Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems

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Particle systems

Consider the evolution of a vector $\mathbf{X}_t$ of N particles/agents indexed by $\omega = 1, 2, \cdots, N$:

$$\mathbf{X}_t = (X_t^1, X_t^2, \cdots, X_t^N) \rightsquigarrow (X_t(1), X_t(2), \cdots X_t(N)) = \big(X_t(\omega)\big)_{\omega=1,2,\cdots,N}$$

$\mathbf{X}$ solves a system of ODEs

$$\boxed{\frac{d}{dt}X_t(\omega) = F(X_t(\omega), \mathbf{X}_t, U_t(\omega))} \quad t \in (0,T),\ X_{t=0}(\omega) = X_0(\omega), \qquad \omega \in \Omega^N = \{1, 2, \cdots, N\},$$

driven by the vector field

$$F = F(x, \mathbf{x}, u) : \mathbb{R}^d \times (\mathbb{R}^d)^N \times \mathbb{U} \to \mathbb{R}^d$$

depending on

- the position $x = X_t(\omega)$ of each particle,
- the vector $\mathbf{x} = \mathbf{X}_t$ of the distribution of the particles
- the control variable $u = U_t(\omega)$ for each particle, varying in the set of controls $\mathbb{U}$.

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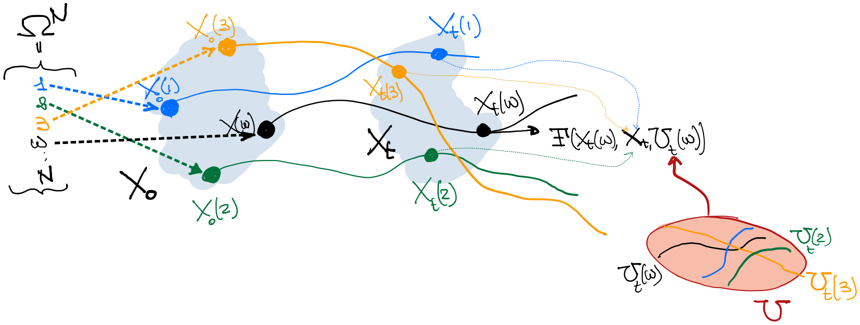

Examples

First order drift-interaction systems:

$$\boxed{F(x,\mathbf{x},u) = f(x,u) + \frac{1}{N}\sum_{\omega=1}^{N} g(x-\mathbf{x}(\omega))}$$

Linear controls: $f(x,u) := f(x) + Lu$.

Second order systems: $x \rightsquigarrow (x,v) \in \mathbb{R}^{2d}$, $\mathbf{x} \rightsquigarrow (\mathbf{x},\mathbf{v}) \in (\mathbb{R}^{2d})^N$

$F(x,v,\mathbf{x},\mathbf{v},u) = \left(v, f(x) + \frac{1}{N}\sum_{\omega=1}^{N} g(x-\mathbf{x}(\omega)) + u\right)$ corresponding to the system for the particles $(X_t(\omega), V_t(\omega))$ in the phase space

$$\begin{cases} \dfrac{d}{dt}X_t(\omega) = V_t(\omega), \\ \dfrac{d}{dt}V_t(\omega) = f(X_t(\omega)) + \dfrac{1}{N}\displaystyle\sum_{\theta=1}^{N} g(X_t(\omega) - X_t(\theta)) + U_t(\omega) \end{cases}$$

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Cost

We introduce the running and final cost functions

$$\mathbf{C} = \mathbf{C}(x, \mathbf{x}, u) : \mathbb{R}^d \times (\mathbb{R}^{\mathbf{d}})^{\mathbf{N}} \times \mathbb{U} \to \mathbb{R}, \qquad \mathbf{C}_T = \mathbf{C}_T(x, \mathbf{x}) : \mathbb{R}^d \times (\mathbb{R}^{\mathbf{d}})^{\mathbf{N}} \to \mathbb{R}$$

and we want to **minimize the total cost**

$$\boxed{\mathcal{J}(\mathbf{X}, \mathbf{U}) := \frac{1}{N} \sum_{\omega=1}^{N} \int_0^T \mathbf{C}(X_t(\omega), \mathbf{X}_t, U_t(\omega))\, dt + \frac{1}{N} \sum_{\omega=1}^{N} \mathbf{C}(X_T(\omega), \mathbf{X}_T).}$$

among all the **admissible pairs** $(\mathbf{X}, \mathbf{U}) \in \mathcal{A}^N(\mathbf{X}_0)$ solving

$$\boxed{\frac{d}{dt} X_t(\omega) = F(X_t(\omega), \mathbf{X}_t, U_t(\omega)), \quad t \in (0, T), \quad X_{t=0}(\omega) = X_0(\omega).}$$

Value function: $\mathcal{V}^N(\mathbf{X}_0) := \inf \left\{ \mathcal{J}(\mathbf{X}, \mathbf{U}) : (\mathbf{X}, \mathbf{U}) \in \mathcal{A}^N(\mathbf{X}_0) \right\}$

Example: $\mathbb{U} := \mathbb{R}^d, \quad \mathbf{C}(x, \mathbf{x}, u) := V(x) + \frac{1}{N} \sum_{\omega=1}^{N} W(x - \mathbf{x}(\omega)) + \psi(u)$

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Invariance and measure interpretation

Every **permutation** $\sigma \in \mathrm{Sym}(N)$ of the set of indices $\Omega^N = \{1, 2, \cdots, N\}$ acts on a vector $\mathbf{x} \in (\mathbb{R}^d)^N$ just by

$$\textbf{a permutation of its components:} \qquad \sigma_\sharp \mathbf{x} = (\mathbf{x}(\sigma(1)), \mathbf{x}(\sigma(2)), \cdots, \mathbf{x}(\sigma(N)))$$

Invariance: $\boxed{F(x, \sigma_\sharp \mathbf{x}, u) = F(x, \mathbf{x}, u), \quad \mathbf{C}(x, \sigma_\sharp \mathbf{x}, u) = \mathbf{C}(x, \mathbf{x}, u), \quad \mathbf{C}_T(x, \sigma_\sharp \mathbf{x}) = \mathbf{C}_T(x, \mathbf{x}).}$

We may interpret a particle system as a **dynamical system in the quotient space** $(\mathbb{R}^d)^N/\sim$,

$$\boxed{x \sim y \Leftrightarrow y = \sigma_\sharp x \quad \text{for some } \sigma \in \mathrm{Sym}(N)}$$

Equivalently, we may associate to x the discrete measure

$$\boxed{\mu[x] := \frac{1}{N} \sum_{\omega=1}^{N} \delta_{x(\omega)} = x_\sharp \mathbb{P}^N} \qquad \mathbb{P}^N = \frac{1}{N} \sum_{\omega=1}^{N} \delta_\omega \text{ uniform discrete measure in } \Omega^N := \{1, \cdots, N\}.$$

$$F(x, x, u) \rightsquigarrow F(x, \mu[x], u), \quad \mathbf{C}(x, x, u) \rightsquigarrow \mathbf{C}(x, \mu[x], u), \quad \mathbf{C}_T(x, x, u) \rightsquigarrow \mathbf{C}_T(x, \mu[x])$$

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From discrete to continuous

We suppose that **we can extend by continuity** the functions F, $\mathbf{C}$, $\mathbf{C}_T$ to $\mathscr{P}_p(\mathbb{R}^d)$:

$$F(x, \mathbf{x}, u) = f(x, u) + \frac{1}{N}\sum_{\omega=1}^{N} g(x - \mathbf{x}(\omega)) = f(x, u) + \int g(x-y)\, d\mu[\mathbf{x}](y)$$

$$\Downarrow$$

$$F(x, \mu, u) = f(x, u) + \int g(x-y)\, d\mu(y)$$

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$$\mu_n \to \mu \quad \Rightarrow \quad F(x, \mu_n, u) \to F(x, \mu, u), \quad \mathbf{C}(x, \mu_n, u) \to \mathbf{C}(x, \mu, u).$$

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Main problems

- Well posedness of the finite particle control problem.
- How to characterize and solve the limit problem: **mean field optimal control** and its Lagrangian, Eulerian, and Kantorovich formulations.
- Can we pass to the limit in the **value** function as $N \to \infty$
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References

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MFOC: **Fornasier-Solombrino** (Mean-field optimal control), **Bongini-Fornasier, Fornasier-Piccoli-Rossi, Caponigro-Fornasier-Piccoli-Trelat, Albi-Choi-Fornasier, Fornasier-Piccoli-Rossi**, ...

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McKean–Vlasov optimal control: **Lacker** '17 (Limit theory for controlled McKean-Vlasov dynamics), **Djete-Possamaï-Tan** '20 (McKean–Vlasov optimal control: The dynamic programming principle)

Models: **Carrillo-Choi-Hauray** (The derivation of swarming models: mean-field limit and Wasserstein distances),

Piccoli-Rossi-Trélat (Control to flocking of the kinetic Cucker-Smale model)

Setting (1)

$\mathbb{S} := (\mathbb{U}, F, \mathbf{C}, \mathbf{C}_T)$.

$\mathbb{U}$ is a subset of some Banach space. Natural metric w.r.t. (x, μ) in $\mathbb{R}^d \times \mathscr{P}_p(\mathbb{R}^d)$:

$$D\big((x_1, \mu_1), (x_2, \mu_2)\big) := \Big(|x_1 - x_2|^p + W_p^p(\mu_1, \mu_2)\Big)^{1/p}, \quad p \in [1, +\infty) \text{ fixed.}$$

- **Continuity:** F, C, C_T are continuous in $\mathbb{R}^d \times \mathscr{P}_p(\mathbb{R}^d) \times \mathbb{U}$ with p-growth.
- (Uniqueness) For every $u \in \mathbb{U}$ the map $(x, \mu) \mapsto F(x, \mu, u)$, are D-**Lipschitz with uniform Lipschitz constant** (it would be possible to relax this condition assuming dissipativity)
- **Coercivity:** $\mathbb{U}$ is compact (or, in the additive case, $C(x, \mu, u) = \tilde{C}(x, \mu) + \psi(u)$, ψ superlinear with compact sublevels).

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Setting (2): convexity and convex relaxation

Convexity: $\mathbb{U}$ is convex, F is affine w.r.t. u, C is convex w.r.t. u.

Convex relaxation: $\mathbb{S} := (\mathbb{U}, F, C, C_T) \quad \rightsquigarrow \quad \mathbb{S}_R := (\mathscr{U}, \mathscr{F}, \mathscr{C}, \mathscr{C}_T)$:

$$\mathscr{U} := \mathscr{P}(\mathbb{U}), \quad \mathscr{F}(x, \mu, \sigma) := \int_{\mathbb{U}} F(x, \mu, u)\, d\sigma(u), \quad \mathscr{C}(x, \mu, \sigma) := \int_{\mathbb{U}} C(x, \mu, u)\, d\sigma(u), \quad \mathscr{C}_T := C_T.$$

$\mathbb{S}_R := (\mathscr{P}(\mathbb{U}), \mathscr{F}, \mathscr{C}, \mathscr{C}_T)$ is a convex system extending $\mathbb{S}$:

$\{\delta_u : u \in \mathbb{U}\} \subset \mathscr{U}$,

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Optimal control for finite particles: convex setting

Recall:

$$\mathcal{J}(\mathbf{X}, \mathbf{u}) := \frac{1}{N}\sum_{\omega=1}^{N}\int_0^T \mathbf{C}(X_t(\omega), \mathbf{X}_t, u_t(\omega))\,dt + \frac{1}{N}\sum_{\omega=1}^{N}\mathbf{C}(X_T(\omega), \mathbf{X}_T).$$

among all the admissible pairs $(\mathbf{X}, \mathbf{u}) \in \mathcal{A}^N(\mathbf{X}_0)$ solving

$$\frac{d}{dt}X_t(\omega) = F(X_t(\omega), \mathbf{X}_t, u_t(\omega)), \quad t \in (0, T), \quad X_{t=0}(\omega) = X_0(\omega).$$

Value function:

$$\mathcal{V}^N(\mathbf{X}_0) := \inf\left\{\mathcal{J}(\mathbf{X}, \mathbf{u}) : (\mathbf{X}, \mathbf{u}) \in \mathcal{A}^N(\mathbf{X}_0)\right\}$$

Theorem

*In the **convex** setting the infimum is attained: there exists an optimal control $\mathbf{u}$ and a corresponding optimal trajectory $\mathbf{X}$ such that*

$$(\mathbf{X}, \mathbf{u}) \in \mathcal{A}^N(X_0), \quad \mathcal{J}(\mathbf{X}, \mathbf{u}) = \mathcal{V}^N(X_0)$$

Chattering and convex relaxation

In the relaxed system $\mathbb{S}_R = (\mathscr{U}, \mathscr{F}, \mathscr{C}, \mathscr{C}_T)$ the controls $\boldsymbol{\sigma}_t(\omega) = \boldsymbol{\sigma}_{t,\omega}$ are probability measures in $\mathscr{U} = \mathscr{P}(\mathbb{U})$

$$\mathcal{J}_R(\mathbf{X}, \boldsymbol{\sigma}) := \frac{1}{N}\sum_{\omega=1}^{N} \int_0^T \boxed{\int_{\mathbb{U}} C(X_t(\omega), \mathbf{X}_t, u)\, d\boldsymbol{\sigma}_{t,\omega}(u)}\, dt + \frac{1}{N}\sum_{\omega=1}^{N} C(X_T(\omega), \mathbf{X}_T).$$

The admissible pairs $(\mathbf{X}, \boldsymbol{\sigma}) \in \mathcal{A}_R^N(\mathbf{X}_0)$ solve

$$\frac{d}{dt}X_t(\omega) = \boxed{\int_{\mathbb{U}} F(X_t(\omega), \mathbf{X}_t, u)\, d\sigma_{t,\omega}(u)}, \quad t \in (0,T), \quad X_{t=0}(\omega) = X_0(\omega).$$

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Theorem

The minimum of the relaxed problem coincides with the Lagrangian value function:

$$\mathcal{V}^N(X_0) = \mathcal{V}_R^N(X_0) = \min\left\{\mathcal{J}_R(\mathbf{X}, \boldsymbol{\sigma}) : (\mathbf{X}, \boldsymbol{\sigma}) \in \mathcal{A}_R^N(X_0)\right\}$$

Mean field control: Lagrangian formulation

Particles are parametrized by an infinite $\boxed{\text{probability parameter space } (\Omega, \mathfrak{B}, \mathbb{P})}$:

$$\{1, 2, \cdots, N\} \rightsquigarrow \Omega, \qquad \frac{1}{N}\sum_{\omega} \delta_\omega \rightsquigarrow \mathbb{P}$$

$(\Omega, \mathfrak{B})$ is a **standard Borel space** (i.e. $\mathfrak{B}$ is the Borel σ-algebra induced by a Polish topology on Ω) and $\mathbb{P}$ **is diffuse** (every point has 0 mass). Canonical example: $\Omega = [0, 1]$, $\mathbb{P} = \mathcal{L}^1$

Trajectories: $X \in L^p(\Omega; W^{1,p}(0, T; \mathbb{R}^d))$, $X_t \in L^p(\Omega; \mathbb{R}^d)$, $\mu_t = (X_t)_\sharp \mathbb{P}$ is the law of X_t.

$$\boxed{\frac{d}{dt} X_t(\omega) = F(X_t(\omega), \mu_t, U_t(\omega))} \quad \mu_t = (X_t)_\sharp \mathbb{P}, \quad X_{t=0} = X_0 \quad \text{in } L^p(\Omega; \mathbb{R}^d).$$

$U : (0, T) \times \Omega \to \mathbb{U}$ is a $\mathcal{L}^1 \times \mathbb{P}$-measurable map, $(X, U) \in \mathcal{A}_L(X_0)$.

$$\boxed{\mathcal{J}(X, U) := \int_\Omega \int_0^T C(X_t(\omega), \mu_t, U_t(\omega))\, dt\, d\mathbb{P}(\omega) + \int_\Omega C_T(X_T(\omega), \mu_T)\, d\mathbb{P}(\omega)}$$

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Recevering the N-particle system

N-particle system:

$\mathfrak{B}^N \subset \mathfrak{B}$ is a finite algebra generated by a **finite partition** $\mathcal{B}^N = (B^1, \cdots, B^N)$ of Ω with $\mathbb{P}(B^i) = 1/N$.

$\mathbf{X}_t$ and $\mathbf{U}_t$ are $\mathfrak{B}^N$-measurable, i.e. **constant on each set** B^i **of the partition**.

$X_t(\omega)$, $U_t(\omega)$ as $\omega \in B^i$, represents the i-th particle driven by its control.

Convex relaxation of Lagrangian problem has the same value function

Starting from $\mathbb{S} = (\mathbb{U}, \mathrm{F}, \mathbf{C}, \mathbf{C}_{\mathrm{T}})$ construct $\mathbb{S}_{\mathrm{R}} = (\mathscr{U}, \mathscr{F}, \mathscr{C}, \mathscr{C}_{\mathrm{T}})$ and the corresponding $\mathcal{A}_{\mathbf{RL}}(X_0)$, $\mathcal{J}_{\mathbf{RL}}(\mathbf{X}, \boldsymbol{\sigma})$ associated to the measure-valued controls $\boldsymbol{\sigma} : (0, T) \times \Omega \to \mathscr{P}(\mathbb{U})$.

$\mathbb{S}_{\mathrm{R}} = (\mathscr{U}, \mathscr{F}, \mathscr{C}, \mathscr{C}_{\mathrm{T}})$ is a **convex** mean field optimal control problem.

Theorem (CLOS)

For every $X_0 \in L^p(\Omega; \mathbb{R}^d)$

$$\boxed{V_{\mathbf{L}}(X_0) = V_{\mathbf{RL}}(X_0)}$$

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Remark

There are cases when the **infimum is not attained, even if the problem is convex!!**

Lack of compactness w.r.t. $\omega \in \Omega$ *of trajectories and controls.*

Analogy with Optimal Transport

A very simple setting: $X_0 \in L^2(\Omega; \mathbb{R}^d)$ with law $\mu_0 = (X_0)_\sharp \mathbb{P}$ and μ_1 in $\mathscr{P}_2(\mathbb{R}^d)$ are given.

$$T = 1, \quad \mathbb{U} = \mathbb{R}^d, \quad F(x, \mu, u) = u, \quad C(x, \mu, u) := |u|^2, \quad C_T(x, \mu) := \begin{cases} 0 & \text{if } \mu = \mu_1, \\ +\infty & \text{otherwise.} \end{cases}$$

ODE System: $\dot{X}_t(\omega) = U_t(\omega)$, the trajectories minimizing $\int_0^1 |U_t(\omega)|^2 \, dt = \int_0^1 |\dot{X}_t(\omega)|^2 \, dt$, are **segments joining** $X_0(\omega)$ to $X_1(\omega)$.

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If μ_0 is diffuse but **concentrated on a set of codimension $\geqslant 1$ in $\mathbb{R}^d$** there are **examples where the infimum is not attained**. Take $\Omega := \mathbb{R}^d$, $\mathbb{P} = \mu_0$, $X_0(\omega) = \omega$

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More regular C_T: $C_t(x, \mu) := W_2^2(\mu, \mu_1)$, *Monge formulation of the barycenter problem.*

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From Monge to Kantorovich and Eulerian formulation of OT

Eulerian (a.k.a. Benamou-Brenier) formulation: look to the evolution of $\mu_t = (\mathbf{X}_t)_\sharp \mathbb{P}$ driven by a control field $\mathbf{u} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$. $\mathcal{A}_E(\mu_0)$ is the set of pairs $(\mu, \mathbf{u})$ solving the continuity equation

$$\boxed{\partial_t \mu_t + \nabla \cdot (u_t \mu_t) = 0} \quad \text{in } (0,T) \times \mathbb{R}^d$$

minimizing the cost

$$\boxed{\mathcal{J}_E(\mu, u) := \int_0^T \int |u_t|^2 \, d\mu_t \, dt}$$

Kantorovich formulation: look to the couplings $\mathbb{G} = (X_0, X_1)_\sharp \mathbb{P}$ of μ_0, μ_1 in $\mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ and minimize the cost

$$\mathcal{J}_K(\mathbb{G}) := \int |x_1 - x_0|^2 \, d\mathbb{G}(x_0, x_1)$$

$$\boxed{\mathcal{V}_E(\mu_0) = \min_{\mathcal{A}_E(\mu_0)} \mathcal{J}_E = \mathcal{V}_K(\mu_0) = \min_{\mathcal{A}_K(\mu_0)} \mathcal{J}_K = \mathcal{V}_L(X_0)}$$

Eulerian (a.k.a. Benamou-Brenier) formulation: look to the evolution of $\mu_t = (X_t)_\sharp \mathbb{P}$ driven by a control field $\mathbf{u} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$. $\mathcal{A}_{\mathbf{E}}(\mu_0)$ is the set of pairs $(\mu, \mathbf{u})$ solving the continuity equation

$$\boxed{\partial_t \mu_t + \nabla \cdot (u_t \mu_t) = 0} \quad \text{in } (0,T) \times \mathbb{R}^d$$

minimizing the cost

$$\boxed{\mathcal{J}_{\mathbf{E}}(\mu, u) := \int_0^T \int |u_t|^2 \, d\mu_t \, dt}$$

Kantorovich formulation: look to the couplings $\mathbb{G} = (X_0, X_1)_\sharp \mathbb{P}$ of μ_0, μ_1 in $\mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ and minimize the cost

$$\mathcal{J}_{\mathbf{K}}(\mathbb{G}) := \int |x_1 - x_0|^2 \, d\mathbb{G}(x_0, x_1)$$

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Variational Γ convergence

Suppose that $X_0^N \to X_0^\infty$ in $L^p(\Omega;\mathbb{R}^d)$, X_0^N are initial parametrizations of a N-particle system. $(\mathbf{X}^N, \mathbf{U}^N) \in \mathcal{A}_L(X_0^N)$ are minimizers, $\mathfrak{B}^N$-measurable. In order to pass to the limit in V_L we need

- **Compactness for trajectories-controls** w.r.t. some notion of convergence: there exists a subsequence $(\mathbf{X}^{N(k)}, \mathbf{U}^{N(k)})$ converging to some limit $(\mathbf{X}^\infty, \mathbf{U}^\infty) \in \mathcal{A}_L(X_0^\infty)$.
- Γ-lim inf **inequality:** along every sequence (X^k, U^k) converging to (X^∞, U^∞) $\liminf_{k\to\infty} \mathcal{J}_L(X^{N(k)}, U^{N(k)}) \geq \mathcal{J}_L(X^\infty, U^\infty)$
- Γ-lim sup **estimate:** if $(X^\infty, U^\infty) \in \mathcal{A}_L(X_0^\infty)$ then there exists $(X^N, U^N) \in \mathcal{A}_L(X_0^N)$ such that $\limsup_{N\to\infty} \mathcal{J}_L(X^N, U^N) \leq \mathcal{J}_L(X^\infty, U^\infty)$.

$$\begin{tikzcd}
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Gain compactness and lower semicontinuity by the Eulerian viewpoint

Look to the evolution of the law of X_t.

$$\mu_0 = (X_0)_\sharp \mathbb{P}, \quad \mu_t = (X_t)_\sharp \mathbb{P} \quad \text{in } \mathscr{P}_p(\mathbb{R}^d)$$

The control is a Borel map $u : (0,T) \times \mathbb{R}^d \to \mathbb{U}$.

When (X_t, U_t) are piecewise constant on the partition $\mathcal{B}^N$ the measure μ_t is discrete:
$\mu_t = \frac{1}{N} \sum_{i=1}^N \delta_{X_t(B^i)}$

$$\begin{tikzcd}
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Mean field control: Eulerian formulation

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μ is a curve in $AC^p(0,T; \mathscr{P}_p(\mathbb{R}^d))$ solving the continuity equation

$$\boxed{\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \quad v_t(x) = F(x, \mu_t, u_t(x))} \qquad \mu_{t=0} = \mu_0.$$

$(\mu, u) \rightsquigarrow \mathcal{A}_E(\mu_0)$.

$$\boxed{\mathcal{J}_E(\mu, u) := \int_0^T \int_{\mathbb{R}^d} \mathcal{C}(x, \mu_t, u_t(x))\, d\mu_t(x)\, dt + \int_{\mathbb{R}^d} \mathcal{C}_T(x, \mu_T)\, d\mu_T(x)}$$

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Compactness in the Eulerian problem

- Given a sequence $(\mu^k, \mathbf{u}^k) \in \mathcal{A}_E(\mu_0^k)$ such that $\mu_0^k \to \mu_0$, we pass to the relaxed formulation via the **Young measure** $\boldsymbol{\sigma}^k := (\mathrm{id} \times \mathbf{u}^k)_\sharp \mu^k$. **Continuity equation:**

$$\iiint \Big(\partial_t \zeta(t,x) + \nabla\zeta(t,x)\cdot F(x,\mu_t,u)\Big)\, d\boldsymbol{\sigma}^k(t,x,u) = 0 \quad \text{for every } \zeta \in C_c^1((0,T)\times\mathbb{R}^d).$$

$$\mathcal{J}_E(\mu^k,\mathbf{u}^k) = \iiint \mathbf{C}(x,\mu_t,u)\,d\boldsymbol{\sigma}^k(t,x,u) + \int \mathbf{C}_T(x,\mu_t)\,d\mu_T(x).$$

- **Young measure solutions (including controls) are tight:**
  the marginal w.r.t. to t is the Lebesgue measure on $(0,T)$,
  the marginal w.r.t. u is supported in the compact set $\mathbb{U}$,
  the marginal w.r.t. x are controlled by momentum estimates.
  Uniform equicontinuity in time for μ_t^k in $\mathscr{P}_p(\mathbb{R}^d)$.
- From a limit measure σ^∞ w.r.t. the topology of $\mathscr{P}((0,T)\times\mathbb{R}^d\times\mathbb{U})$ we recover
  μ^∞ as the marginal of σ w.r.t. (t,x)
  $\mathbf{u}^\infty$ as the barycenter w.r.t. u, i.e. by disintegrating $\sigma^\infty = \int \sigma_{t,x}\,d\mu^\infty$ and
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Closure and lower semicontinuity

The limit **continuity equation:**

$$\iiint \Big(\partial_t \zeta(t,x) + \nabla\zeta(t,x)\cdot F(x,\mu_t,u)\Big)\,d\boldsymbol{\sigma}^\infty(t,x,u) = 0 \quad \text{for every } \zeta \in C^1_c((0,T)\times\mathbb{R}^d).$$

yields

$$\boxed{\partial_t \mu^\infty + \nabla\cdot(\mathbf{u}^\infty\,\mu^\infty) = 0, \quad (\mu^\infty, \mathbf{u}^\infty) \in \mathcal{A}_E(\mu_0)}$$

. **Convexity and Jensen inequality** yield

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Well posedness of the Eulerian problem

Theorem

In the convex setting there exists a minimizer of the Eulerian MFOC

$$V_E(\mu_0) = \min\left\{\mathcal{J}_E(\mu, \mathbf{u}) : (\mu, \mathbf{u}) \in \mathcal{A}_E(\mu_0)\right\}$$

and

$$\liminf_{N\to\infty} V_E(\mu_0^N) \geqslant V_E(\mu_0).$$

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and

$$\liminf_{N\to\infty} V_E(\mu_0^N) \geqslant V_E(\mu_0).$$

Moreover, if $X_0^N \to X_0$ *strongly in* $L^p(\Omega; \mathbb{R}^d)$ *then* $\mu_0^N \to \mu_0$ *in* $\mathscr{P}_p(\mathbb{R}^d)$ *and*

$$\liminf_{N\to\infty} V_L(X_0^N) \geqslant V_E(\mu_0).$$

Lagrangian$^\infty$ vs Eulerian$^\infty$ 1

Theorem

Let $X_0 \in L^p(\Omega;\mathbb{R}^d)$ *be given with* $\mu_0 = (X_0)_\sharp\mathbb{P}$ *and let* $(\mathbf{X},\mathbf{U}) \in \mathcal{A}_{\mathbf{L}}(X_0)$*. if* $\boldsymbol{\sigma} := (\mathbf{X},\mathbf{U})_\sharp(\mathcal{L}^1 \times \mathbb{P})$ *and* $\mu = \mathbf{X}_\sharp(\mathcal{L}^1 \times \mathbb{P})$*, the barycenter* $\mathbf{u}$ *of* $\boldsymbol{\sigma}$ *w.r.t. its marginal* μ *satisfies*

$$(\mu,\mathbf{u}) \in \mathcal{A}_{\mathbf{E}}(\mu_0), \qquad J_{\mathbf{L}}(\mathbf{X},\mathbf{U}) \geqslant J_{\mathbf{E}}(\mu,\mathbf{u})$$

In particular

$$\boxed{\mathcal{V}_{\mathbf{L}}(\mathbb{X}_0) = \mathcal{V}_{\mathbf{E}}(\mu_0)}$$

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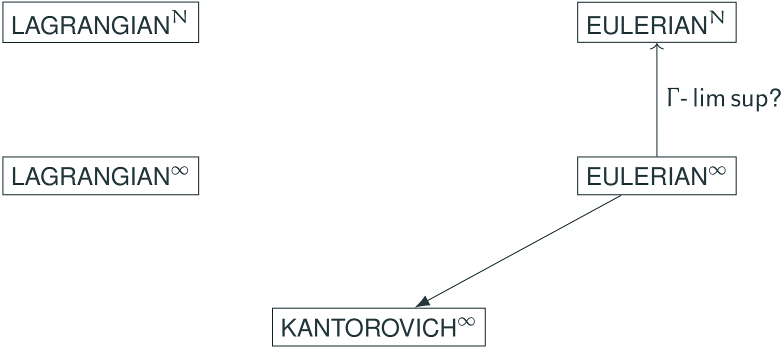
Γ-$\limsup$ inequality: the Kantorovich formulation

It is not clear how to prove directly the Γ-$\limsup$ **inequality** at the Eulerian level. Moreover, it would be interesting to improve the inqualities between the Lagrangian and the Eulerian formulations.

As in Optimal Transport, it is useful to introduce a new formulation:



Kantorovich formulation

Path space: $\Gamma := C([0,T];\mathbb{R}^d)$, curves in Γ are denoted by ω.

X_t is the evaluation map at time t : $X_t(\omega) := \omega(t)$.

Controls depend on time t and on curves ω: $U(t,\omega)$.

Given a probability measure $\mathbb{G}$ on Γ, its time marginals $\mu_t = (X_t)_\sharp\mathbb{G}$ are evolving probability laws.

$\mathcal{A}_K(\mu_0)$ is given by pairs $(\mathbb{G}, U)$ such that $\mathbb{G}$-almost every curve ω solves

$$\boxed{\frac{d}{dt}\omega(t) = F(\omega(t), \mu_t, U_t(\omega)), \quad (X_0)_\sharp\mathbb{G} = \mu_0}$$

The corresponding cost can be computed as

$$\boxed{\mathcal{J}_K(\mathbb{G}, U) = \int_\Gamma \int_0^T C(\omega(t), \mu_t, U_t(\omega))\, dt\, d\mathbb{G}(\omega) + \int_{\mathbb{R}^d} C_T(x, \mu_T)\, d\mu_T(x)} \quad \mu_t = (X_t)_\sharp\mathbb{G}$$

$$\mathcal{V}_K(\mu_0) = \inf\left\{\mathcal{J}_K(\mathbb{G}, U) : (\mathbb{G}, U) \in \mathcal{A}_K(\mu_0)\right\}.$$

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Kantorovich is a particular Lagrangian

Once $\mathbb{G}$ is fixed, the **Kantorovich formulation yields a a particular version of a Lagrangian problem** with

$$\boxed{\Omega := \Gamma, \quad \mathfrak{B} = \left\{\text{Borel } \sigma\text{-algebra of } \Gamma\right\}, \quad \mathbb{P} = \mathbb{G}}$$

However, in this formulation, **we fix the maps** X_t (the evaluation maps) and **we allow for modifications of the reference measure** $\mathbb{P}$.

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Eulerian vs Kantorovich (1)

Theorem

For every μ_0 *and* $(\mu, \mathbf{u}) \in \mathcal{A}_{\mathbf{E}}(\mu_0)$ *there exists* $(\mathbb{G}, \mathrm{U}) \in \mathcal{A}_{\mathbf{K}}(\mu_0)$ *such that*

$$\mu_t = (X_t)_\sharp \mathbb{G}, \quad \mathcal{J}_{\mathbf{K}}(\mathbb{G}, \mathrm{U}) = \mathcal{J}_{\mathbf{E}}(\mu, \mathbf{u}).$$

In particular $\mathcal{V}_{\mathbf{K}}(\mu_0) \leqslant \mathcal{V}_{\mathbf{E}}(\mu_0)$.

It is a consequence of the **superposition theorem**: given μ solving

$$\partial_t \mu + \nabla \cdot (v\mu) = 0, \quad \int_0^T \int |v|^p \, d\mu_t \, dt < +\infty,$$

there exists a probability measure $\mathbb{G}$ on Γ such that

$$\boxed{\Gamma \text{ is concentrated on solutions of } \omega'(t) = v_t(\omega_t), \quad \text{and } (X_t)_\sharp \mathbb{G} = \mu_t \quad \text{in } [0, T].}$$

Since $v_t(\omega(t)) = F(\omega(t), \mu_t, u_t(\omega(t))$ we can define $U_t(\omega) := u_t(\omega(t))$ and we obtain a pair $(\mathbb{G}, U) \in \mathcal{A}_K(\mu_0)$ with $\mathcal{J}_K(\mathbb{G}, U) = \mathcal{J}_E(\mu, u)$.

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Eulerian vs Kantorovich (1)

Theorem

For every μ_0 *and* $(\mu, \mathbf{u}) \in \mathcal{A}_E(\mu_0)$ *there exists* $(\mathbb{G}, \mathbb{U}) \in \mathcal{A}_K(\mu_0)$ *such that*

$$\mu_t = (X_t)_\sharp \mathbb{G}, \quad \mathcal{J}_K(\mathbb{G}, \mathbb{U}) = \mathcal{J}_E(\mu, \mathbf{u}).$$

In particular $\mathcal{V}_K(\mu_0) \leqslant \mathcal{V}_E(\mu_0)$.

It is a consequence of the **superposition theorem**: given μ solving

$$\partial_t \mu + \nabla \cdot (\boldsymbol{v}\mu) = 0, \quad \int_0^T \int |\boldsymbol{v}|^p \, d\mu_t \, dt < +\infty,$$

there exists a probability measure $\mathbb{G}$ on Γ such that

$$\boxed{\Gamma \text{ is concentrated on solutions of } \omega'(t) = \boldsymbol{v}_t(\omega_t), \quad \text{and } (X_t)_\sharp \mathbb{G} = \mu_t \quad \text{in } [0, T].}$$

Since $\boldsymbol{v}_t(\omega(t)) = F(\omega(t), \mu_t, \mathbf{u}_t(\omega(t))$ we can define $\mathbb{U}_t(\omega) := \mathbf{u}_t(\omega(t))$ and we obtain a pair $(\mathbb{G}, \mathbb{U}) \in \mathcal{A}_K(\mu_0)$ with $\mathcal{J}_K(\mathbb{G}, \mathbb{U}) = \mathcal{J}_E(\mu, \mathbf{u})$.

Eulerian vs Kantorovich (2)

Since every solution $(\mathbb{G}, \mathfrak{U}) \in \mathcal{A}_{\mathbf{K}}(\mu_0)$ is a Lagrangian solution, we know that $V_{\mathbf{K}}(\mu_0) \geqslant V_{\mathbf{E}}(\mu_0)$.

Corollary

$V_{\mathbf{K}}(\mu_0) = V_{\mathbf{E}}(\mu_0)$ *and* $V_{\mathbf{K}}$ *is attained.*

$$\begin{tikzcd}
\boxed{\text{LAGRANGIAN}^{N}} & & \boxed{\text{EULERIAN}^{N}} \\
\boxed{\text{LAGRANGIAN}^{\infty}} & & \boxed{\text{EULERIAN}^{\infty}} \\
& \boxed{\text{KANTOROVICH}^{\infty}} \arrow[ur, leftrightarrow, "="'] &
\end{tikzcd}$$

Kantorovich vs Lagrangian vs Eulerian

Every Lagrangian pair $(\mathbf{X}, \mathbf{U}) \in \mathcal{A}_{\mathbf{L}}(X_0)$ induces a Kantorovich pair with lower cost. Conversely, every Kantorovich pair can be approximated by Lagrangian pairs.

Theorem

For every $X_0 \in L^p(\Omega; \mathbb{R}^d)$ *with* $\mu_0 = (X_0)_\sharp \mathbb{P}$ *and every* $(\mathbb{G}, \mathbf{U}) \in \mathcal{A}_{\mathbf{K}}(\mu_0)$ *there exists* $(\mathbf{X}^n, \mathbf{U}^n) \in \mathcal{A}_{\mathbf{L}}(X_0)$ *such that* $\mathcal{J}_{\mathbf{L}}(\mathbf{X}^n, \mathbf{U}^n) \to \mathcal{J}_{\mathbf{K}}(\mathbb{G}, \mathbf{U})$. *In particular* $\mathcal{V}_{\mathbf{L}}(X_0) = \mathcal{V}_{\mathbf{K}}(\mu_0) = \mathcal{V}_{\mathbf{E}}(\mu_0)$.

$$
\begin{tikzcd}
\boxed{\text{LAGRANGIAN}^\infty} \arrow[rr, leftrightarrow, "="] \arrow[dr, leftrightarrow, "="'] & & \boxed{\text{EULERIAN}^\infty} \arrow[dl, leftrightarrow, "="] \\
& \boxed{\text{KANTOROVICH}^\infty} &
\end{tikzcd}
$$

Finite particle approximations

Theorem

Suppose that X_0^N *is a family of* N *particles such that* $\mu_0^N = (X_0^N)_\sharp \mathbb{P}$ *converges to* μ_0^∞ *in* $\mathscr{P}_p(\mathbb{R}^d)$ *and let* X_0^∞ *such that* $(X_0^\infty)_\sharp \mathbb{P} = \mu_0$. *Then*

$$\boxed{\lim_{N\to\infty} \mathcal{V}_{\mathbf{L}}(X_0^N) = \lim_{N\to\infty} \mathcal{V}_{\mathbf{E}}(\mu_0^N) = \lim_{N\to\infty} \mathcal{V}_{\mathbf{K}}(\mu_0^N) = \mathcal{V}_{\mathbf{L}}(X_0^\infty) = \mathcal{V}_{\mathbf{E}}(\mu_0^\infty) = \mathcal{V}_{\mathbf{K}}(\mu_0^\infty)}$$

$$\begin{tikzcd}
\boxed{\text{LAGRANGIAN}^N} \arrow[rr, leftrightarrow, "="] & & \boxed{\text{EULERIAN}^N} \arrow[d, "\text{Compactness} \;\; \Gamma\text{-}\liminf \geqslant"] \\
\boxed{\text{LAGRANGIAN}^\infty} \arrow[u, "\Gamma\text{-}\limsup"] \arrow[rr, leftrightarrow, "="] \arrow[dr, leftrightarrow, "="'] & & \boxed{\text{EULERIAN}^\infty} \arrow[dl, leftrightarrow, "="] \\
& \boxed{\text{KANTOROVICH}^\infty} &
\end{tikzcd}$$