JNCF 2024

Topology and periods of elliptic surfaces

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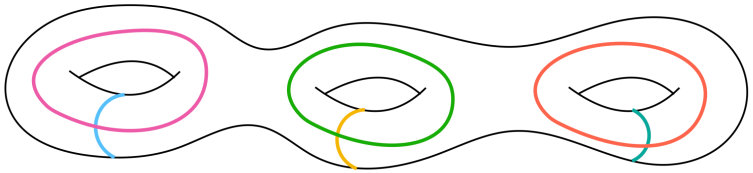
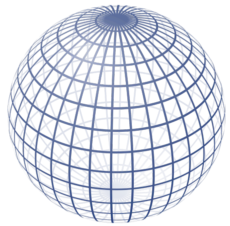
Elliptic curves

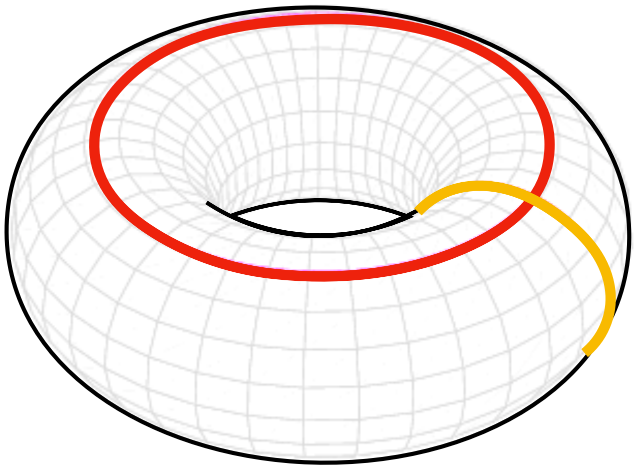
Smooth projective complex algebraic curves are closed oriented real surfaces. Their topology is thus determined by an integer, the **genus** of the curve.

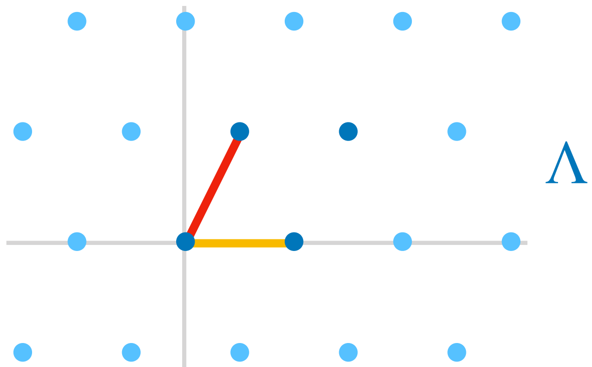


A curve of genus 3



A curve of genus 0
The **projective line** $\mathbb{P}^1$





$$X \simeq \mathbb{C}/\Lambda$$

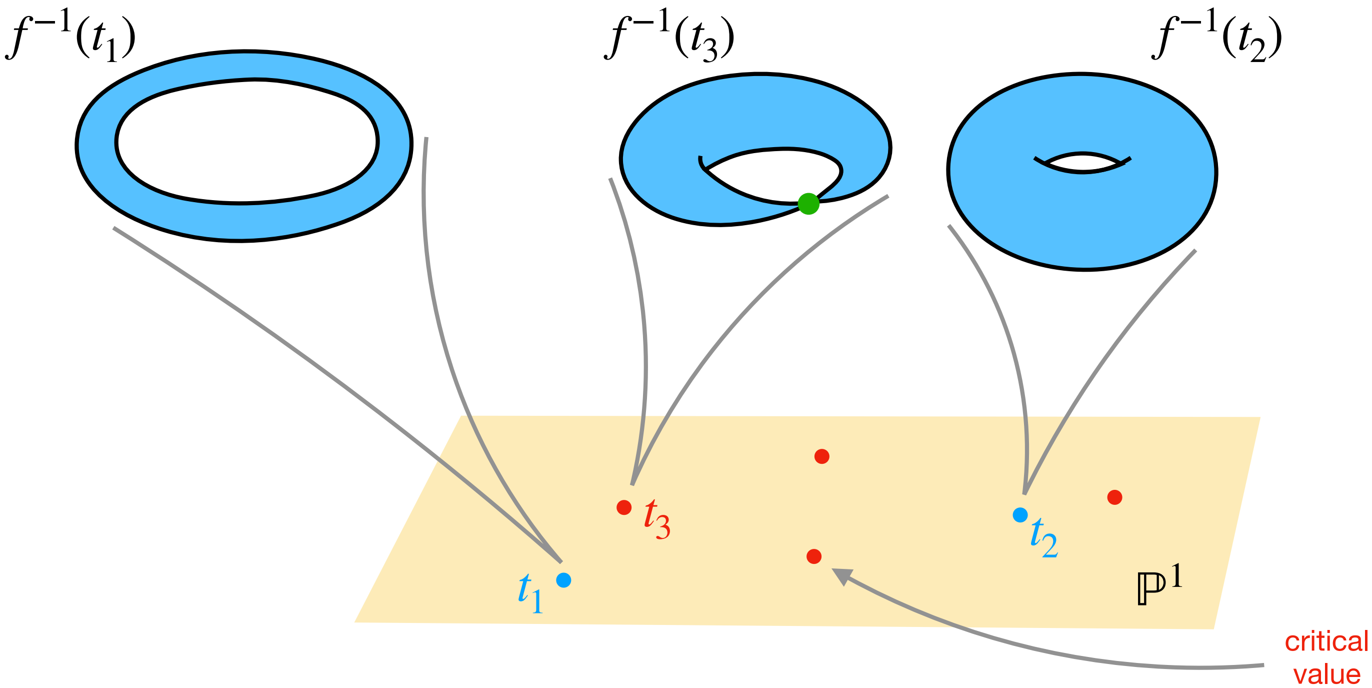
An **elliptic curve** is a smooth algebraic curve of genus 1.

Elliptic surfaces

An **elliptic surface** S is a smooth algebraic surface equipped with a map to the projective line

$$f : S \to \mathbb{P}^1$$

such that all but finitely many fibres $f^{-1}(t)$ are elliptic curves.



Elliptic surfaces

Example: The **small Apéry numbers** is a sequence that was introduced by Apéry to prove the irrationality of $\zeta(2)$.

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \qquad f(t) = \sum_{n=0}^{\infty} u_n t^n = 1 + 3t + 19t^2 + 147t^3 + 1251t^4 + \dots$$

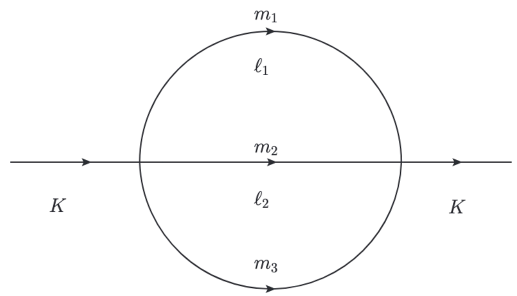
They are characterised by an integral solution to the differential operator

$$\left(t^3 + 11t^2 - t\right)\partial t^2 + \left(3t^2 + 22t - 1\right)\partial t + t + 3$$

which can be linked to the geometry of the **rational elliptic surface**

$$y^2 + (t-1)xy + ty = x^3 - tx^2 .$$

Example: The **sunset graph** (a Feynman graph) is associated to an **elliptic K3 surface**.

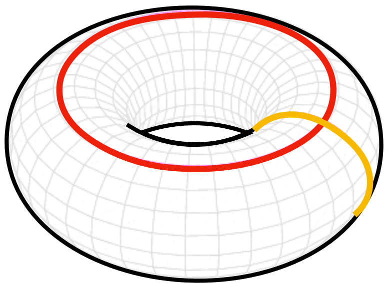


[Bloch&Vanhove, 2014]

$$\frac{\pi^D \Gamma(3-D)}{(\mu^2)^{D-3}} \int_0^\infty \int_0^\infty \frac{(x+y+xy)^{\frac{3(2-D)}{2}}\, dx dy}{((m_1^2 x + m_2^2 y + m_3^2)(x+y+xy) - K^2 xy)^{3-D}} .$$

Periods and classifying varieties

A **period** of an algebraic variety is the integral of a form of the variety on a cycle.



Some integration domain without boundary

$$\int_{T(\gamma)} \frac{A}{P^k}$$

A is a polynomial

P defines a smooth variety $V(P) = \{z \mid P(z) = 0\}$

They encode the comparison between **topological data** (cycles) and **algebraic data** (algebraic De Rham forms).

$$H_n(S, \mathbb{Z}) \times H^n_{DR}(S) \to \mathbb{C} \qquad \gamma, \omega \mapsto \int_\gamma \omega$$

[Stiller 1987]

Reach in **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

e.g. **Torelli-type theorems:**

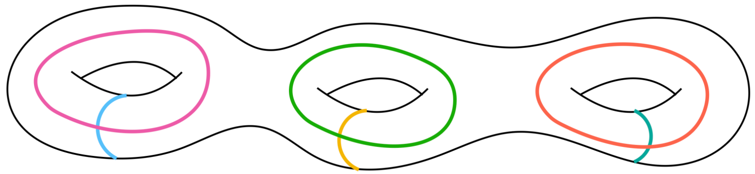
$V(P_1)$ is isomorphic to $V(P_2)$ if and only if their periods are “the same”.

Goal: compute numerical approximations of these integrals with large precision.

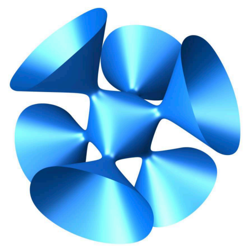
For this we need a representation of the integration cycles that is well suited for integration.

Previous works

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
Algebraic curves (Riemann surfaces)

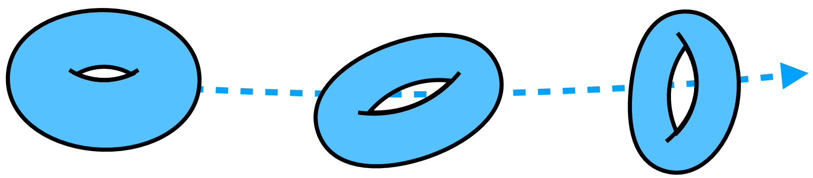


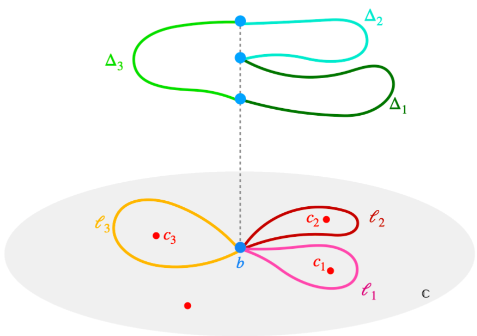
[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:
Higher dimensional varieties (double covers of $\mathbb{P}^2$ ramified along 6 lines / of $\mathbb{P}^3$ ramified along 8 planes)



Picture by
Alessandra Sarti

[Sertöz 2019]: compute the period matrix of hypersurfaces by **deformation.**





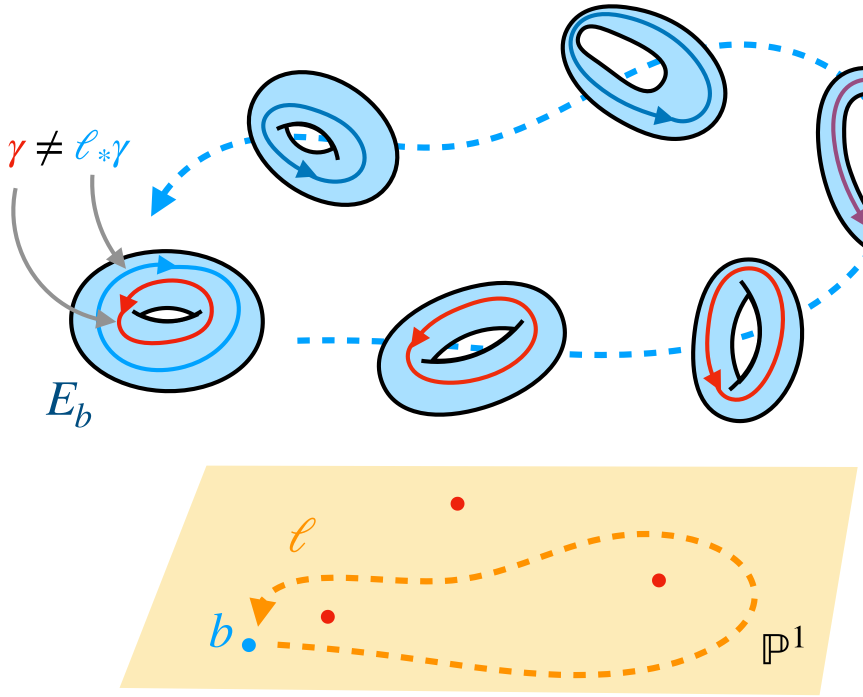
[Lairez, PP, Vanhove 2023]: compute the period matrix of hypersurfaces by **integration**.
(Efficient enough to compute periods of quartic surfaces)

relies on an effective description of the topology

Monodromy

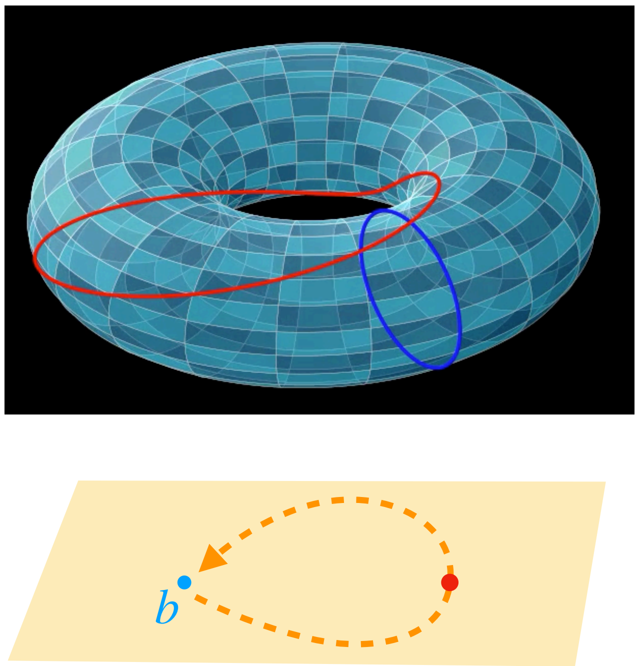
The fibre $E_t = f^{-1}(t)$ above t, which is an elliptic curve, deforms as t moves in $\mathbb{P}^1$ (Ehresmann's fibration theorem).

The map $\ell_* : H_1(E_b) \to H_1(E_b)$ induced by this deformation along a loop ℓ is called the **monodromy along ℓ**.



The monodromy is encoded in a differential operator: the **Picard-Fuchs equation**.

A Dehn twist



When the monodromy is a Dehn twist, the singular fibre is said to be of **Lefschetz type**.
$\ell_* - \mathrm{id}$ has **rank 1** and its image is **primitive**.

Computing the homology of elliptic surface

We can recover integration 2-cycles for the periods of elliptic surfaces as **extensions** of 1-cycles of the fibre.

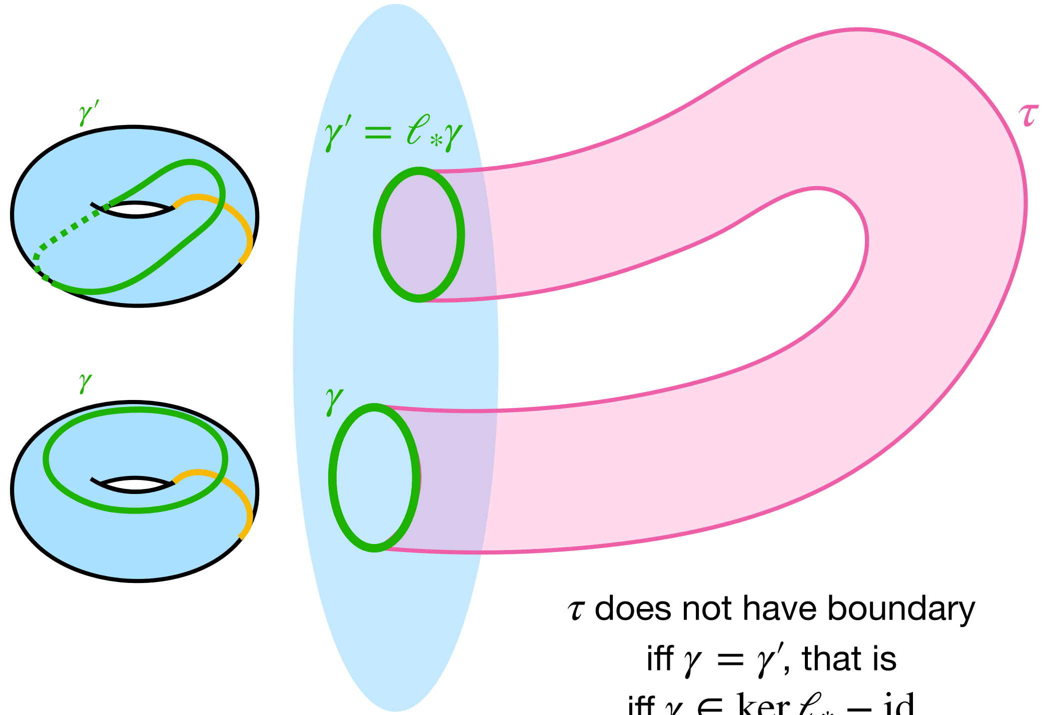
$$\pi_1(\mathbb{P}^1 \backslash \Sigma, b) \times H_1(E_b) \to H_2(S, E_b)$$
$$\ell, \gamma \mapsto \tau_\ell(\gamma)$$

This description of cycles is well-suited for integrating the periods:

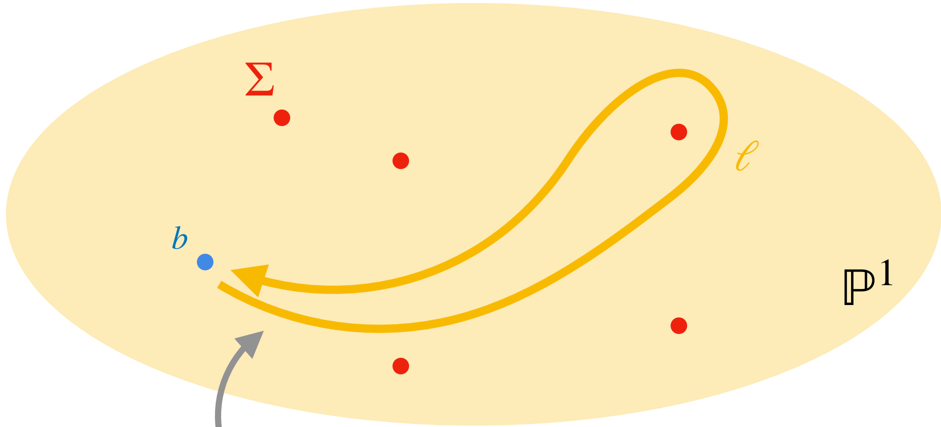
$$\int_{\tau_\ell(\gamma)} f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_\gamma f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Two line integrals:
we know how to compute these efficiently!
[Chudnovsky[2], Van der Hoeven, Mezzarobba]

$$\partial \tau_\ell(\gamma) = \gamma' - \gamma$$



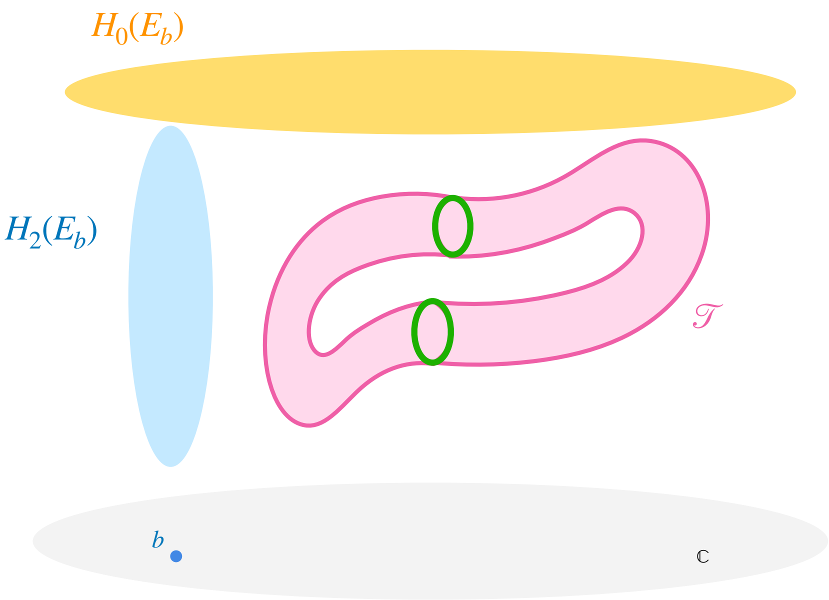
τ does not have boundary
iff $\gamma = \gamma'$, that is
iff $\gamma \in \ker \ell_* - \mathrm{id}$



such a path is called a **simple loop**

The Lefschetz case

When all fibres are of Lefschetz type, each **simple loop** ℓ contributes a single nontrivial relative homology class, called the **thimble**.



Thimbles serve as building blocks for extensions: we can glue thimbles together in a way that matches their **boundary** to obtain closed cycles.

Obtained from the monodromy :
$\partial\tau_\ell(\gamma) = \ell_*\gamma - \gamma$

Furthermore $H_2(S)$ is generated by **extensions**, **a generic fibre**, and a **section**.

Their periods are zero.
We only need to compute periods of extensions.

Algorithm for the Lefschetz case

1. Compute the set Σ of **critical values**.
2. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \backslash \Sigma, b)$.
3. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
4. Glue thimbles together to obtain **extension cycles**.
5. Integrate the **periods** on these extensions.

Computing monodromy

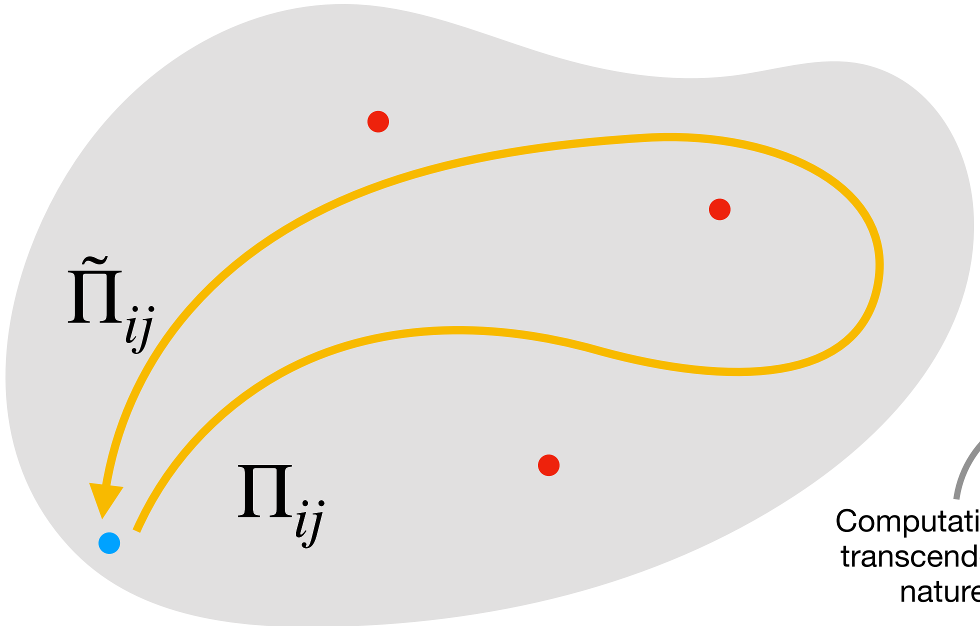
Globally defined
$\Longrightarrow$ no monodromy

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow[\text{[Chudnovsky}^2\text{ 90, Van der Hoeven 99, Mezzarobba 2010]}]{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\color{red}{\sum_k c_{kj}\gamma_k}} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

Solution to
Picard-Fuchs
equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\boxed{\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_2(\mathbb{Z})}$$

Computation of
transcendental
nature

It is sufficient to carry out this
computation with precision $< 1/2$
to recover C exactly.

The general case: an example

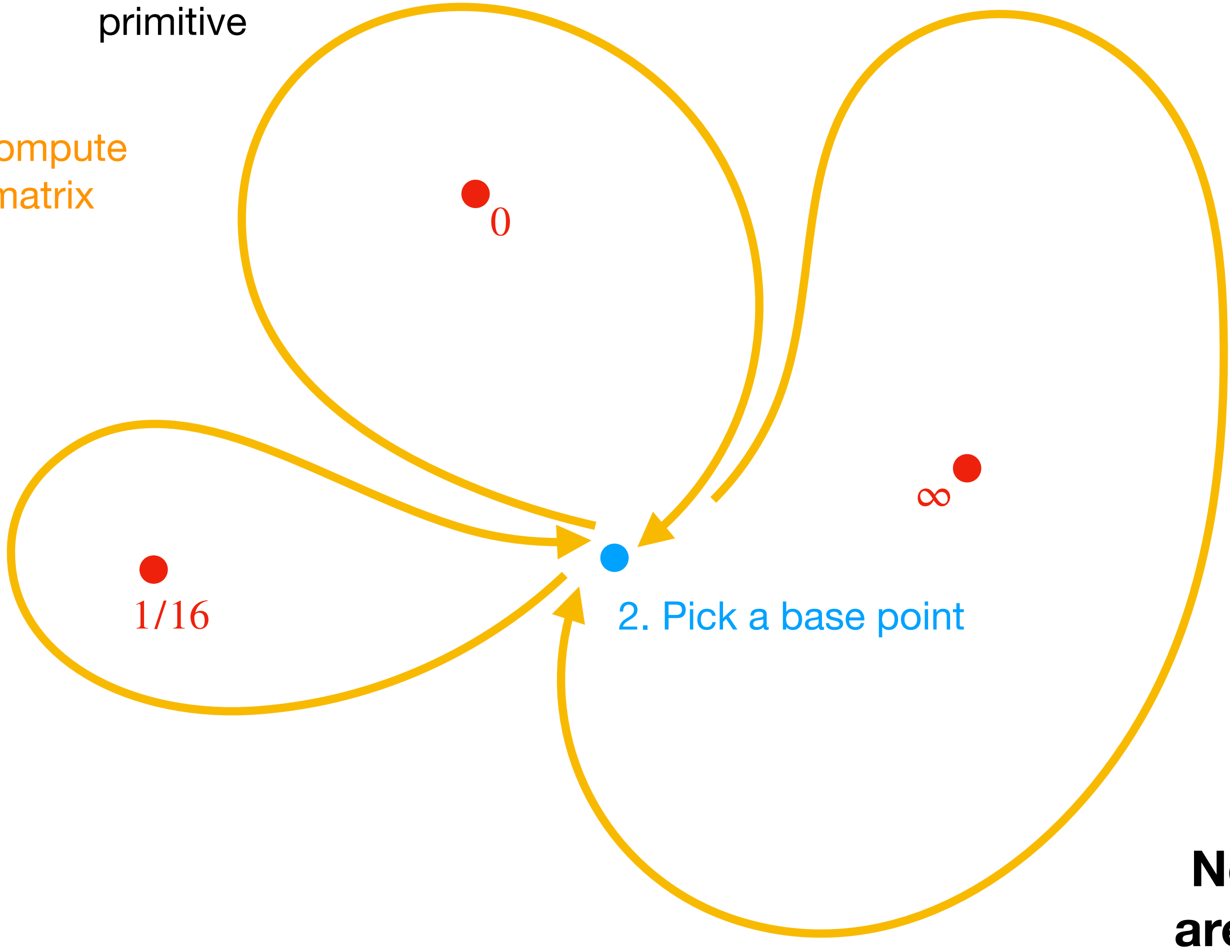
The Apéry surface S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

1. Compute the set Σ of critical values

i.e., the roots of the discriminant $t^4(t - \frac{1}{16})$

$(4{,}0)$ not primitive $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

4. For each loop, compute the monodromy matrix



$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

$\operatorname{rank}\left(\ell_* - \operatorname{id}\right) \neq 1$

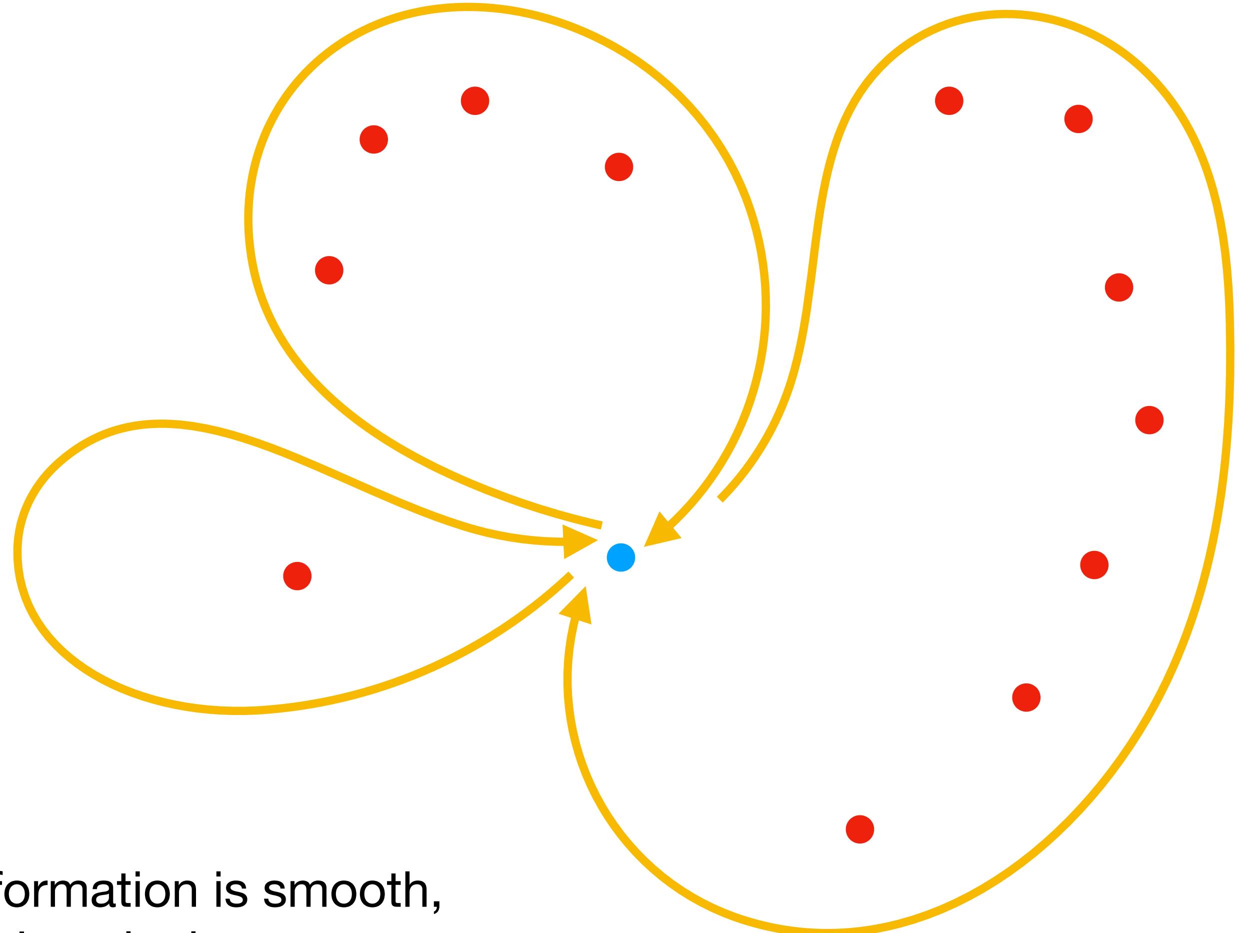
3. Compute a basis of simple loops of $\pi_1(\mathbb{C}\backslash\Sigma)$

Not all fibres are Lefschetz!

We have to find a workaround …

The general case: an example

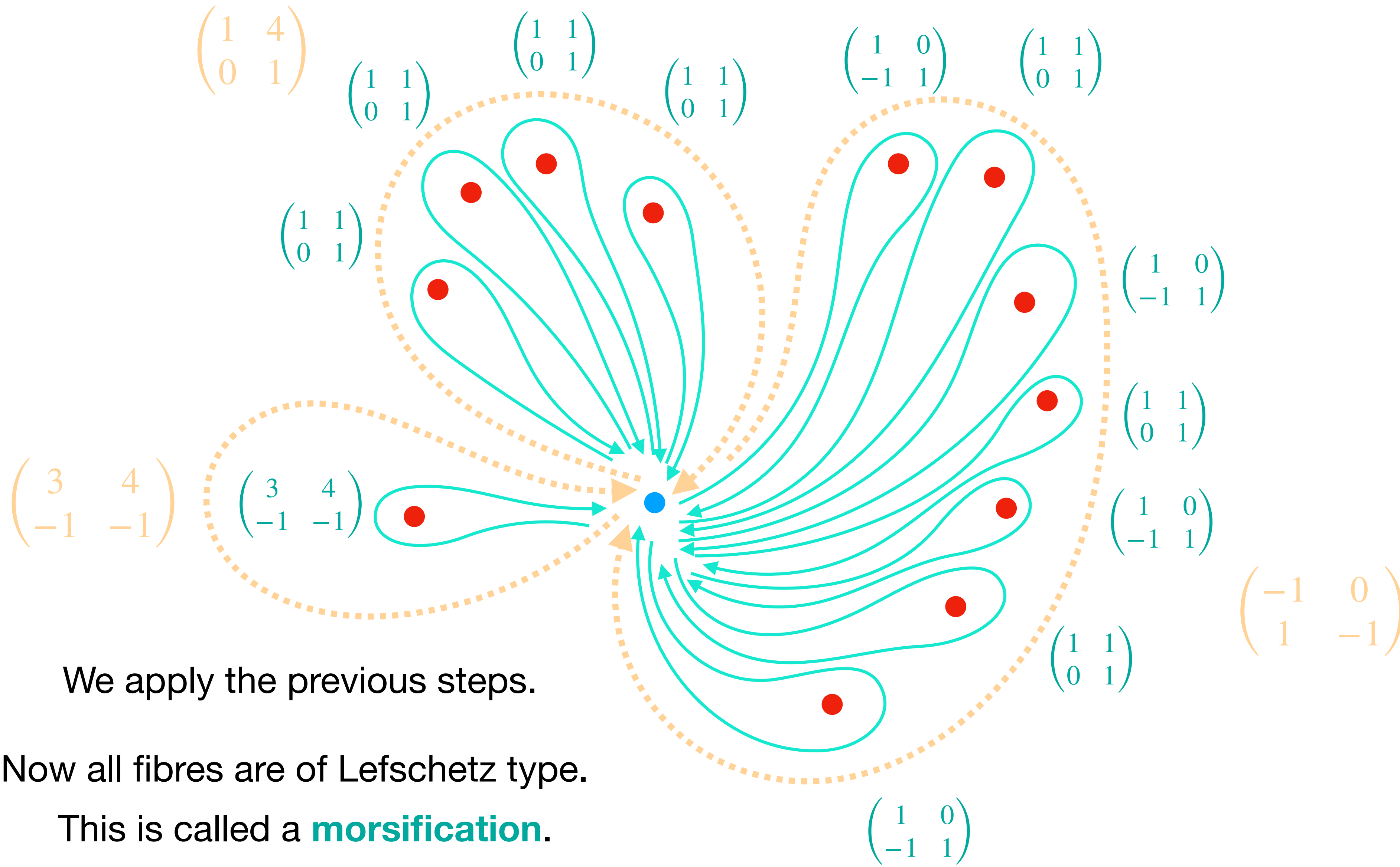
We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



As the deformation is smooth,
the topology is the same:
$H_2(S) \simeq H_2(\tilde{S})$.

The general case: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



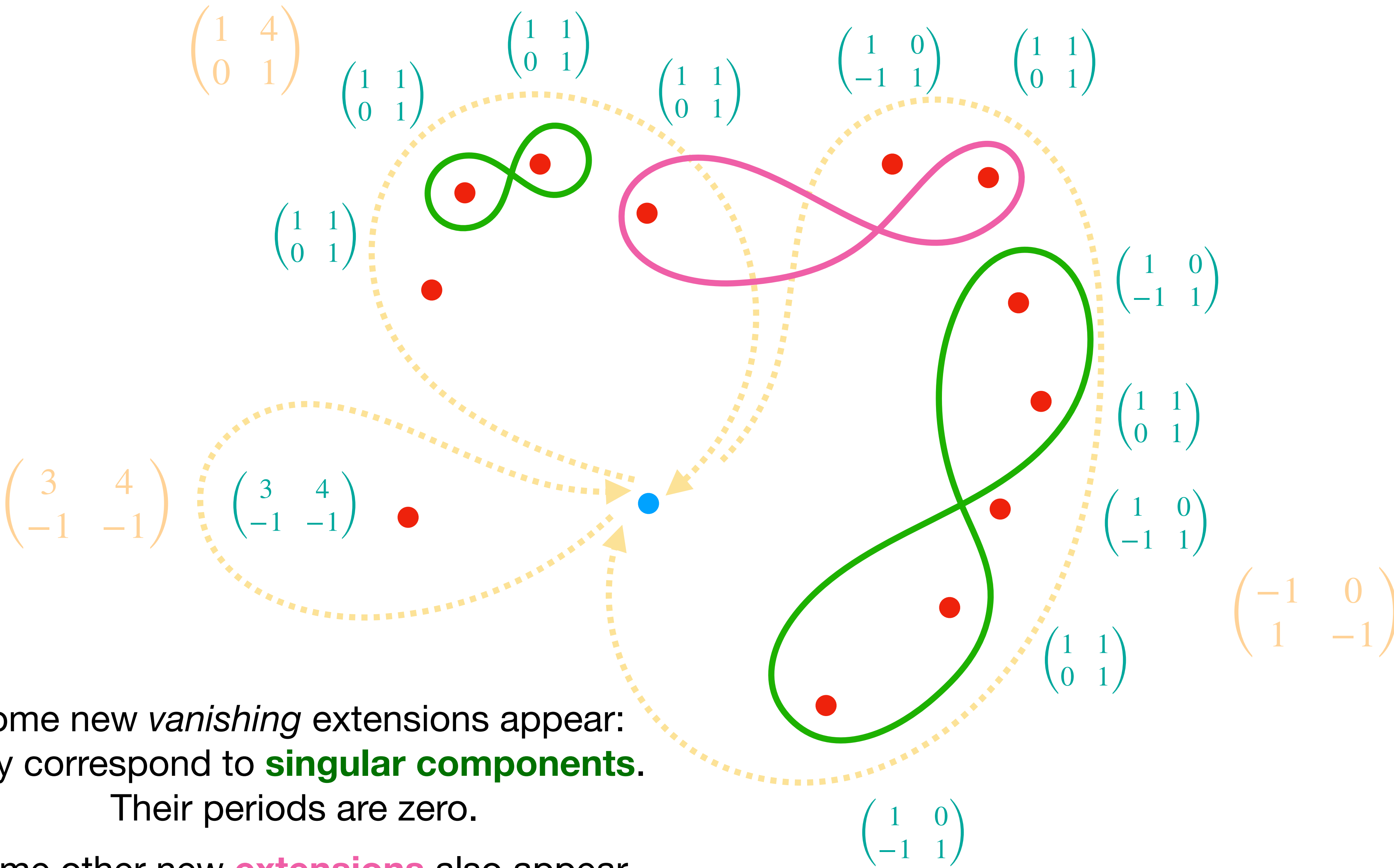
We apply the previous steps.

Now all fibres are of Lefschetz type.

This is called a **morsification**.

The general case: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



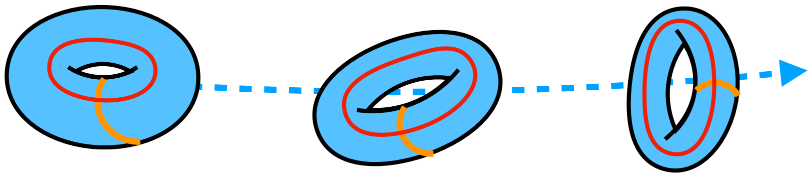
Some new *vanishing* extensions appear:
they correspond to **singular components**.
Their periods are zero.

Some other new **extensions** also appear.

The general case

Theorem [Moishezon 1977]: Morsifications always exist.

Monodromy preserves the intersection product



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\iff \quad ad - bc = 1$$

The monodromy matrix is in $\mathrm{SL}_2(\mathbb{Z})$.

Kodaira classification [1963]

$I_\nu, \nu \geq 1$	$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$	U^ν
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	VU
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	VUV
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$(VU)^2$

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

...

Theorem [Cadavid, Vélez 2009]:
The monodromy of the morsification is determined by the monodromy of S.

only cycles with nonzero periods

Theorem: The sublattice of $H_2(S)$ generated by **extensions** of S, the **section**, the **fibre** and **singular components** has full rank.

Theorem [Cadavid, Vélez 2009]:

The monodromy of the morsification is determined by the monodromy of S.

$$\begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \\ I_1 : U & I_4 : U^4 & I_1^* : UVUVUV \end{array}$$

morsification

$$\begin{array}{cccccccccccc} \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ \ell'_1 & \ell'_{21} & \ell'_{22} & \ell'_{23} & \ell'_{24} & \ell'_{31} & \ell'_{32} & \ell'_{33} & \ell'_{34} & \ell'_{35} & \ell'_{36} & \ell'_{37} \end{array}$$

In particular **we do not need to find an explicit realisation of the morsification**!

The algorithm

1. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \setminus \Sigma, b)$
2. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
3. Glue thimbles together to obtain **extension cycles** of $H_2(S)$.
4. Integrate the **periods** on these cycles.

5. From the monodromy type of ℓ_{i*}, recover the monodromy matrices of a **morsification** $\tilde{S}$.
6. Glue thimbles together to obtain **extension cycles** of $H_2(\tilde{S})$.
7. Recover the homology $H_2(\tilde{S})$ of the morsification (**extensions** + **fibre** + **section**).
8. Describe the extensions of $H_2(S)$ in terms of the extensions of $H_2(\tilde{S})$.

9. Recover the periods of all of $H_2(S) \simeq H_2(\tilde{S})$.

This allows for the (heuristic) computation of certain algebraic invariants of the elliptic surface (Néron-Severi group, Mordell-Weil group, …)

Implemented in the **lefschetz-family** Sagemath package, available on my webpage.

Thank you for listening!