Solving parameter-dependent semi-algebraic systems with Hermite matrices

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Solving parametric polynomial systems with inequalities

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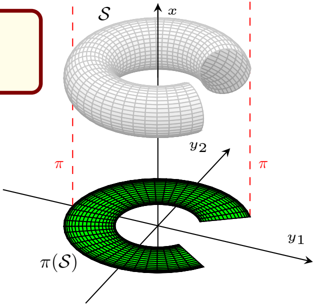
- $\boldsymbol{y} = (y_1, \ldots, y_t)$ are parameters
- $\boldsymbol{x} = (x_1, \ldots, x_n)$ are unknowns
- $\pi \colon (\boldsymbol{y},\boldsymbol{x}) \mapsto \boldsymbol{y}$ the $\boldsymbol{y}$-coordinate projection



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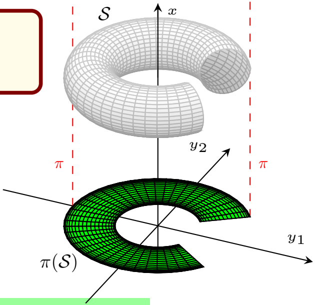

Assumption: For a *generic* choice of parameters $\eta \in \mathbb{C}^t$, $\boldsymbol{f}(\eta, \cdot) = 0$ is zero-dim

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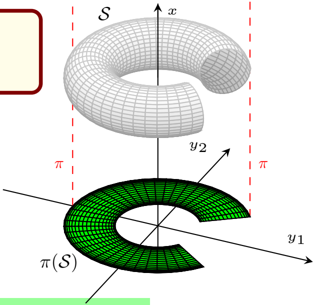

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Goals

- Classify the possible number of real solutions depending on the values of the parameters
- Describe regions of the parameter space $\mathbb{R}^t$ over which the number of solutions is invariant

→ **Applications** in Robotics, Computer Vision, Physics,...

Real Solution Classification

Given a semi-algebraic (s.a) set $\mathcal{S} \subseteq \mathbb{R}^{t+n}$ defined by

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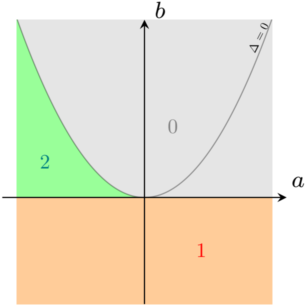

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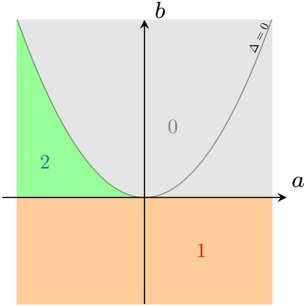

Valid Classification:

r	η	Φ
0	$(1, 1)$	$\Delta < 0 \vee (a > 0 \wedge b > 0)$
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Compute $(\Phi_i, \eta_i, r_i)_{1 \leq i \leq \ell}$ with Φ_i a s.a formula in $\mathbb{Q}[\boldsymbol{y}]$ defining the s.a set $\mathcal{T}_i \subseteq \mathbb{R}^t$, $\eta_i \in \mathcal{T}_i$ and $r_i \geq 0$ st,

- for all $\eta \in \mathcal{T}_i$, $\sharp \mathcal{S} \cap \pi^{-1}(\eta) = r_i$
- $\bigcup_{i=1}^{\ell} \mathcal{T}_i$ is dense in $\mathbb{R}^t$

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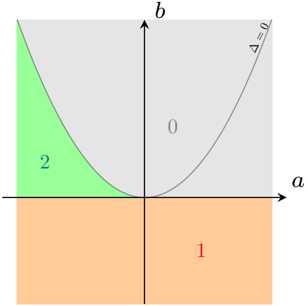

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Question: Can we achieve a simply exponential complexity **with inequalities**?

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- Implementation that solves instances that were previously out of reach

Hermite's quadratic forms

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Theorem (Hermite 1853)

$\mathrm{Sign}(\mathrm{Herm}(\boldsymbol{f}, g)) = \mathrm{TaQ}(g, \boldsymbol{f})$

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Remark:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c(g = 0) \\ c(g > 0) \\ c(g < 0) \end{bmatrix} = \begin{bmatrix} \mathrm{TaQ}(1, \boldsymbol{f}) \\ \mathrm{TaQ}(g, \boldsymbol{f}) \\ \mathrm{TaQ}(g^2, \boldsymbol{f}) \end{bmatrix}$$

where $c(g \diamondsuit 0) := \sharp\{x \mid \boldsymbol{f}(x) = 0 \wedge g(x) \diamondsuit 0\}$

Parametric Hermite matrices

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→ The set B of all monomials in $\boldsymbol{x}$ that are not reducible by $\mathrm{lm}(G)$ is a basis of $A_{\mathbb{K}}$

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Let H be a real symmetric matrix of rank r s.t its r first leading principal minors $M_1, \ldots, M_r \neq 0$. Then, $\mathrm{Sign}(H) = r - 2v$ *with* $v = \mathrm{Var}(1, M_1, \ldots, M_r)$*.*

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Key idea

- Over a **connected component** of the s-a set defined by the **non-vanishing locus** of the leading principal minors of $\mathcal{H}_1, \mathcal{H}_g, \mathcal{H}_{g^2}$, c_η is invariant

→ Sample one point in every **connected component** using [Le, Safey El Din 2022]

→ Deduce formulas for the classification from the sign patterns of these minors

Algorithm for $s = 1$

Algorithm 1: Real Solution Classification for 1 inequality

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3 $\mathrm{Minors} \leftarrow$ **LeadPrincMinors**$(U^t \mathcal{H}_1 U, U^t \mathcal{H}_g U, U^t \mathcal{H}_{g^2} U)$

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for $\eta \in L$ **do**

$\quad T_\eta \leftarrow \big(\mathrm{Sign}(\mathcal{H}_1(\eta)), \mathrm{Sign}(\mathcal{H}_g(\eta)), \mathrm{Sign}(\mathcal{H}_{g^2}(\eta))\big)^t$

$\quad$ Solve $M \cdot c_\eta = T_\eta$ to compute $r_\eta := c(g(\eta, \cdot) > 0)$

$\quad \Phi_\eta \leftarrow$ sign pattern of Minors evaluated in η

end

return $(\Phi_\eta, \eta, r_\eta)_{\eta \in L}$

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Problem

Need to compute 3^s Hermite matrices $\rightarrow$ exceed our target complexity (polynomial in s)

How to avoid unrealizable sign conditions

$d := \max(\deg f_i, \deg g_j)$

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- Adaptation of the Sign Determination algorithm of [Basu, Pollack, Roy 2006]
- At each step, add a new inequality g_i and use the sample points routine to determine the unrealizable sign conditions
- The size of the system is maintained to be less than ρ

$\rightarrow$ Control the needed number of Hermite matrices.

Practical Results

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				Hermite	RF	RRC
n	t	s	d			
2	2	2	2			
2	2	3	2			
3	2	1	2			
3	2	2	2			
2	3	2	2			
3	3	1	2			
2	2	1	3			
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Table: Generic dense system

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				Hermite		RF	RRC
n	t	s	d	**hm**	**det**	**dv**	**bp**
2	2	2	2	0.15 s	0.1 s	0.14 s	0.11 s
2	2	3	2	0.7 s	0.1 s	0.9 s	1 s
3	2	1	2	0.5 s	0.4 s	10 mn	7 mn
3	2	2	2	3 s	0.4 s	10 mn	14 mn
2	3	2	2	0.3 s	0.1 s	0.7 s	0.2 s
3	3	1	2	1 s	6 s	>50 h	>50 h
2	2	1	3	0.9 s	0.8 s	52 mn	47 s
2	2	2	3	5 s	1 s	57 mn	2 mn

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n	t	s	d	**hm**	**det**	**min**	**sp**	**dv**	**cad**	**bp**
2	2	2	2	0.15 s	0.1 s	0.4 s	5 s	0.14 s	2 s	0.11 s
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3	2	1	2	0.5 s	0.4 s	9 s	33 s	10 mn	11 mn	7 mn
3	2	2	2	3 s	0.4 s	1 mn	57 s	10 mn	13 mn	14 mn
2	3	2	2	0.3 s	0.1 s	4 s	18mn	0.7 s	>50 h	0.2 s
3	3	1	2	1 s	6 s	4 mn	>50 h	>50 h	>50 h	>50 h
2	2	1	3	0.9 s	0.8 s	30 s	3mn	52 mn	57 mn	47 s
2	2	2	3	5 s	1 s	5 mn	6 mn	57 mn	1h 16 mn	2 mn

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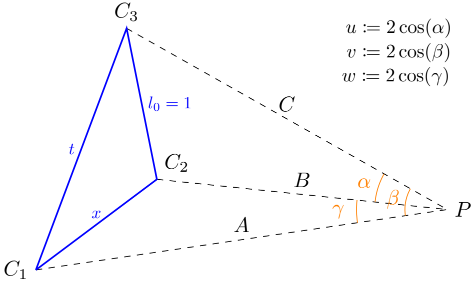
Perspective-3-Point Problem



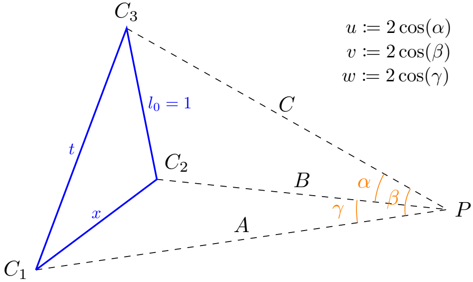
Perspective-3-Point Problem

$$\begin{cases} 1 &= A^2 + B^2 - ABu \\ t &= B^2 + C^2 - BCv \\ x &= A^2 + C^2 - ACw \end{cases}, \quad A, B, C > 0$$

with the constraints:

$$x, t > 0, \quad -2 < u, v, w < 2$$

- 3 unknowns : A, B, C
- 5 parameters : x, t, u, v, w



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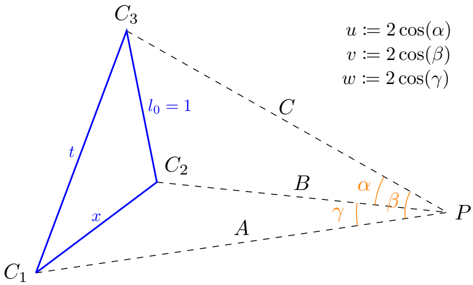

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Results

- A complete classification in less than **one hour** in the isosceles case ($t = 1$)
- In the general case: able to compute the Hermite matrices and derive the semi-algebraic conditions from their minors.

→ **Next step**: compute all the possible number of solutions and determine which conditions are feasible using the sample points routine