Grobner Bases for polytopal affinoid algebras

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Joint work with M. Barkatou and T. Vaccon

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Outline

1. Introducing polytopal affinoid algebras (PAA)

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2. Grobner theory for ideals in PAA
    - Definition of Grobner bases
    - Multivariate division algorithm
    - S-pairs criterion
    - Buchberger-like algorithm

Non-archimedean setting

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Notations :

- $n \in \mathbb{N}^*$
- $\mathbf{X} := (X_1, \ldots, X_n)$
- $u \in \mathbb{Z}^n, \mathbf{X}^u := X_1^{u_1} \cdots X_n^{u_n}$

Reminder: affinoid algebras

- Tate's 1971 famous paper: "Rigid analytic spaces"

Let $r = (r_1, \cdots, r_n) \in \mathbb{Q}^n$

Definition (Tate Algebra)

$$K_r\{\mathbf{X}\} := \{ \sum_{u \in \mathbb{N}^n} a_u \mathbf{X}^u \ : \ a_u \in K, \ \mathrm{val}(a_u) - r \cdot u \xrightarrow[|u| \to +\infty]{} +\infty \}$$

Elements of $K_r\{X\}$ are power series converging on the product of closed balls $B(0, |\pi|^{r_1}) \times \cdots \times B(0, |\pi|^{r_n})$.
Gauss valuation : $\mathrm{val}_r(\sum a_u \mathbf{X}^u) := \min_u (val(a_u) - r \cdot u)$

Definition (Affinoid Algebra)

Any quotient of a Tate algebra equipped with the induced norm.

Polytopal affinoid algebras

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Elements of $K\{\mathbf{X}; P\} =$ "series $\sum_{u \in \mathbb{Z}^n} a_u \mathbf{X}^u$ converging on $\mathrm{val}^{-1}(P) \subset (K^\times)^n$".

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$$\forall r \in P, \quad \mathrm{val}_{\mathbf{r}}\left(\sum_{u \in \mathbb{Z}^n} a_u \mathbf{X}^u\right) := \min_{u \in \mathbb{Z}^n} \mathrm{val}(a_u) - r \cdot u,$$

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$$\forall f \in K\{\mathbf{X}; P\}, \quad \text{val}_P(f)$$

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$$\forall f \in K\{\mathbf{X}; P\}, \quad \mathrm{val}_P(f) := \min_{r \in Vert(P)} \mathrm{val}_{\mathbf{r}}(f)$$

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$$\forall f \in K\{\mathbf{X}; P\}, \quad \mathrm{val}_P(f) := \min_{r \in Vert(P)} \mathrm{val}_\mathbf{r}(f)$$

$\mathrm{val}_P(f)$ is the minimum valuation reached by f on $\mathrm{val}^{-1}(P)$.

Motivations behind PAA

- Polytopal **affinoid** algebras are **affinoid** algebras !
- Introduced in :
  - Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. "Non-archimedean amoebas and tropical varieties". In: (2006)
  - Walter Gubler. "Tropical varieties for non-Archimedean analytic spaces". In: *Inventiones mathematicae* 169.2 (2007), pp. 321–376
- Tropical analytic geometry

Related previous work

Grobner bases for ideals in affinoid algebras (local team):

1. Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "Gröbner bases over Tate algebras". In: ISSAC 2019 - Beijing, China
2. Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "Signature-Based Algorithms for Gröbner Bases over Tate Algebras". In: ISSAC 2020 - Kalamata, Greece
3. Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "On FGLM Algorithms with Tate Algebras". In: ISSAC 2021 - Virtual Event, Russian Federation
4. Xavier Caruso, Tristan Vaccon, and Thibaut Verron. "On Polynomial Ideals and Overconvergence in Tate Algebras". In: ISSAC 2022 - Villeneuve-d'Ascq, France
5. Tristan Vaccon and Thibaut Verron. "Universal Analytic Gröbner Bases and Tropical Geometry". In: ISSAC 2023 - Tromsø, Norway

Main idea: use the valution to order terms

*Tate algebra setting

Fix $\leq_m$ a monomial order on $\mathbb{N}^n$.
Order terms $a\mathbf{X}^u, b\mathbf{X}^v \in K_r\{\mathbf{X}\}$ by the following:

$$a\mathbf{X}^u \leq b\mathbf{X}^v \iff (\mathrm{val}_r(a\mathbf{X}^u) > \mathrm{val}_r(b\mathbf{X}^v)) \text{ or } (\mathrm{val}_r(a\mathbf{X}^u) = \mathrm{val}_r(b\mathbf{X}^v) \text{ and } \mathbf{X}^u \leq_m \mathbf{X}^v)$$

Main idea continued

There is no "monomial order on $\mathbb{Z}^n$".
Recall that a monomial order is a total order that satisfies, for all r, s, t:

1. $0 \leq t$
2. $r < s \implies r + t < s + t$

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Proof.
Assume such an order exists. Take any $a \neq 0$. By (1), $0 < a$, and by (2) $-a < 0$, contradiction. □

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But we can use "generalized monomial order" as defined by Pauer-Unterkircher in:
Franz Pauer and Andreas Unterkircher. "Gröbner bases for ideals in Laurent polynomial rings and their application to systems of difference equations". In: (1999)

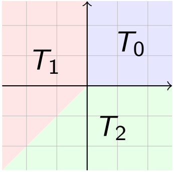
Generalized monomial order

Put $T := \{\mathbf{X}^u,\ u \in \mathbb{Z}^n\}$.

Definition

A conic decomposition of T is a finite family $(T_i)_{i \in I}$ of finitely generated submonoids of T such that

1. for each i, the only invertible element in the monoid T_i is 1 and the group generated by T_i is T.
2. the union of all the T_i's equal T



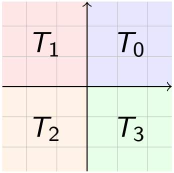


Figure: Conic decompositions for $n = 2$

Generalized monomial order

Definition

Let $(T_i)_{i\in I}$ be a conic decomposition of T. A generalized monomial order (or g.m.o) on T for the decomposition $(T_i)_{i\in I}$ is a total order $<$ on T such that

1. $\forall t \in T, 1 \leq t$
2. $\forall r \in T, \forall i \in I, (s, t \in T_i \text{ and } r < s) \implies rt < st$

For $f \in K[\mathbf{X}^{\pm 1}], K\{\mathbf{X}; P\}\ldots$ and $t \in T$, we generally have :

$$\mathrm{lt}(tf) \neq t\mathrm{lt}(f)$$

Definition

For $i \in I$ define $T_i(f) := \{t \in T,\ \mathrm{lm}(tf) \in T_i\}$.

The compatibility condition implies ...

Take $i \in I$, $f \in K\{\mathbf{X}; P\}$, $u, v \in T_i(f)$.

Lemma

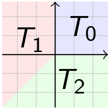
Write $\mathrm{Im}(uf) = ut_u \in T_i$ *and* $\mathrm{Im}(vf) = vt_v \in T_i$ *for some monomials* t_u, t_v *of* f*. Then* $t_u = t_v$*.*

Definition

For $t \in T_i(f)$, define $\mathrm{Im}_i(f) := \mathrm{Im}(tf)t^{-1}$.

An example

Take the conic decomposition:



Define a g.m.o for this decomposition by the formula :

$$(x_1, y_1) \leq (x_2, y_2) \iff -\min(0, x_1, y_1) \leq -\min(0, x_2, y_2) \text{ or }$$
$$-\min(x_1, y_1) = -\min(x_2, y_2) \text{ and } (x_1, y_1) \leq_{lex} ($$

For $f = xy^{-2} + x^{-2}y^{-2} + x^{-1}y^{-2} + y^2$:

- $xy^{-2} > x^{-1}y^{-2} > x^{-2}y^{-2} > y^2$,
- $\mathrm{lm}_0(f) = \mathrm{lm}_2(f) = \mathrm{lm}(f) = xy^{-2}$
- $\mathrm{lm}_1(f) = x^{-2}y^{-2}$
- $T_0(f) = x^2y^2T_0$
- $T_1(f) = xy^2T_1, \; T_2(f) = x^2y^2T_2$

Main idea revisited

*polytopal algebra setting

Fix $\leq_\omega$ a generalized monomial order on $\mathbb{Z}^n$.
Order terms $a\mathbf{X}^u, b\mathbf{X}^v \in K\{\mathbf{X}; P\}$ by the following:

$$a\mathbf{X}^u \leq b\mathbf{X}^v \iff (\mathrm{val}_P(a\mathbf{X}^u) > \mathrm{val}_P(b\mathbf{X}^v)) \text{ or } (\mathrm{val}_P(a\mathbf{X}^u) = \mathrm{val}_P(b\mathbf{X}^v) \text{ and } \mathbf{X}^u \leq_\omega \mathbf{X}^v)$$

Grobner basis

Fix a g.m.o $\leq$ for a conic decomposition $(T_i)_{i\in I}$.
Let J be an ideal in $K\{\mathbf{X}; P\}$ and G be a finite subset of $J \setminus \{0\}$.
We always have the containment:

$$\{\mathrm{lm}(f), f \in J\} \supset \bigcup_{g\in G} \{\mathrm{lm}(tg), t \in T\} = \bigcup_{g\in G, i\in I} T_i(g)\mathrm{lm}_i(g)$$

Definition

We say that G is a Gröbner basis of J when:

$$\{\mathrm{lm}(f), f \in J\} = \bigcup_{g\in G} \{\mathrm{lm}(tg), t \in T\} = \bigcup_{g\in G, i\in I} T_i(g)\mathrm{lm}_i(g)$$

Multivariate division algorithm in $K\{\mathbf{X}; P\}$

input : $f, g_1, \ldots, g_m \in K\{\mathbf{X}; P\}$
output: $q_1, \ldots, q_m, r$

$q_1, \ldots, q_m, r \leftarrow 0$;
while $f \neq 0$ **do**
  while $\exists (i,j) \in I \times [1, m]$ *such that* $\operatorname{lm}\left(\frac{\operatorname{lm}(f)}{\operatorname{lm}_i(g_j)} g_j\right) = \operatorname{lm}(f)$ **do**
    $t \leftarrow \frac{\operatorname{lt}(f)}{\operatorname{lt}_i(g_j)}$;
    $q_j \leftarrow q_j + t$;
    $f \leftarrow f - tg_j$;
  $r \leftarrow r + \operatorname{lt}(f)$
  $f \leftarrow f - \operatorname{lt}(f)$;
return $q_1, \ldots, q_m, r \in K\{\mathbf{X}; P\}$

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q_1, ..., q_m, r ← 0;
while f ≠ 0 do
    while ∃(i, j) ∈ I × [1, m] such that lm( (lm(f) / lm_i(g_j)) g_j ) = lm(f) do
        t ← lt(f) / lt_i(g_j);
        q_j ← q_j + t;
        f ← f − t g_j;
    r ← r + lt(f)
    f ← f − lt(f);
return q_1, ..., q_m, r ∈ K{X; P}
```

- $f = \sum q_j g_j + r$
- for all monomial t in r, $t \notin \bigcup_{i \in I, g \in G} T_i(g) \mathrm{lm}_i(g)$
- for all g_j and all monomial t in q_j, $\mathrm{lt}(t g_j) \leq \mathrm{lt}(f)$.

S-pairs: definition

Take $i \in I, f, g \in K\{\mathbf{X}; P\}$.

Lemma

The T_i-module

$$\mathrm{lm}_i(f)\, T_i(f) \cap \mathrm{lm}_i(g)\, T_i(g)$$

is finitely generated.

Let $U(i, f, g)$ be a finite system of generator.

Definition (S-pair)

For $v \in U(i, f, g)$:

$$S(i, f, g, v) := \mathrm{lc}_i(g) \frac{v}{\mathrm{lm}_i(f)} f - \mathrm{lc}_i(f) \frac{v}{\mathrm{lm}_i(g)} g.$$

S-pairs criterion

Let $h_1, \dots, h_m \in K\{\mathbf{X}; P\}$ and $i \in I$. For $1 \le j \le m-1$, let $U(i, h_j, h_{j+1})$ be a finite system of generators. Suppose that there are terms $\{t_1, \dots, t_m\}$, $u \in T_i$ and $c \in \mathrm{val}(K^\times)$ such that

- for all $j \in \{1, \dots, m\}$, $\mathrm{lt}(t_j h_j) = c_j u$ with $\mathrm{val}(c_j) = c$
- $\mathrm{lt}(\sum_{i=1}^m t_j h_j) < c_1 u$.

Then there are elements $d_j \in K$, $v_j \in U(i, h_j, h_{j+1})$ for $1 \le j \le m-1$ and $t'_m \in K\{\mathbf{X}; P\}$ such that:

1. $\sum_{j=1}^m t_j h_j = \sum_{j=1}^{m-1} d_j \frac{u}{v_j} S(i, h_j, h_{j+1}, v_j) + t'_m h_m$.
2. $\mathrm{val}_P(t'_m h_m) > \mathrm{val}_P(u c_1)$.
3. $\frac{u}{v_j} \in T_i$ for all $j < m$.
4. For all $j < m$, $\mathrm{val}\left(d_j \mathrm{lc}_i(h_j) \mathrm{lc}_i(h_{j+1})\right) \ge c$.

Buchberger algorithm in $K\{\mathbf{X}; P\}$

input : $J = (h_1, \dots, h_m)$ an ideal of $K\{\mathbf{X}; P\}$
output: a Gröbner basis of J

$H \leftarrow \{h_1, \dots, h_m\}$; $B \leftarrow \{(h_i, h_j),\ 1 \leq i < j \leq m\}$
while $B \neq \emptyset$ **do**
  $(f, g) \leftarrow$ element of B; $B \leftarrow B \setminus \{(f, g)\}$;
  for $i \in I$ **do**
    $U(i, f, g) \leftarrow$ finite set of generators of $\mathrm{lm}_i(f)T_i(f) \cap \mathrm{lm}_i(g)T_i(g)$;
    for $v \in U(i, f, g)$ **do**
      $\_, r \leftarrow$ division$(S(i, f, g, v), H)$;
      if $r \neq 0$ **then**
        $B \leftarrow B \cup \{(h, r),\ h \in H\}$;
        $H \leftarrow H \cup \{r\}$
return H