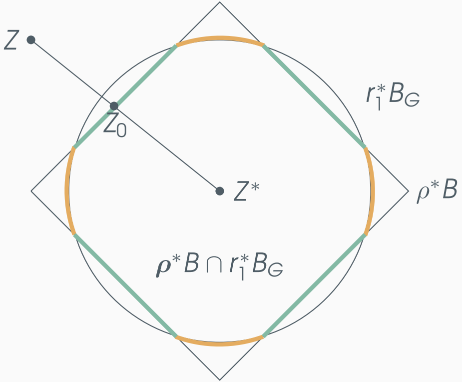
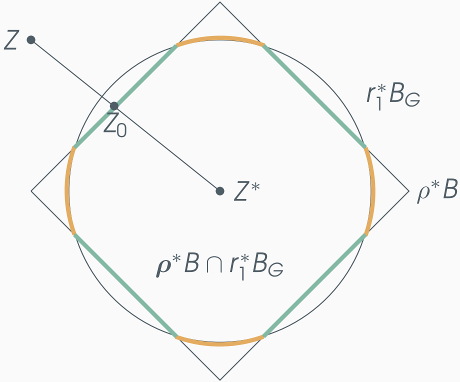
- From (4), (5) and (6) with $b=2$, we conclude that (2) holds for any $\eta < (134/400)(r_2^*)^2$



- Consider $Z \in \mathcal{B}_1$. On $\Omega_{K,G}$, we find at least $K/2$ blocks on which

$$P_{B_k} \mathcal{L}_Z^\lambda \geq -13(r_2^*)^2/40 \tag{7}$$



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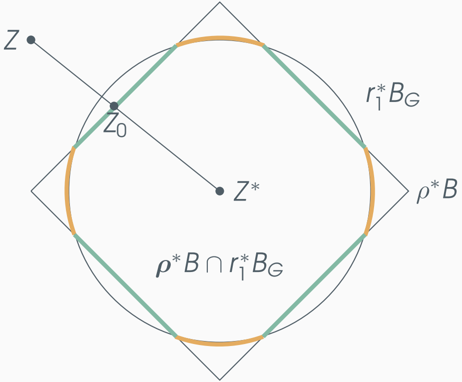
$$P_{B_k} \mathcal{L}_Z^\lambda \geq -13(r_2^*)^2/40 \tag{7}$$
- From (4), (5) and (6) with $b = 1$, we conclude that (3) holds with $\eta = (13/40)(r_2^*)^2$



- Consider $Z \in \mathcal{B}_1$. On $\Omega_{K,G}$, we find at least $K/2$ blocks on which

$$P_{B_k} \mathcal{L}_Z^\lambda \geq -13(r_2^*)^2/40 \tag{7}$$

- From (4), (5) and (6) with $b = 1$, we conclude that (3) holds with $\eta = (13/40)(r_2^*)^2$

Finally, both (2) and (3) hold for $\eta = (33/100)(r_2^*)^2$.

$\rightarrow$ We conclude from Lemma 1. that $\|\hat{Z} - Z^*\| \leq 2\rho^*$ and $G(\hat{Z} - Z^*) \leq r^*_{\mathrm{RMOM},G}(\gamma, 2\rho^*)^2$.

Bound on $P\mathcal{L}_{\hat{Z}}$

- From the above: $\hat{Z} \in \mathcal{B}_2 \to$ on $\Omega_{K,G}$ there exist stricly more than $K/2$ blocks on which

$$P\mathcal{L}_{\hat{Z}} \leq P_{B_k}\mathcal{L}_{\hat{Z}} + (r_2^*)^2/(20). \tag{8}$$

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$$P\mathcal{L}_{\hat{Z}} \leq P_{B_k}\mathcal{L}_{\hat{Z}} + (r_2^*)^2/(20). \tag{8}$$

- From (3) with $\eta = (33/100)(r_2^*)^2$ and the definition of $\hat{Z}$, we find at least $K/2$ blocks B_k on which

$$P_{B_k}\mathcal{L}_{\hat{Z}} \leq \frac{88}{100}(r_2^*)^2. \tag{9}$$

Bound on $P\mathcal{L}_{\hat{Z}}$

- From the above: $\hat{Z} \in \mathcal{B}_2 \to$ on $\Omega_{K,G}$ there exist stricly more than $K/2$ blocks on which

$$P\mathcal{L}_{\hat{Z}} \leq P_{B_k}\mathcal{L}_{\hat{Z}} + (r_2^*)^2/(20). \tag{8}$$

- From (3) with $\eta = (33/100)(r_2^*)^2$ and the definition of $\hat{Z}$, we find at least $K/2$ blocks B_k on which

$$P_{B_k}\mathcal{L}_{\hat{Z}} \leq \frac{88}{100}(r_2^*)^2. \tag{9}$$

- Then, there exists at least one on which (8) and (9) hold simultaneously, which gives

$$P\mathcal{L}_{\hat{Z}} \leq \frac{93}{100}(r_2^*)^2.$$

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$\rightarrow$ On $\Omega_{K,G}$, $\hat{Z}$ has the announced properties.

Step 2 |

Lemma

Assume that $K \geq 100|\mathcal{O}|$, *and let* $\rho^* > 0$ *be such that it satisfies the sparsity equation. Then,* $\Omega_{K,G}$ *holds with probability at least* $1 - 2\exp(-72K/625)$.

Definitions

- For $k \in [K]$:
  $W_k := \{X_i : i \in B_k\}$ and $F_Z(W_k) = (P_{B_k} - P)\mathcal{L}_Z$.
  $\widetilde{W}_k := \left\{\widetilde{X}_i : i \in B_k\right\}$ and $F_Z(\widetilde{W}_k) = (\widetilde{P_{B_k}} - \widetilde{P})\mathcal{L}_Z$, their non-corrupted counterparts.
- For $b \in \{1, 2\}$:

$$Z \to \psi_b(Z) = \sum_{k \in [K]} 1_{\left\{|F_Z(W_k)| \leq \frac{(r_b^*)^2}{20}\right\}}.$$

- $\phi : t \in \mathbb{R} \to 1_{\{t \geq 1\}} + 2(t - 1/2)1_{\{1/2 \leq t \leq 1\}}$.
- We show that

$$(\Omega_{K,G} \text{ holds w.h.p}) \iff \left(\psi_b(Z) \leq \frac{49K}{400} \text{ w.h.p for } Z \in \mathcal{B}_b \text{ and } b \in \{1, 2\}\right).$$

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$\rightarrow$ We bound $\psi_b(Z)$ with high probability.

Bounding $\psi_b(Z)$

In what follows, we consider $b \in \{1, 2\}$ and $Z \in \mathcal{B}_b$.

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Basic manipulations lead to

$$\psi_b(Z) \leq \sup_{Z \in \mathcal{B}_b} \left(\sum_{k \in [K]} \phi\left(\frac{20|F_Z(\widetilde{W}_k)|}{(r_b^*)^2} \right) - \mathbb{E}\left[\phi\left(\frac{20|F_Z(\widetilde{W}_k)|}{(r_b^*)^2} \right) \right] \right) + \sum_{k \in [K]} \mathbb{P}\left(|F_Z(\widetilde{W}_k)| > \frac{(r_b^*)^2}{40} \right)$$
$$= S_b^{(1)}(Z) + S_b^{(2)}(Z). \tag{10}$$

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We bound each term separately.

Bounding $\psi_b(Z)$ | Bounding $S_b^{(2)}(Z)$

For each $k \in [K]$, Markov's inequality and the definition of r_b^* yield to

$$\mathbb{P}\left(|F_Z(\widetilde{W}_k)| > \frac{(r_b^*)^2}{40}\right) \leq \frac{1600^2}{(r_b^*)^4}\left(V_K(r_b^*)\right)^2 \leq \frac{1}{200},$$

so that

$$S_b^{(2)}(Z) \leq \frac{K}{200}. \tag{11}$$

Bounding $\psi_b(Z)$ | Bounding $S_b^{(1)}(Z)$

- Mc Diarmind inequality with $t = 12/25 \to$ w.p.a.l $1 - e^{-72K/625}$

$$S_b^{(1)}(Z) \leq \frac{12K}{25} + \mathbb{E}\left[\sup_{Z \in \mathcal{B}_b} \sum_{k\in[K]} \phi\left(\frac{20|F_Z(\widetilde{W}_k)|}{(r_b^*)^2}\right) - \mathbb{E}\phi\left(\frac{20|F_Z(\widetilde{W}_k)|}{(r_b^*)^2}\right)\right]$$
$$:= \frac{12K}{25} + T_b^{(1)}(Z). \tag{12}$$

- $\epsilon_1, \ldots, \epsilon_K$, Rademacher variables independant from the $\widetilde{X}_i$'s. By the symmetrization Lemma:

$$T_b^{(1)}(Z) \leq 2\mathbb{E}\left[\sup_{Z \in \mathcal{B}_b} \sum_{k\in[K]} \epsilon_k \phi\left(\frac{20|F_Z(\widetilde{W}_k)|}{(r_b^*)^2}\right)\right] := 2T_b^{(2)}(Z). \tag{13}$$

- ϕ is Lipschitz with $\phi(0) = 0 \rightarrow$ by the contraction Lemma:

$$T_b^{(2)}(Z) \leq \frac{40}{(r_b^*)^2}\mathbb{E}\left[\sup_{Z\in\mathcal{B}_b}\sum_{k\in[K]}\epsilon_k(\widetilde{P_{B_k}} - \widetilde{P})\mathcal{L}_Z\right] := \frac{40}{(r_b^*)^2}T_b^{(3)}(Z) \tag{14}$$

- $(\sigma_i)_{i=1,\dots,N}$ Rademacher variables independant from the $\widetilde{X}_i$'s and the ϵ_i's. $\rightarrow$ By the symmetrization Lemma:

$$T_b^{(3)}(Z) \leq 2KE_G(r_b^*, b\rho^*) \leq 2K\gamma(r_b^*)^2. \tag{15}$$

Combining (12), (13), (14) and (15), we get that w.p.a.l $1 - e^{-72K/625}$

$$S_b^{(1)}(Z) \leq \left(\frac{12}{25} + 160\gamma\right)K. \tag{16}$$

Bounding $\psi_b(Z)$

Finally, combining (10), (11) and (16), we get that with probability at least $1 - \exp(-72K/625)$:

$$\psi_b(Z) \leq \frac{49}{400}K.$$

This must be verified for both $b = 1$ and 2, so the final probability is $1 - 2\exp(-72K/625)$, which concludes the proof.