Cryptanalysis of multivariate signatures: Singular points of UOV and VOX

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Context

NIST Post-quantum competition

- First NIST post-quantum standards: 2022
  - 2 lattice-based signatures (Dilithium, Falcon)
  - a hash-based signature (SPHINCS+)

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Our approach

Study UOV to derive results on schemes related to UOV.

Building cryptography from (quantum-)hard problems

Multivariate Quadratic Problem - MQ(n, m, q)

Find **a** solution (if any) $\boldsymbol{x} \in \mathbb{F}_q^n$ to a system of m quadratic equations in n variables

$$\mathcal{P}(\boldsymbol{x}) = 0 \in \mathbb{F}_q^m$$

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- The public key $\mathcal{P}$ is an instance of MQ(n, m, q), $n > m$.
- The secret key $\mathcal{S}$ enables, for all $\boldsymbol{t} \in \mathbb{F}_q^m$, to efficiently find $\boldsymbol{x} \in \mathbb{F}_q^n$ s.t. $\mathcal{P}(\boldsymbol{x}) = \boldsymbol{t}$

UOV: Original formulation

Unbalanced Oil and Vinegar [Kipnis, Patarin, Goubin, 1999]

Secret key: - m quadratic polynomials $\boldsymbol{x}^T F_i \boldsymbol{x} \in \mathbb{F}_q[x_1, \ldots, x_n]$
linear in $x_1, \ldots, x_m$.
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Figure 1: UOV key pair in $\mathbb{F}_{257}$

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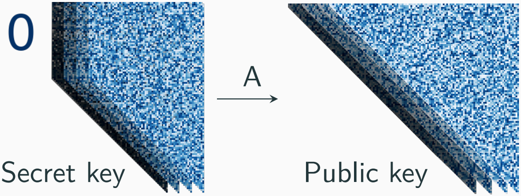

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Naming conventions and parameters

With $I = \langle p_1(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x}) \rangle$, define the UOV variety:

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Characterisation of the secret key **[Kipnis, Shamir 1998]**

Trapdoor: linear subspace $\mathcal{O} \subset \mathbb{F}_q^n$ of dimension m such that

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Security level	I	III	V
Classical gates	2^{143}	2^{207}	2^{272}

Contributions

One vector to full key recovery in polynomial time [P. 2023]

From **one vector** in $\mathcal{O}$, return a basis of $\mathcal{O}$ in polynomial time.

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Subfield attack on QR-UOV$\hat{\dagger}$ [P. 2024]

Identified a weakness in a **structured variant** of UOV$\hat{\dagger}$submitted to the additional NIST call for signature schemes [1]:

- Broken on a laptop in $0.3s, 1.35s, 0.56s$ (level I, III, V).

[1] [Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud, Patarin, 2023]

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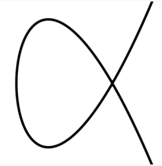
Singular points



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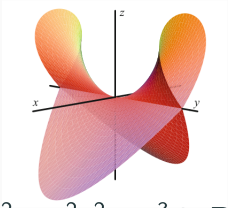

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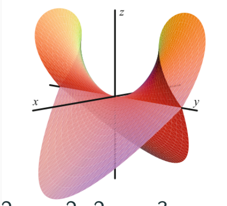

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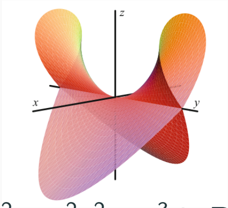

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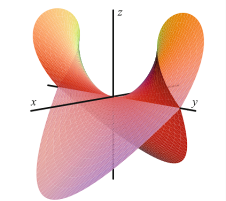

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Let $I = \langle p_1, \ldots, p_m \rangle$ be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$.
$\boldsymbol{x} \in V(I) \setminus \{0\}$ is singular if $\mathrm{Jac}_{\mathcal{P}}(\boldsymbol{x})$ has rank less than $n - m$.

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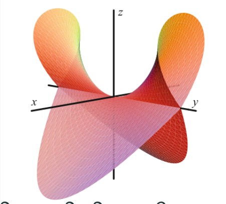

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$$\mathrm{Jac}_{\mathcal{P}}(\boldsymbol{x}) = \left(\frac{\partial}{\partial x_j} p_i(\boldsymbol{x}) \right) \in \mathbb{K}[x_1, \dots, x_n]^{m \times n}$$

Structured equations yield a structured Jacobian

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Where $J_1 \in \mathbb{F}_q[x_{m+1}, \ldots, x_n]^{m \times m}$ and $J_2 \in \mathbb{F}_q[x_1, \ldots, x_n]^{m \times n-m}$.

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Determinantal ideal

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Dimension of the singular locus

Under a genericity assumption, [FSS13][1] yields

$$\dim\left(\mathrm{Sing}(V(I)) \cap \mathcal{O}\right) = 3m - n - 1 > 0$$

[1] Faugère, Safey El Din, Spaenlehauer, 2013, Theorem 10

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These systems may be solved with Gröbner bases computations.

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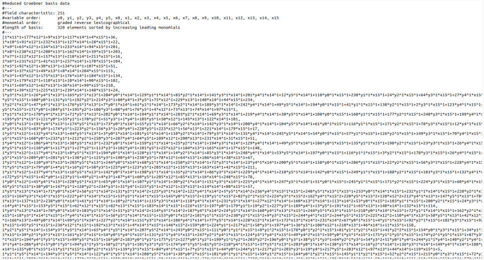

```
#Reduced Groebner basis data
#---
#field characteristic: 251
#variable order:      y0, y1, y2, y3, y4, y5, x0, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14, x15
#monomial order:      graded reverse lexicographical
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1*x7^1+112*x12^1+157*x13^1+218*x14^1+211*x15^1+16,
1*x6^1+231*x12^1+41*x13^1+237*x14^1+170*x15^1+104,
1*x5^1+92*x12^1+30*x13^1+134*x14^1+107*x15^1+51,
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Figure 2: First 30 polynomials (out of 320) in a grevlex Gröbner basis for the system $\mathcal{B}(\mathcal{P}), m = 7, n = 17, q = 251$ obtained with **msolve**.

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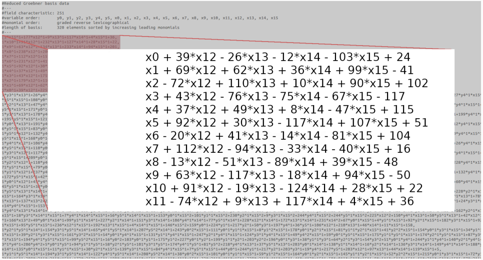

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Are Gröbner bases overkill for this problem?

Self-diagnosis

If one or more of the below applies to you:

- I am terrified by polynomial systems!
- I have been traumatized by the $F4/F5$ algorithms!
- I really really love linear algebra!
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Motivation

Small field: Gröbner basis computation improved by enumeration.

An enumerative approach

Bihomogeneous modeling

$$\boldsymbol{x} \in \mathrm{Sing}(V(I)) \iff \begin{cases} \boldsymbol{x} \in \mathbb{F}_q^n, \boldsymbol{y} \in \mathbb{F}_q^m \\ \mathcal{P}(\boldsymbol{x}) = 0 \\ \boldsymbol{y}^T \mathrm{Jac}_{\mathcal{P}}(\boldsymbol{x}) = 0 \end{cases}$$

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$$\implies \boldsymbol{x} \text{ is an eigenvector of } P_m^{-1}\sum_{i=1}^{m-1} y_i P_i.$$

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Kipnis-Shamir revisited

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$\boldsymbol{x}$ is an eigenvector of $P_m^{-1} \sum_{i=1}^{m-1} y_i P_i$ and $\boldsymbol{x} \in V(I)$.

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Expected cost **[P. 2024]**

If $\dim \operatorname{Sing}(V(I)) = d$, find $\mathbb{F}_q$-rational singular points by enumerating all $(y_1, \ldots, y_{m-1}) \in \mathbb{F}_q^{m-1}$ in time $O(q^{m-1-d} mn^2)$

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What did we bring to the table ?

- Highlight heuristics and limits of Kipnis-Shamir.
- Gröbner bases attack works if solutions are not $\mathbb{F}_q$-rational
- Framework enables attacks on "perturbed" keys

  $\Longrightarrow$ we can attack other schemes.

The $\hat{+}$ perturbation

UOV$^{\hat{+}}$ **[Faugère, Macario-Rat, Patarin, Perret 2022]**

Take a UOV secret key, replace t equations by uniformly random equations, and mix the equations.

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Methodology of the security analysis

Let $\mathcal{P}$ be a UOV$^{\hat{+}}$public key defining an ideal $I = \langle p_1, \ldots, p_m \rangle$.
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Motivation

This methodology justifies an aggressive choice of parameters for improved efficiency compared with UOV.

Singular points attack and asymptotic result **[P. 2024]**

Singular points of $\hat{\mathcal{F}} \circ A$ leak the trapdoor without inverting $\mathcal{S}$:
Our attack requires $\boldsymbol{O(q^{2t} n^{\omega})}$ operations versus claimed $\boldsymbol{q^{3t}}$.

[3] [Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud, Patarin, 2023]

New attack on VOX/UOV$\hat{+}$

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For parameters submitted to NIST for VOX[3]:

Parameters	I	III	V
Target (classical gates)	2^{143}	2^{207}	2^{272}
This work (classical gates)	$\mathbf{2^{121}}$	$\mathbf{2^{167}}$	$\mathbf{2^{221}}$

3 [Cogliati, Faugère, Fouque, Goubin, Larrieu, Macario-Rat, Minaud, Patarin, 2023]

Thank you for your attention!

One vector to full key recovery in polynomial time [P. 2023]

From **one vector** in $\mathcal{O}$, return a basis of $\mathcal{O}$ in polynomial time.

Singular points of UOV and $\text{UOV}^{\hat{+}}$ [P. 2024]

- $V(I)$ has a large singular locus.
- Singular points of $\text{UOV}^{\hat{+}}$ yield faster attacks.
- One vector to full key recovery on $\text{UOV}^{\hat{+}}$ in $O(q^t n^\omega)$.

Recap of the attack

- Find a weakness using determinantal ideals.
- Solve bihomogeneous polynomial systems.

Thank you for your attention!

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Subfield attack on QR-UOV$^{\hat{\dagger}}$ [P. 2024]

Weakness in a structured variant of UOV$^{\hat{\dagger}}$ submitted to NIST:

- Broken on a laptop in $\mathbf{0.3}s$, $\mathbf{1.35}s$, $\mathbf{0.56}s$ (level I, III, V).
- Attack new parameters by factoring the degree of extension.

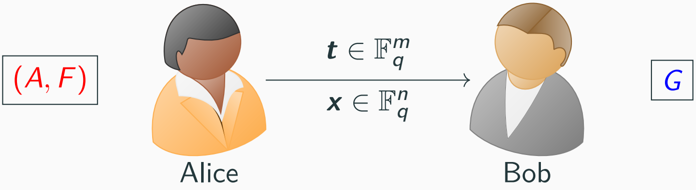
Bonus

UOV: Signing process

Signing

A signature for the message $\boldsymbol{t} \in \mathbb{F}_q^m$ is a vector $\boldsymbol{x} \in \mathbb{F}_q^n$ such that

$$1 \leq i \leq m, G_i(\boldsymbol{x}) = t_i$$



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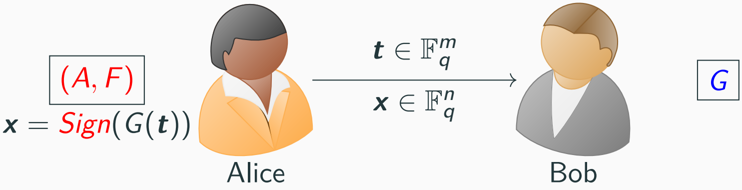

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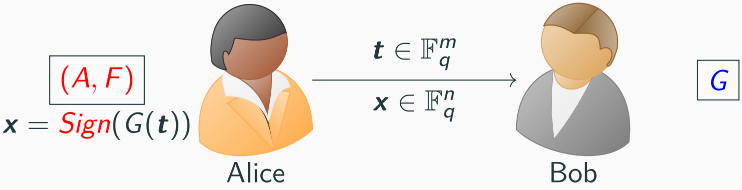
  Alice returns $\boldsymbol{x} = A^{-1}y$



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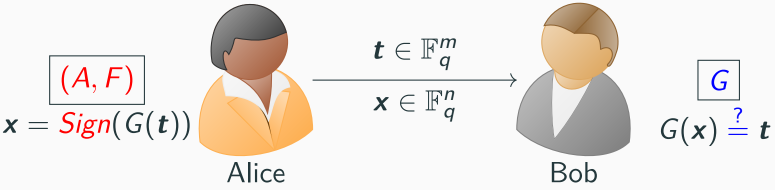

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Hash-and-sign

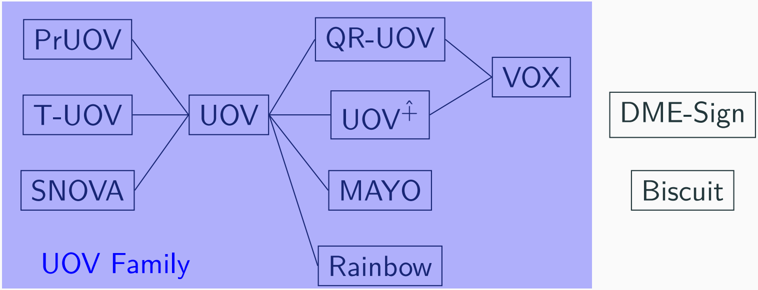
In practice, $\boldsymbol{t} = \mathcal{H}(M), M \in \{0,1\}^*$

UOV: Parameters

	NIST SL	n	m	$\mathbb{F}_q$	$\lvert$pk$\rvert$ (bytes)	$\lvert$sk$\rvert$ (bytes)	$\lvert$cpk$\rvert$ (bytes)	$\lvert$sig+salt$\rvert$ (bytes)
`ov-Ip`	1	112	44	$\mathbb{F}_{256}$	278 432	237 912	43 576	128
`ov-Is`	1	160	64	$\mathbb{F}_{16}$	412 160	348 720	66 576	96
`ov-III`	3	184	72	$\mathbb{F}_{256}$	1 225 440	1 044 336	189 232	200
`ov-V`	5	244	96	$\mathbb{F}_{256}$	2 869 440	2 436 720	446 992	260

Figure 3: Modern UOV[Beullens, Chen, Hung, Kannwischer, Peng, Shih, Yang 2023]

Multivariate Post-Quantum Zoo at NIST



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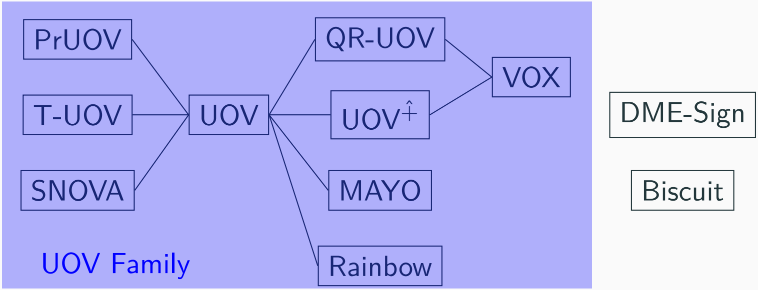

The UOV family

- "Multi-layer structure": Rainbow

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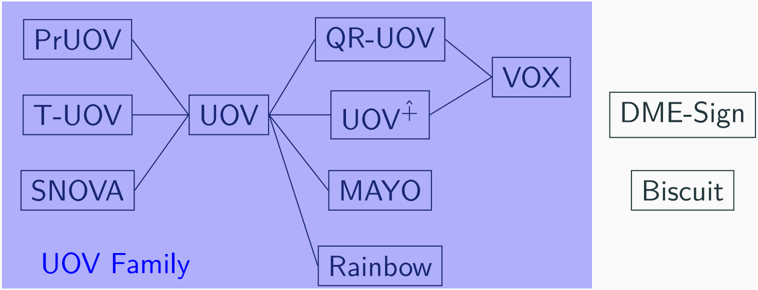

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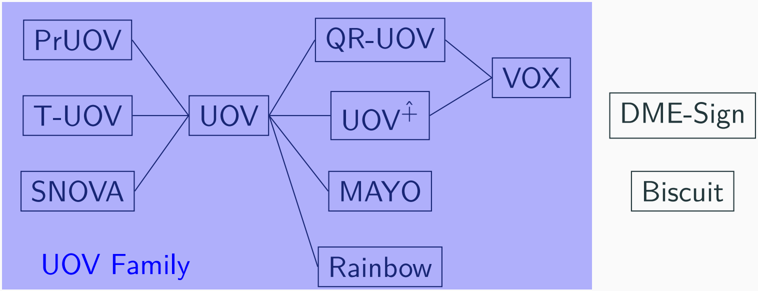

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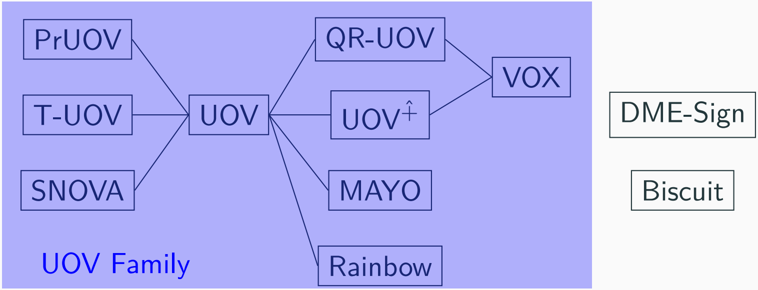

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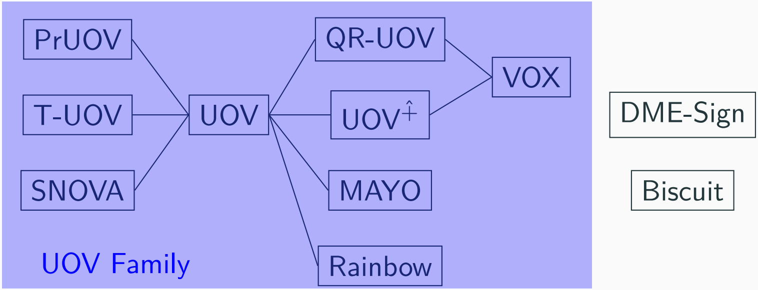

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- Formal security proof: T-UOV, PrUOV