Suppression of chemotactic blow-up by buoyancy

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joint work with Zhongtian Hu and Alexander Kiselev

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The Keller-Segel equation

- The Keller-Segel equation models the collective motion of cells attracted by a self-emitted chemical substance. The parabolic-elliptic Keller-Segel equation in $\mathbb{R}^2$ is

$$\begin{cases} \rho_t = \Delta \rho - \nabla \cdot (\rho \nabla c), \\ \Delta c + \rho = 0. \end{cases}$$

(Patlak '53, Keller-Segel '71)

- In $\mathbb{R}^2$, there is a "critical mass" $M_c = 8\pi$ such that:
  - If the mass satisfies $M > M_c$, all solutions blow-up in finite time.
  - If $M < M_c$, all solutions remains globally bounded, and dissipate according to heat equation scaling as $t \to \infty$.
  - If $M = M_c$, no blow-up, but solutions may aggregate as $t \to \infty$.

(Jäger-Luckhaus '92, Dolbeault-Perthame '04, Blanchet-Dolbeault-Perthame '06, Blanchet-Carrillo-Masmoudi '08, Blanchet-Carlen-Carrillo '12, Carlen-Figalli '13)

- For $\Omega \subset \mathbb{R}^2$ with smooth boundary, let $-\Delta c = \rho - \rho_M$ with $\frac{\partial c}{\partial n} = 0$ on $\partial\Omega$. The critical mass becomes $M_c = 4\pi$.

(Biler '98, Nagai '01)

Adding advection into the picture

- Chemotactic processes often take place in ambient fluid.
- Question: Can blow-up be suppressed if the density is advected by a divergence-free velocity field u?

$$\rho_t + u \cdot \nabla \rho = \Delta \rho - \nabla \cdot (\rho \nabla c)$$

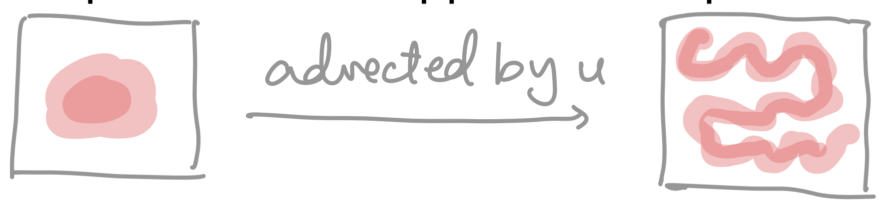
Keller-Segel equation with passive advection

Consider the equation

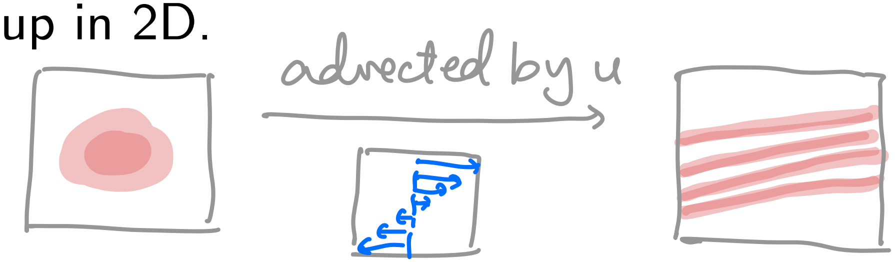
$$\rho_t + Au \cdot \nabla\rho = \Delta\rho - \nabla\cdot(\rho\nabla(-\Delta)^{-1}\rho),$$

where u is a given divergence-free vector field, with amplitude $A \in \mathbb{R}$.

- Given ρ_0, there exist flows can prevent blow-up for sufficiently large amplitude (depending on ρ_0):
  - Kiselev-Xu '16 showed that flows that are mixing or diffusion-enhancing with sufficiently large amplitudes can suppress blowup.



  - Bedrossian-He '17, He '18: sufficiently strong monotone shear flow also prevents blow-up in 2D.



- However, any passive flow would not change the critical mass:
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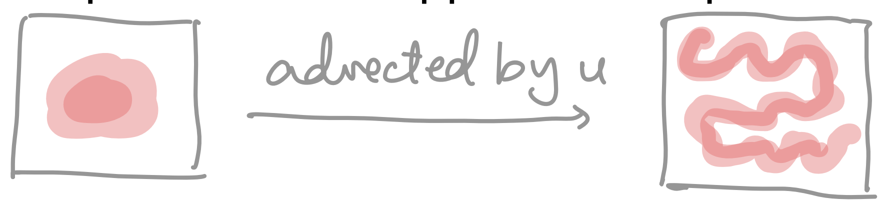

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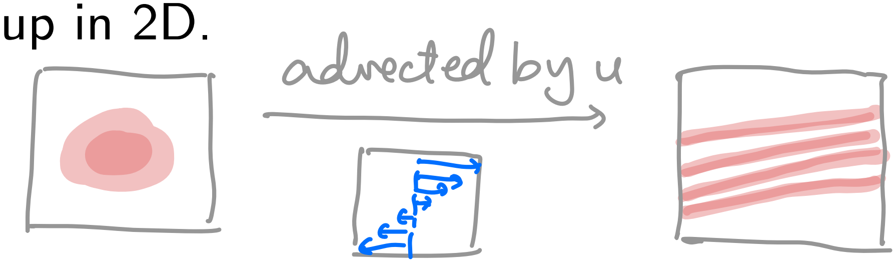

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Keller-Segel equation with active advection

Since the past decade, there has been a growing interest in chemotaxis equations coupled with fluid equations (usually via gravity force):

- Di Francesco–Lorz–Markowich '10: global existence for chemotaxis-Stokes system with nonlinear diffusion
- Duan–Lorz–Markowich '10: global existence for chemotaxis-Navier-Stokes near constant steady state
- Liu–Lorz '11, Winkler '12: global existence of chemotaxis-Navier-Stokes when chemical consumed by bacteria
- Lorz '12: global existence for Keller-Segel-Stokes model with small data
- Chae–Kang–Lee '14: Global existence for Keller-Segel–Navier–Stokes with small data
- Tao–Winkler '16: Global existence of Keller-Segel–Navier–Stokes with a reaction term $-\mu\rho^2$
- Winkler '21 Suppressing blow-up in Keller-Segel–Navier–Stokes by flux limitation
- Zeng–Zhang–Zi '21: Suppression of blow-up by a sufficiently strong Couette flow

In all the above results, the global existence results either requires smallness of initial data / strength of flow, or still hold with $u \equiv 0$.

Coupling with Darcy's law for incompressible porous media

- We study the following equation in $\Omega := \mathbb{T} \times [0, \pi]$:

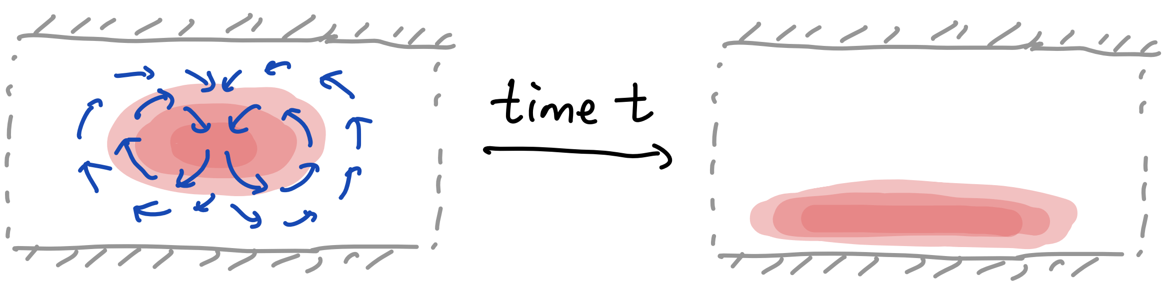
$$\rho_t + u \cdot \nabla \rho - \Delta \rho + \nabla \cdot (\rho \nabla (-\Delta_N)^{-1} (\rho - \rho_M)) = 0.$$

- Here u obeys Darcy's law for incompressible porous media via gravity:

$$\begin{cases} u = -\nabla p - g \rho e_2 \text{ in } \Omega, \\ \nabla \cdot u = 0 \text{ in } \Omega, \quad u \cdot n = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g \in \mathbb{R}$ is the gravitational constant.

- When such u is coupled with $\rho_t + u \cdot \nabla \rho = 0$:
  - Córdoba–Gancedo–Orive '07: Local well-posedness and blow-up criteria.
  - Whether smooth initial data leads to a blow-up is still open.



- Heuristically: the flow tend to make the solution stratified (heavier density falls down and lighter density goes up). Since 1D Keller-Segel doesn't blow-up, maybe a sufficiently large g should prevent blow-up?

Any $g \neq 0$ prevents blow-up!

Surprisingly, it turns out that blow-up is prevented for any non-zero $g \in \mathbb{R}$!

Theorem (Hu–Kiselev–Y., preprint, 2023)

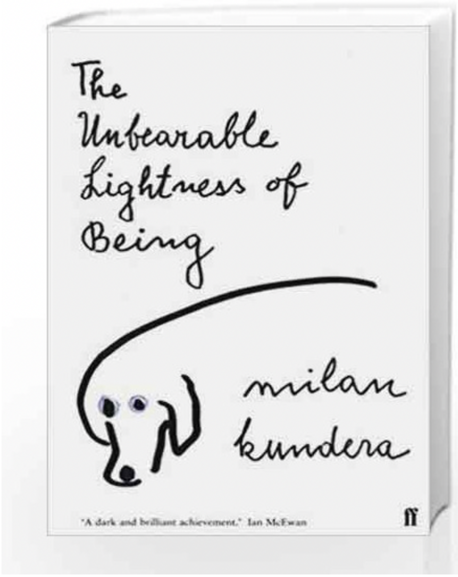
Consider the coupled system

$$\begin{cases} \rho_t + u \cdot \nabla\rho - \Delta\rho + \nabla \cdot (\rho\nabla(-\Delta_N)^{-1}(\rho - \rho_M)) = 0, \\ u = -\nabla p - g\rho e_2 \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u \cdot n = 0 \text{ on } \partial\Omega, \end{cases}$$

with smooth initial condition $\rho_0 \geq 0$ in $\Omega = \mathbb{T} \times [0, \pi]$. Then for any $g \neq 0$, the solution is regular globally in time.

Main ideas:

- Tracking the evolution of $\|\rho(t) - \rho_M\|_{L^2}^2$ and the potential energy $E(t) := \int_\Omega \rho(x,t) x_2 dx$ simultaneously.
- Goal: if $\lim_{t \to T} \|\rho(t) - \rho_M\|_{L^2} = \infty$, we have $\lim_{t \to T} E(t) = -\infty$.
- But this causes a "unbearable heaviness of being" since $E(t) \geq 0$ for all times!



Act I: the L^2 norm enters

- Towards a contradiction, assume $\lim_{t\to T}\|\rho(t)-\rho_M\|_{L^2}=\infty$.
- A naive energy estimate:

$$\frac{d}{dt}\|\rho-\rho_M\|_{L^2}^2 = -\|\nabla\rho\|_{L^2}^2 + C\|\rho-\rho_M\|_{L^2}^4 + \rho_M^2.$$

So it takes at least $\sim 2^{-N}$ time for $\|\rho-\rho_M\|_{L^2}^2$ to grow from 2^N to 2^{N+1}.

- Let's decompose $\rho(x_1,x_2) := \bar\rho(x_2)+\tilde\rho(x_1,x_2)$, where $\bar\rho(x_2)$ is the average of ρ on each horizontal slice.
- Gagliardo-Nirenberg inequalities:

$$\|\nabla\bar\rho\|_{L^2}^2 \ge c(\Omega)\|\tilde\rho\|_{L^1}^{-4}\|\bar\rho-\rho_M\|_{L^2}^6 \ge c(\Omega)\rho_M^{-4}\|\bar\rho-\rho_M\|_{L^2}^{\mathbf{6}},$$

$$\|\nabla\tilde\rho\|_{L^2}^2 \ge c(\Omega)\|\tilde\rho\|_{L^1}^{-2}\|\tilde\rho\|_{L^2}^4 \ge c(\Omega)\rho_M^{-2}\|\tilde\rho\|_{L^2}^{\mathbf{4}},$$

So the culprit of blow-up must be $\tilde\rho$ and the energy estimate becomes

$$\frac{d}{dt}\|\rho-\rho_M\|_{L^2}^2 = -\|\nabla\tilde\rho\|_{L^2}^2 + C\|\tilde\rho\|_{L^2}^4 + C(\rho_M).$$

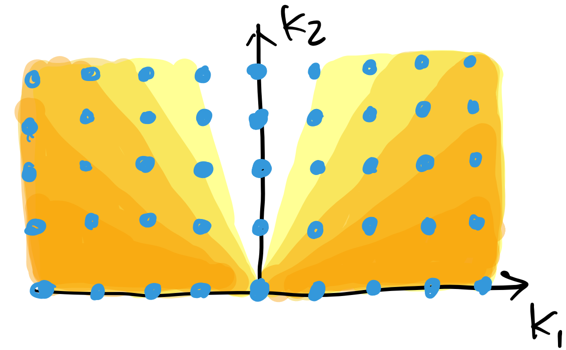
Act II. The potential energy enters

- Recall the potential energy is $E(t) := \int_\Omega \rho(x,t) x_2 \, dx$.
- A simple computation gives

$$E'(t) = \underbrace{-g\|\partial_{x_1}\rho\|^2_{\dot{H}_0^{-1}}}_{\text{Advection}} + \overbrace{\underbrace{\int_\Omega \partial_{x_2}\rho \, dx}_{\text{Diffusion}} - \underbrace{\int_\Omega \rho \partial_{x_2}(-\Delta_N)^{-1}(\rho - \rho_M)}_{\text{Keller-Segel nonlinearity}}}^{\text{ignore these for now...}}.$$

- Expanding ρ into basis $\{k_1 \in \mathbb{Z}, k_2 \in \mathbb{N} : e^{ik_1x_1}\cos(k_2x_2)\}$, we have

$$\|\partial_{x_1}\rho\|^2_{\dot{H}_0^{-1}} = \sum_{k \in \mathbb{Z}\times\mathbb{N}} \left| \frac{k_1}{|k|}\hat{\rho}(k) \right|^2.$$



- This is clearly bounded above by $\|\tilde{\rho}\|^2_{L^2}$.

  Question: Is it comparable to $\|\tilde{\rho}\|^2_{L^2}$, or much smaller?

Act III. Their first meeting

The two main characters are linked by the following key lemma:

Lemma (an "anisotropic" Nash's inequality)

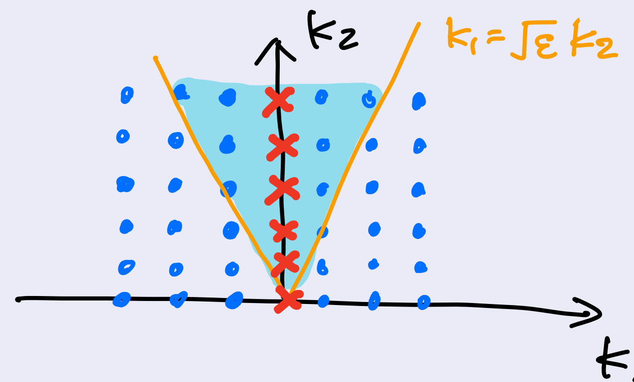
Assume that $\rho \in \dot{H}^1(\Omega)$. *If*

$$\|\partial_{x_1}\rho\|^2_{\dot{H}_0^{-1}} \le \epsilon \|\tilde{\rho}\|^2_{L^2}$$

for some $0 < \epsilon < 1$, *then*

$$\|\tilde{\rho}\|^2_{L^2} \le C\epsilon^{1/4}\|\tilde{\rho}\|_{L^1}\|\nabla\tilde{\rho}\|_{L^2}$$

for some universal constant C.



If $\|\rho - \rho_M\|^2_{L^2}$ monotone increases to $+\infty$:

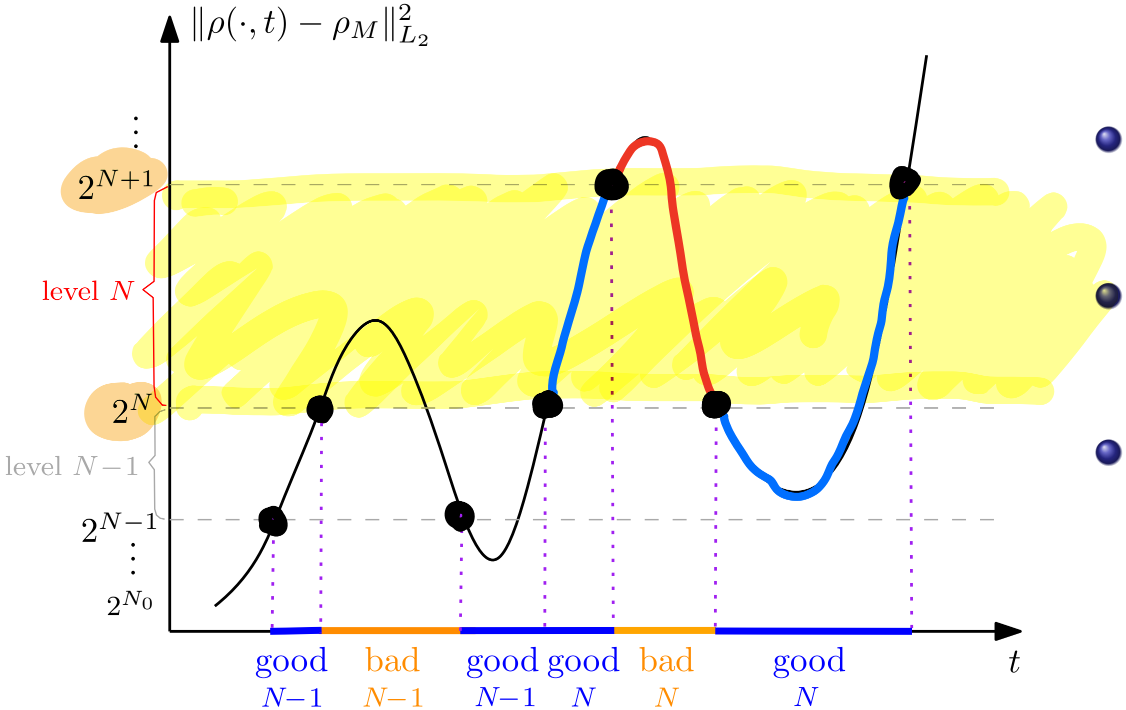
- In order for the L^2 norm to increase, one needs $\|\tilde{\rho}\|^2_{L^2} \ge c(\Omega)\|\tilde{\rho}\|_{L^1}\|\nabla\tilde{\rho}\|_{L^2}$.
- So the lemma gives $\|\partial_{x_1}\rho\|^2_{\dot{H}_0^{-1}} \ge c(\Omega)\|\tilde{\rho}\|^2_{L^2}$
- As $\|\rho - \rho_M\|^2_{L^2}$ increases from 2^N to 2^{N+1}, it takes time at least $\sim 2^{-N}$, and in this time interval we have $\|\partial_{x_1}\rho\|^2_{\dot{H}_0^{-1}} \sim \|\tilde{\rho}\|^2_{L^2} \sim 2^N$.
- So $E(t)$ has to drop by order 1 in this time interval!

Act IV. Dancing together

The above sketch captures the main idea, but it's sloppy in the following ways:

- What if the L^2 norm goes up-and-down before it blows up?
- What about the rest of the terms in the time derivative of $E(t)$?

In order to fix them, we introduce "good" and "bad" time intervals:



- On a good interval of level N: $E(t)$ decreases at least cgN^{-1}.
- On a bad interval of level N: $E(t)$ increases at most $2^{-N/3}$.
- Note that for every level N, the number of good interval exceeds the bad intervals by 1.

Since $\sum_N N^{-1}$ diverges, for any $g > 0$, the potential energy still becomes $-\infty$, a contradiction!

Open questions

- What about the full-domain case $\mathbb{R}^2$?

  (We know it's impossible to blow-up with a finite potential energy. But is it possible that the potential energy also goes to $-\infty$ as $t \to T$?)

- What about the parabolic-parabolic Keller-Segel equation?

- What about Keller-Segel coupled with Stokes, or Navier-Stokes?
- What about the long-time behavior?

  (We are not able to find a monotone-in-time Lyapunov functional.)

Thank you!