Probabilistic Methods

Part I. Lovász Local Lemma

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Introduction

Probabilistic methods

- prove the existence of combinatorial objects
- using probabilistic tools and arguments
  - First moment principles: linearity of expectation
  - Second moment inequalities
  - Lovász Local Lemma
  - Entropy Compression
  - Concentration inequalities
  - ...

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Outline of today's talk

- a warmup example
- hypergraph coloring problem
- statement of the Lovász Local Lemma
- application in hypergraph coloring
- application in acyclic graph coloring

Warmup example

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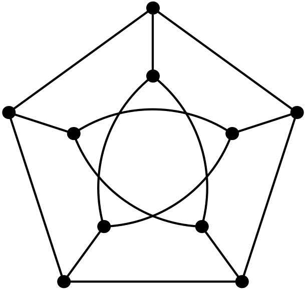

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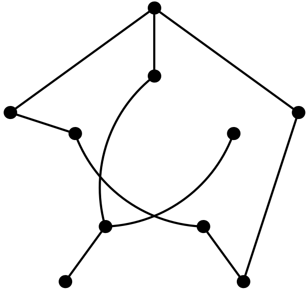

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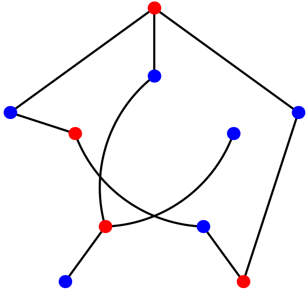

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The best we can hope for is $\sim \frac{m}{2}$:

- a complete graph on n vertices has $\binom{n}{2} \sim \frac{n^2}{2}$ edges
- a complete bipartite graph on $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$ vertices has $\sim \frac{n^2}{4}$ edges

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- Remove monochromatic edges

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$\mathbb{E}(\sum_{e \in E(G)} X_e) = \sum_{e \in E(G)} \mathbb{E}(X_e) = \frac{m}{2}$.

Therefore, there exists a coloring with at least $\frac{m}{2}$ bichromatic edges.

Hypergraph coloring

A hypergraph $H = (V, E)$ is a couple of sets with

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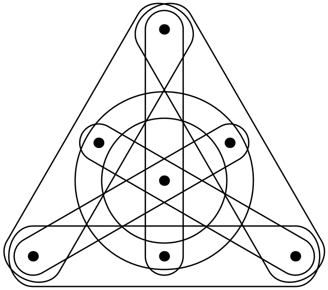

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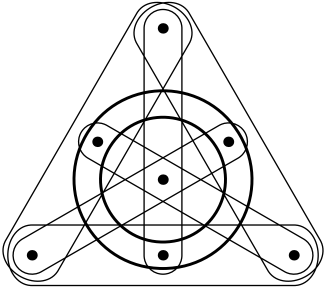

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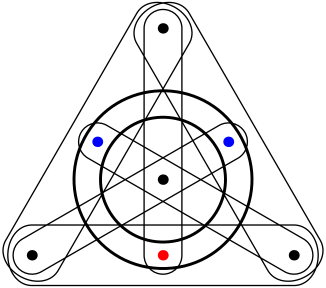

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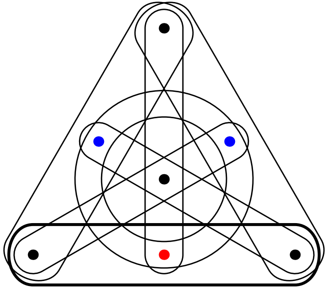

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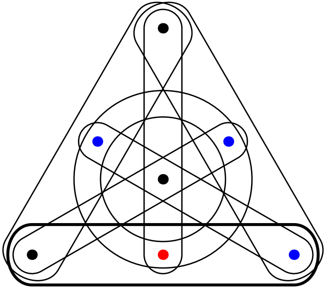

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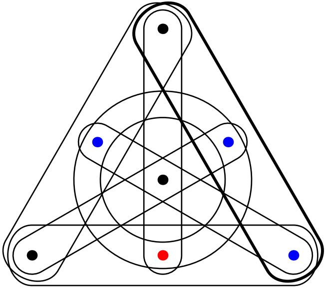

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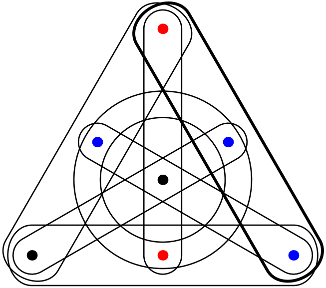

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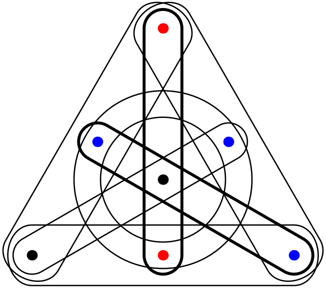

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If $\mathscr{A} := (A_e, e \in E)$ were independent, we would have

$$\mathbb{P}\left(\bigcap_{e \in E} \overline{A_e}\right) = \left(1 - \frac{1}{2^{k-1}}\right)^m > 0$$

Mutually independent events

Definition

Let A be an event and let $\mathscr{B}$ be a set of events in a probability space. We say that A is <u>*mutually independent*</u> *of $\mathscr{B}$ if*

$$\mathbb{P}\left(A \mid \bigcap_{B_i \in S} B_i\right) = \mathbb{P}(A)$$

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For example, in the context of random hypergraph coloring, A_e is mutually independent of

$$\{A_{e'} : e \cap e' = \emptyset\}.$$

Lovász Local Lemma

If a set of bad events that are mostly mutually independent happen with low probability, then with positive probability none of them happen.

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Theorem (Lovász Local Lemma, Symmetric version)

Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ *be a set of events such that for each* $i = 1, 2 \ldots, n$

- $\mathbb{P}(A_i) \leq p$ *and*
- $\exists \mathscr{D}_i \subset \mathscr{A}$ *of size at most* d *such that* A_i *is mutually independent of* $\mathscr{A} \setminus \mathscr{D}_i$.

If

$$e \cdot p \cdot (d+1) \leq 1$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

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If a set of bad events that are mostly mutually independent happen with low probability, then with positive probability none of them happen.

Theorem (LLL)

If $\mathbb{P}(A_i) \leq p$, A_i is mutually independent of $\mathscr{A} \setminus \mathscr{D}_i$ with $|\mathscr{D}_i| \leq d$, and $ep(d+1) \leq 1$, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

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In the context of random coloring of a k-regular k-uniform hypergraph, $p = \frac{1}{2^{k-1}}$ and each A_e is mutually independent of all but at most k^2 other edges, so $d = k^2$.

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In the context of random coloring of a k-regular k-uniform hypergraph, $p = \frac{1}{2^{k-1}}$ and each A_e is mutually independent of all but at most k^2 other edges, so $d = k^2$.

There exists a coloring without a monochromatic edge whenever

$$\frac{e}{2^{k-1}} \cdot k^2 \leq 1.$$

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Theorem (Alon and Bregman 1988, Henning and Yeo 2013)

Let $k \geq 4$. Then every k-regular k-uniform hypergraph is 2-colorable.

Acyclic graph coloring

Definition

Let $G = (V, E)$ *be a graph. A coloring* $\varphi : V(G) \to \{1, 2, \ldots, k\}$ *is an* <u>*acyclic*</u> *coloring of* G *if*

- $\varphi(u) \neq \varphi(v) \quad \forall uv \in E(G), \quad$ *(*φ *is a proper coloring)*
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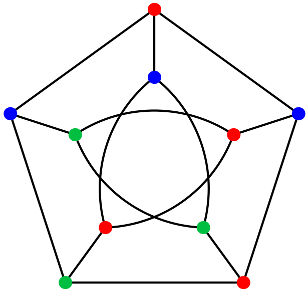

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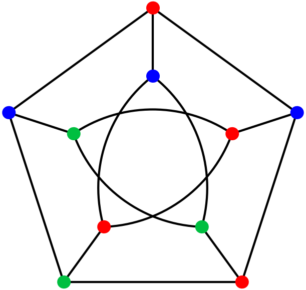

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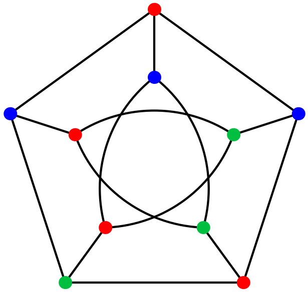

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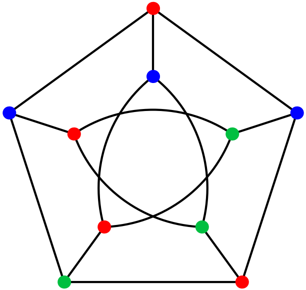

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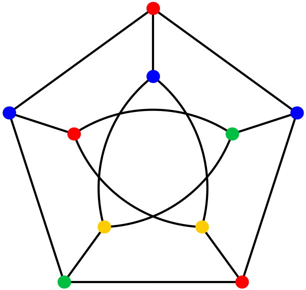

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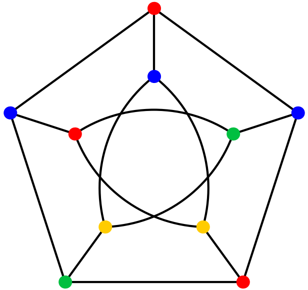

an acyclic coloring with 4 colors

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Greedy bound

Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of G?

If we color every vertex with a color distinct from all the colors of its neighbors and the neighbors of its neighbors, surely we will not create any bichromatic cycle.

This is always possible provided we have at least

$$\Delta + \Delta(\Delta - 1) + 1 = \Delta^2 + 1$$

colors. Hence,

$$\chi_a(G) \leq \Delta^2 + 1$$

for every graph G.

Using Lovász Local Lemma

Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of G?

Theorem (Alon, McDiarmid, Reed 1991)

Let G be a graph with maximum degree Δ. Then

$$\chi_a(G) \leq 50\Delta^{4/3}.$$

On the other hand, there are graphs for which

$$\chi_a(G) = \Omega\left(\frac{\Delta^{4/3}}{(log\,\Delta)^{1/3}}\right).$$

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Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of G?

Theorem

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Let C be a set of $K \geq 7\Delta^{3/2}$ colors.
Randomized procedure : For each vertex v, let $F(v)$ be the set of colors forbidden at v – the colors of the neighbors already colored, and let $C(v) = C \setminus F(v)$ be the set of available colors at v. Clearly, $|F(v)| \leq \Delta$.

- Choose an integer $i \leq K - \Delta$ uniformly randomly and color v with i-th available color.

This procedure gives a proper coloring of G.

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Let A_P be the event that a 4-vertex path $P = v_1 v_2 v_3 v_4$ gets only two colors.

$$\mathbb{P}(A_P) \leq \frac{1}{(K-\Delta)^2}.$$

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A_P is independent of all $A_{P'}$ with $P \cap P' = \emptyset$.

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- Choose an integer $i \leq K - \Delta$ uniformly randomly and color v with i-th available color.

Let A_P be the event that a 4-vertex path $P = v_1 v_2 v_3 v_4$ gets only two colors.

$$\mathbb{P}(A_P) \leq \frac{1}{(K - \Delta)^2}.$$

A_P is independent of all $A_{P'}$ with $P \cap P' = \emptyset$.
The dependency degree is (less than)

$$d < 4 \cdot 4 \cdot \Delta^3 = 16\Delta^3.$$

Using Lovász Local Lemma

Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of G?

Theorem

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Theorem (LLL)

If $\mathbb{P}(A_i) \leq p$, A_i is mutually independent of $\mathscr{A} \setminus \mathscr{D}_i$ with $|\mathscr{D}_i| \leq d$, and $ep(d+1) \leq 1$, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

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and so

$$ep(d+1) \leq \frac{e(16\Delta^3)}{(7\Delta^{3/2} - \Delta)^2} < \frac{0.89\Delta}{(\Delta^{1/2} - \frac{1}{7})^2} < 1$$

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□

Conclusion

LLL: If a set of bad events that are mostly mutually independent happen with low probability, then with positive probability none of them happen.

Applications in graphs, hypergraphs, coloring, transversals, satisfiability, combinatorics of words, etc.

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Thank you for your attention!