Global existence and boundedness for chemotaxis models with local sensing

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Joint work with :
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Outline

A chemotaxis system with density-mediated motility and nutrient consumption

Model for self-induced periodic stripe pattern formation:

$$\partial_t u = \Delta\big(u\gamma(v)\big) + \theta u f(n),$$
$$\partial_t v = D_v \Delta v - \beta v + \alpha u,$$
$$\partial_t n = D_n \Delta n - k_s \theta u f(n),$$

where

- u: cell density
- v: signal/chemical concentration
- n: nutrient level
- $(\theta, D_v, \alpha, D_n, k_s) \in (0, \infty)^5$, $\beta \geq 0$.

Liu *et al.* (2011)

A chemotaxis system with density-mediated motility and nutrient consumption

- The cell motion is mediated by the chemical and includes a chemical-dependent diffusion coefficient and a chemotactic bias:

$$\Delta\big(u\gamma(v)\big) = \mathrm{div}\big(\gamma(v)\nabla u + u\gamma'(v)\nabla v\big).$$

  Chemoattraction corresponds to $\gamma' < 0$ (and chemorepulsion to $\gamma' > 0$);
- The chemical is produced by the cells, spreads according to Fick's law, and is possibly degraded ($\beta \geq 0$):
- The nutrient is consumed by the cells and spreads according to Fick's law.

A chemotaxis system with density-mediated motility and nutrient consumption

- Cell motility $\gamma > 0$: almost a step function. Low motility for large values of v, large motility for small values of v, with a sharp transition;
- Consumption rate $f \geq 0$:

$$f(s) = \frac{s^2}{s^2 + K_n}, \qquad s \geq 0,$$

for some $K_n > 0$.

A chemotaxis system with density-mediated motility and nutrient consumption

Aim: well-posedness of and boundedness of solutions to

$$
\begin{aligned}
\partial_t u &= \Delta\big(u\gamma(v)\big) + \theta u f(n), \\
\partial_t v &= D_v \Delta v - \beta v + \alpha u, \\
\partial_t n &= \Delta n - k_s \theta u f(n),
\end{aligned}
$$

including its reduced version $f \equiv 0$

$$
\begin{aligned}
\partial_t u &= \Delta\big(u\gamma(v)\big), \\
\partial_t v &= D_v \Delta v - \beta v + \alpha u,
\end{aligned}
$$

Keller-Segel's model (1971)

$$\partial_t u = \mathrm{div}\big(D_2 \nabla u - D_1 \nabla v\big),$$
$$\partial_t v = D_v \Delta v + uS(v) - k(v)v.$$

- Let η be the ratio of effective body length (i.e. distance between receptors) to the size of the elementary step of an amobae.
- $D_2 = \gamma(v) > 0$, $D_1 = (\eta - 1)u\gamma'(v)$
- Both diffusion and chemotactic bias are mediated by v

$$\partial_t u = \mathrm{div}\big(\gamma(v)\nabla u - (\eta - 1)u\gamma'(v)\nabla v\big)$$
$$= \Delta(u\gamma(v)) - \eta \mathrm{div}\big(u \nabla \gamma(v)\big).$$

We recover the previous system for $\eta = 0$: local sensing.

Alternative derivation through local sensing
($\neq$ gradient sensing) Othmer & Stevens (1997)

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A chemotaxis system with density-mediated motility and nutrient consumption

After rescaling:

$$\begin{aligned} \partial_t u &= \Delta\big(u\gamma(v)\big) + uf(n), \qquad (t,x) \in (0,\infty) \times \Omega, \\ \tau\partial_t v &= \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty) \times \Omega, \\ \partial_t n &= \Delta n - uf(n), \qquad (t,x) \in (0,\infty) \times \Omega, \end{aligned}$$

supplemented with no-flux boundary conditions

$$\nabla\big(u\gamma(v)\big) \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = \nabla n \cdot \mathbf{n} = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega,$$

and non-negative initial conditions

$$(u,v,n)(0) = (u^{in}, v^{in}, n^{in}), \qquad x \in \Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^d$, $d \geq 1$.

Reduced model: $f \equiv 0$

$$\begin{aligned} \partial_t u &= \Delta\big(u\gamma(v)\big), && (t,x) \in (0,\infty)\times\Omega, \\ \tau\partial_t v &= \Delta v - \beta v + u, && (t,x) \in (0,\infty)\times\Omega, \end{aligned}$$

supplemented with no-flux boundary conditions and non-negative initial conditions.

- Quasilinear equation for u;
- Degenerate diffusion when $\gamma(0) = 0$ or $\gamma(\infty) = 0$;
- Comparison with the classical Keller-Segel model

$$\mathrm{div}\big(\nabla u - u\nabla v\big) = \mathrm{div}\big(u\nabla\big[\ln u - v\big]\big) = \mathrm{div}\big(e^{v}\nabla\big[ue^{-v}\big]\big)$$

Some properties

- Non-negativity: $u \geq 0$, $v \geq 0$;
- Conservation of matter: $\|u(t)\|_1 = \|u^{in}\|_1$;
- Regularity for v from linear parabolic/elliptic theory with right-hand side in $L^\infty((0,T), L^1(\Omega))$;
- Duality estimate: control on $\|u\|_{(H^1)'}$ and

$$\int_0^t \int_\Omega u^2 \gamma(v) \, \mathrm{d}x\mathrm{d}s.$$

Trouble with the possible degeneracy of $\gamma(v)$ ($\gamma(s) = e^{-s}$ for instance).

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Outline

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Chemotaxis model with local sensing: $\gamma(v) = e^{-v}$

$$
\begin{aligned}
\partial_t u &= \Delta\big(ue^{-v}\big), \qquad (t,x) \in (0,\infty) \times \Omega, \\
\tau\partial_t v &= \Delta v + u - v, \qquad (t,x) \in (0,\infty) \times \Omega,
\end{aligned}
$$

supplemented with no-flux boundary conditions and non-negative initial conditions.

$$
\begin{aligned}
\operatorname{div}\big(ue^{-v}\nabla\big[\ln u - v\big]\big) &= \Delta\big(ue^{-v}\big), \\
\operatorname{div}\big(\nabla u - u\nabla v\big) = \operatorname{div}\big(u\nabla\big[\ln u - v\big]\big) &= \operatorname{div}\big(e^{v}\nabla\big[ue^{-v}\big]\big)
\end{aligned}
$$

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Chemotaxis model with local sensing: $\gamma(v) = e^{-v}$

Conservation of matter: $\|u(t)\|_1 = \|u^{in}\|_1$

A Liapunov functional:

$$\frac{d}{dt}\mathcal{L}(u,v) + \int_\Omega u e^{-v} \left|\nabla(\ln u - v)\right|^2 \,\mathrm{d}x + \tau\|\partial_t v\|_2^2 = 0,$$

$$\mathcal{L}(u,v) = \int_\Omega \left(u \ln u - u - uv\right) \,\mathrm{d}x + \frac{\|\nabla v\|_2^2}{2} + \frac{\beta\|v\|_2^2}{2}.$$

The functional $\mathcal{L}(u,v)$ features positive and negative terms: boundedness from below ?

Chemotaxis model with local sensing: $\gamma(v) = e^{-v}$

A duality estimate:

$$-\Delta U = u - u_\Omega \ \text{ in } \ \Omega, \quad \nabla U \cdot \mathbf{n} = 0 \ \text{ on } \ \Omega, \quad U_\Omega = 0,$$

where

$$z_\Omega = \frac{1}{|\Omega|} \int_\Omega z(x)\, \mathrm{d}x, \qquad z \in L^1(\Omega).$$

$$\frac{d}{dt} \|\nabla U\|_2^2 + 2 \int_\Omega u^2 e^{-v}\, \mathrm{d}x = 2 u_\Omega^{in} \int_\Omega u e^{-v}\, \mathrm{d}x$$

$$\leq \int_\Omega u^2 e^{-v}\, \mathrm{d}x + \left(u_\Omega^{in}\right)^2 \int_\Omega e^{-v}\, \mathrm{d}x,$$

$$\frac{d}{dt} \|\nabla U\|_2^2 + \int_\Omega u^2 e^{-v}\, \mathrm{d}x \leq |\Omega| \left(u_\Omega^{in}\right)^2$$

Chemotaxis model with local sensing: $\gamma(v) = e^{-v}$

$$-\Delta U = u - u_\Omega \text{ in } \Omega, \quad \nabla U \cdot \mathbf{n} = 0 \text{ on } \Omega, \quad U_\Omega = 0.$$

$$\begin{aligned} -\int_\Omega uv \,\mathrm{d}x &= \int_\Omega v(\Delta U - u_\Omega^{in}) \,\mathrm{d}x \\ &= -\int_\Omega \nabla v \cdot \nabla U \,\mathrm{d}x - |\Omega| u_\Omega^{in} v_\Omega \\ &\geq -\frac{\varepsilon}{2}\|\nabla v\|_2^2 - \frac{1}{2\varepsilon}\|\nabla U\|_2^2 - |\Omega| u_\Omega^{in} v_\Omega. \end{aligned}$$

$\longrightarrow$ $\mathcal{L}(u, v)$ is bounded from below on finite time intervals $\longrightarrow$ global existence of weak solutions for $d \geq 1$ $\neq$ classical Keller-Segel system

Burger, L & Trescases (2021)

Chemotaxis model with local sensing: $\gamma(v) = e^{-v}$

$\tau \geq 0$

- Existence and uniqueness of a global classical solution, $d \geq 1$

  Fujie & Jiang (2020,2021), Jiang & L (2021), Jiang, L & Zhang (2022)

- $d = 2$. Bounded solutions if $\|u_0\|_1 < 4\pi$ (or $\|u_0\|_1 < 8\pi$ in a ball with radial symmetry)

  Fujie & Jiang (2020,2021), Jin & Wang (2020)

- $d = 2$. Unbounded solutions for some initial data satisfying $\|u_0\|_1 \in (4\pi, \infty) \setminus 4\pi\mathbb{N}$

  Fujie & Jiang (2020,2021,2022), Jin & Wang (2020)

- $d \geq 3$. Unbounded solutions whatever the value of $\|u_0\|_1$

  Fujie & Senba (2022)

Outline

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Chemotaxis model with local sensing: $\tau = 0$

$$\partial_t u = \Delta\big(u\gamma(v)\big), \qquad (t,x) \in (0,\infty)\times\Omega,$$
$$0 = \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty)\times\Omega,$$

supplemented with no-flux boundary conditions and non-negative initial conditions

- Existence and uniqueness of a global classical solution when $\gamma \in C^3((0,\infty))$, $\gamma > 0$ in $(0,\infty)$

  Jiang & L (2021,2024)

- Previous results when $\gamma' \leq 0$

  Ahn & Yoon (2019), Fujie & Jiang (2020, 2021), Wang (2021)

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Chemotaxis model with local sensing: $\tau = 0$, $\gamma(v) = (1+v)^{-k}$, $k > 0$

$$\partial_t u = \Delta\big(u(1+v)^{-k}\big), \qquad (t,x) \in (0,\infty) \times \Omega,$$
$$0 = \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty) \times \Omega,$$

supplemented with no-flux boundary conditions and non-negative initial conditions

Bounded solutions: $k < d/(d-2)$ Jiang & L (2021) (improving Ahn & Yoon (2019), Fujie & Jiang (2021), Jiang (2022))

Chemotaxis model with local sensing: $\tau = 0$, $\gamma(v) = (1+v)^{-k}$, $k > 0$, $\beta = 1$

$$\partial_t u = \Delta\big(u(1+v)^{-k}\big), \qquad (t,x) \in (0,\infty) \times \Omega,$$
$$0 = \Delta v - v + u, \qquad (t,x) \in (0,\infty) \times \Omega,$$

Duality technique: Fujie & Jiang (2020)

$$\begin{aligned}(\mathrm{id} - \Delta)\partial_t v &= \partial_t u = \Delta\big(u(1+v)^{-k}\big) \\ &= -(\mathrm{id}-\Delta)\big(u(1+v)^{-k}\big) + u(1+v)^{-k} \\ \partial_t v + u(1+v)^{-k} &= (\mathrm{id}-\Delta)^{-1}\big(u(1+v)^{-k}\big)\big) \\ &\le (\mathrm{id}-\Delta)^{-1} u = v\end{aligned}$$

$\longrightarrow$ L^∞-bound on v in $(0,T) \times \Omega$

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Chemotaxis model with local sensing: $\tau = 0$, $\gamma(v) = (1+v)^{-k}$, $k > 0$, $\beta = 1$

- Evolution equation for v:

$$\partial_t v + u(1+v)^{-k} = (\mathrm{id} - \Delta)^{-1}\left(u(1+v)^{-k}\right)$$
$$\partial_t v - (1+v)^{-k}\Delta v + v(1+v)^{-k} = (\mathrm{id} - \Delta)^{-1}\left(u(1+v)^{-k}\right)$$

- L^∞-bound on v in $(0,T) \times \Omega$
- C^α-bound on v in $(0,T) \times \Omega$ via local energy estimates

  Ladyzhenskaya, Solonnikov & Ural'tseva (1968)
- $L^\infty((0,T), W^{1,p}(\Omega))$-bounds on v via regularity for non-autonomous linear parabolic equations of the form $\partial_t v - a(t,x)\Delta v = f$ Amann (1990-1995)
- L^∞-bounds on u by Moser's iteration technique

Chemotaxis model with local sensing: $\tau = 0$, $\gamma(v) = (1+v)^{-k}$, $k > 0$, $\beta = 1$

Boundedness: Evolution equation for v:

$$\partial_t v - (1+v)^{-k}\Delta v + v(1+v)^{-k} = (\mathrm{id} - \Delta)^{-1}\left(u(1+v)^{-k}\right)$$

$$\begin{aligned} u(1+v)^{-k} &= (1+v)^{-k}\left(v - \Delta v\right) \\ &= v(1+v)^{-k} - \mathrm{div}\left((1+v)^{-k}\nabla v\right) + \nabla(1+v)^{-k}\cdot\nabla v \\ &\le v(1+v)^{-k} - \frac{(1+v)^{1-k}}{1-k} + \left(\mathrm{id} - \Delta\right)\left[\frac{(1+v)^{1-k}}{1-k}\right] \\ &\le C + \left(\mathrm{id} - \Delta\right)\left[\frac{(1+v)^{1-k}}{1-k}\right] \end{aligned}$$

Chemotaxis model with local sensing: $\tau = 0$, $\gamma(v) = (1+v)^{-k}$, $k > 0$, $\beta = 1$

Boundedness: Evolution equation for v:

$$\partial_t v - (1+v)^{-k}\Delta v + v(1+v)^{-k} = (\mathrm{id} - \Delta)^{-1}\left(u(1+v)^{-k}\right),$$

$$\left(\mathrm{id} - \Delta\right)^{-1}\left[u(1+v)^{-k}\right] \leq C + \frac{(1+v)^{1-k}}{1-k}$$

$\longrightarrow$ sublinear right-hand side

$$\partial_t v - (1+v)^{-k}\Delta v + v(1+v)^{-k} \leq C + \frac{(1+v)^{1-k}}{1-k}$$

Chemotaxis model with local sensing: $\tau = 0$

$$\begin{aligned} \partial_t u &= \Delta\big(u\gamma(v)\big), \qquad (t,x) \in (0,\infty)\times\Omega, \\ 0 &= \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty)\times\Omega, \end{aligned}$$

supplemented with no-flux boundary conditions and non-negative initial conditions

Bounded solutions

- $d = 2$: for all $\chi > 0$, $\lim_{s\to\infty} e^{\chi s} = \infty$. Fujie & Jiang (2021)
- $d = 2$: $\lim_{s\to\infty} e^{\alpha s} = \infty$ and $\|u_0\|_1 < \frac{4\pi}{\chi}$. Fujie & Jiang (2021)

Chemotaxis model with local sensing: $\tau = 0$

$$\begin{aligned} \partial_t u &= \Delta\big(u\gamma(v)\big), \qquad (t,x) \in (0,\infty)\times\Omega, \\ 0 &= \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty)\times\Omega, \end{aligned}$$

supplemented with no-flux boundary conditions and non-negative initial conditions

Bounded solutions

- $d \geq 3$: there are $k \geq l \geq 0$ such that

$$\liminf_{s\to\infty} s^k\gamma(s) > 0, \quad \limsup_{s\to\infty} s^l\gamma(s) < \infty$$

with $k < \frac{d}{d-2}$ and $k - l < \frac{2}{d-2}$ Jiang & L (2021)

- $d \geq 1$: if either $\gamma' \geq 0$ or $\limsup_{s\to\infty} \gamma(s) = \infty$ Jiang & L (2024)

Chemotaxis model with local sensing: $\tau = 0$

- A Liapunov functional with $\gamma(s) = s^{-k}$ for $k \in (0,1)$: no pattern formation Ahn & Yoon (2019)
- A Liapunov functional when $s \mapsto s\gamma(s)$ is non-decreasing: no pattern formation Jiang & L (2024)

Outline

A chemotaxis system with density-mediated motility and nutrient consumption

$$
\begin{aligned}
\partial_t u &= \Delta\big(u\gamma(v)\big) + uf(n), && (t,x) \in (0,\infty) \times \Omega, \\
\tau\partial_t v &= \Delta v - \beta v + u, && (t,x) \in (0,\infty) \times \Omega, \\
\partial_t n &= \Delta n - uf(n), && (t,x) \in (0,\infty) \times \Omega,
\end{aligned}
$$

supplemented with no-flux boundary conditions and non-negative initial conditions, where

- $\gamma \in C^3((0,\infty))$, $\gamma > 0$, $\gamma' < 0$ in $(0,\infty)$ and $\lim_{s\to\infty} \gamma(s) = 0$
- $f \in C^1([0,\infty))$, $f(0) = 0$, $f \geq 0$ in $(0,\infty)$ $[f \equiv 0]$

Global existence of a classical solution

Fujie & Senba (2022), Jiang, L & Zhang (2022), Lyu & Wang (2022)

A chemotaxis system with density-mediated motility and nutrient consumption

- Global existence of a classical solution
- $d = 2$: bounded if $\lim_{s\to\infty} e^{\chi s} = \infty$ for all $\chi > 0$ or if $\lim_{s\to\infty} e^{\alpha s} = \infty$ and $\|u_0\|_1 < \frac{4\pi}{\chi}$
- $d \geq 3$: bounded if there are $k \geq l \geq 0$ such that

$$\liminf_{s\to\infty} s^k \gamma(s) > 0, \quad \limsup_{s\to\infty} s^l \gamma(s) < \infty$$

with $k < \frac{d}{d-2}$ and $k - l < \frac{2}{d-2}$

Fujie & Senba (2022), Jiang, L & Zhang (2022), Lyu & Wang (2022)

Chemotaxis model with local sensing: $\tau > 0$

$$\begin{aligned} \partial_t u &= \Delta\big(u\gamma(v)\big), && (t,x) \in (0,\infty) \times \Omega, \\ \tau\partial_t v &= \Delta v - \beta v + u, && (t,x) \in (0,\infty) \times \Omega, \end{aligned}$$

supplemented with no-flux boundary conditions and non-negative initial conditions

Global weak solutions:

- $d \geq 2$: $\gamma \in W^{1,\infty}(0,\infty)$, $\frac{1}{\gamma} \in L^\infty(0,\infty)$ (bounded classical solutions if $d = 2$) Tao & Winkler (2017)
- $\gamma(s) = (1+s)^{-k}$: $k > 0$ when $d = 1$, $k \in (0,2)$ when $d = 2$, $k \in (0, 4/3)$ when $d = 3$ Desvillettes, Kim, Trescases & Yoon (2019)

Chemotaxis model with local sensing: $\tau > 0$, $\gamma \in L^\infty(0,\infty)$

$$\partial_t u = \Delta\big(u\gamma(v)\big), \qquad (t,x) \in (0,\infty) \times \Omega,$$
$$\tau\partial_t v = \Delta v - \beta v + u, \qquad (t,x) \in (0,\infty) \times \Omega,$$

supplemented with no-flux boundary conditions and non-negative initial conditions

Auxiliary function: $-\Delta w + \beta w = u$ in Ω, $\nabla w \cdot \mathbf{n} = 0$ on $\partial\Omega$
Evolution equation for w:

$$\begin{aligned}\partial_t w + u\gamma(v) &= \big(\beta\mathrm{id} - \Delta\big)^{-1}\big[u\gamma(v)\big] \\ &\le \|\gamma\|_\infty \big(\beta\mathrm{id} - \Delta\big)^{-1}[u] = \|\gamma\|_\infty w\end{aligned}$$

$\longrightarrow$ L^∞-bound on w in $(0,T) \times \Omega$ (but not on v!)

Chemotaxis model with local sensing: $\tau > 0$, $\gamma \in L^\infty(0,\infty)$

Introducing $z := \left(\beta \mathrm{id} - \Delta\right)^{-1}[v]$,

$$\tau \partial_t z = \Delta z - \beta z + w, \qquad (t,x) \in (0,\infty) \times \Omega,$$

supplemented with no-flux boundary conditions and non-negative initial condition and the previously derived L^∞-bound on w, along with parabolic regularity, gives a bound on z in $L^q((0,T), W^{2,q}(\Omega))$ for any $q \in (1,\infty)$; that is,

$$v \in L^q((0,T) \times \Omega).$$

$\longrightarrow$ weak solutions if $\inf_{s \geq 0} \left\{(1+s)^k \gamma(s)\right\} > 0$ for some $k > 0$ Desvillettes, L, Trescases & Winkler (2023)

Chemotaxis model with local sensing: $\tau > 0$, $\gamma \in L^\infty(0,\infty)$

Boundedness of v: one-sided comparison with w.
$w = \big(\beta \mathrm{id} - \Delta\big)^{-1}[u]$ and v solve

$$\partial_t w + u\gamma(v) = \big(\beta \mathrm{id} - \Delta\big)^{-1}\big[u\gamma(v)\big],$$
$$\mathcal{L}v = \partial_t v - \Delta v + \beta v = u.$$

$$\begin{aligned}\mathcal{L}v &= -\Delta w + \beta w = \mathcal{L}w - \tau\partial_t w \\ &= \mathcal{L}w + \tau u\gamma(v) - \tau\big(\beta \mathrm{id} - \Delta\big)^{-1}\big[u\gamma(v)\big] \\ &\leq \mathcal{L}w + \tau\|\gamma\|_\infty u = \mathcal{L}[w + \tau\|\gamma\|_\infty v]\end{aligned}$$

$$\mathcal{L}[(1-\tau\|\gamma\|_\infty)v - w] \leq 0$$

If $\tau\|\gamma\|_\infty < 1$ then $v \leq \dfrac{w}{1-\tau\|\gamma\|_\infty} \longrightarrow L^\infty$-bound for v

Chemotaxis model with local sensing: $\tau > 0$, $\gamma \in L^\infty(0,\infty)$

Boundedness of v: one-sided comparison with w.
If $\tau\|\gamma\|_\infty < 1$ then

$$v \leq \frac{w}{1-\tau\|\gamma\|_\infty}$$

$\longrightarrow$ L^∞-bound for v on $(0, T)$

- Global classical solution if $\gamma' \leq 0$ Fujie & Senba (2022), Jiang, L & Zhang (2022)
- Global weak solution if $\tau\|\gamma\|_\infty < 1$ Li & Jiang (2021)

- Boundedness of (u, v): $w \leq Kv$
- Unbounded γ: in progress

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