Symplectic singularities and diagonal invariants

Singularités symplectiques et invariants diagonaux

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based on joint work in progress with Ulrich Thiel*

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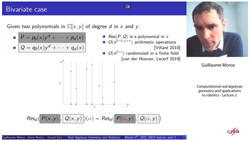

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Content

1. Complex reflection groups
2. Symplectic singularities
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TRR 195 SYMBOLIC TOOLS

Complex reflection groups

Let $\mathfrak{h}$ be a finite dimensional $\mathbb{C}$–vector space.

Definition

A **reflection** is an element $s \in \mathrm{GL}(\mathfrak{h})$, whose fixed point space is a hyperplane, that is,

$$\mathrm{codim}(\ker(\mathrm{id}_{\mathfrak{h}} - s)) = 1.$$

A **complex reflection group** over $\mathfrak{h}$ is a finite subgroup $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h})$, which is generated by reflections.

Classification (Shephard–Todd 1954)

- G1 $\mathfrak{S}_n$: the symmetric groups
- G2 $\mathfrak{C}_m$: the cyclic groups
- G2 $\mathfrak{G}(m, p, n)$: normal subgroups of $\mathfrak{S}_n \wr \mathfrak{C}_m$ with index p
- G4 … G37 exceptional groups

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Invariants

Let $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h})$ be a finite group.
$\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*)$ is the coordinate ring of $\mathfrak{h}$.

Definition

A polynomial $f \in \mathbb{C}[\mathfrak{h}]$ is called **invariant** under $\mathcal{W}$,
if $f(s(x)) = f(x)$ for all $s \in \mathcal{W}$, $x \in \mathfrak{h}$.
The ring of invariants is denoted by $\mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$.

$\mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$ is the coordinate ring of the orbit space $\mathfrak{h}/\mathcal{W}$.

Theorem (Shephard–Todd, Chevalley, Serre)

The following are equivalent:

- $\mathcal{W}$ is a complex reflection group.
- $\mathbb{C}[\mathfrak{h}]^{\mathcal{W}}$ is a polynomial algebra.
- $\mathfrak{h}/\mathcal{W}$ is a smooth variety.

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A *symplectic singularity* is obtained by replacing $\mathfrak{h}$ with $\mathfrak{h} \oplus \mathfrak{h}^*$.

W. R. Hamilton
(1805-1865)

The Hamiltonian equations of motion for a mechanical system with configuration space $\mathfrak{h}$ are deduced by studying energy functions on the cotangent bundle $\mathfrak{h} \oplus \mathfrak{h}^*$.

When the system has symmetries, given by a finite group $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h})$, those can be translated to $\mathfrak{h} \oplus \mathfrak{h}^*$ and exploited.

More motivations:
P. Etingof (2006) Lectures on Calogero–Moser systems. arXiv.

Symplectic singularities

A. Beauville[1]

A symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$ is defined by

$$\omega((y,x),(x',y')) := x'(y) - x(y').$$

$\mathcal{W} \hookrightarrow \mathrm{GL}(\mathfrak{h} \oplus \mathfrak{h}^*)$ has a **diagonal action**, where the matrices are represented as

$$\mathcal{W} \ni s \mapsto \begin{pmatrix} s & 0 \\ 0 & (s^t)^{-1} \end{pmatrix}.$$

Lemma

The orbit space $\mathfrak{h} \oplus \mathfrak{h}^*/\mathcal{W}$ is not smooth ($\Leftrightarrow$ has a singularity).

(Indeed, suppose that $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is generated by reflections $S_i = \begin{pmatrix} s_i & 0 \\ 0 & (s_i^t)^{-1} \end{pmatrix}$. Then $\det(S_i) = 1$, but the determinant of a reflection is a non–trivial root of unity. Conclude with Theorem.)

[1] A. Beauville (2000) Symplectic singularities. *Inv. Math.* 139, 541–549

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Computing diagonal invariants

The coordinate ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathcal{W}}$ of $\mathfrak{h} \oplus \mathfrak{h}^*/\mathcal{W}$ is not polynomial, but one can still compute generators, see for example[2].

From now on, we fix a basis $\{x_j\}$ for $\mathfrak{h}$ and dual basis $\{y_i\}$ for $\mathfrak{h}^*$.

Example (for $\mathfrak{h}$)

Let $a \in \mathbb{C}$ with $a^2 + a + 1 = 0$, that is, $a^3 = 1$. The exceptional reflection group $\mathcal{W} = G4$ has order 24 and is generated by

$$s_1 = \begin{pmatrix} a & 0 \\ -a-1 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & a+1 \\ 0 & a \end{pmatrix}.$$

The invariant ring $\mathbb{C}[x_1, x_2]^{\mathcal{W}}$ is polynomial with 2 generators.

[2] H. Derksen, G. Kemper (2002) Computational Invariant Theory. *Springer*.

Computing diagonal invariants

Example (for $\mathfrak{h} \oplus \mathfrak{h}^*$)

With King's algorithm in OSCAR[3], one obtains 8 fundamental invariants for $\mathbb{C}[x_1, x_2, y_1, y_2]^{\mathcal{W}}$:

$$f_1 = x_1\, y_1 + x_2\, y_2$$

$$f_2 = y_1\, y_2\, (y_1^2 - y_2^2 + (2\,a + 1)\, y_1\, y_2)$$
$$f_3 = (x_1^2 - x_2^2)\, (x_1^2 - x_2^2 + 4/3\, (2\,a + 1)\, x_1\, x_2)$$

$$f_4 = x_1\, y_1^3 + x_2\, y_2^3 - 3\, (x_2\, y_1 + x_1\, y_2) + (2\,a + 1)\, y_1\, y_2\, (x_1\, y_1 - x_2\, y_2) - (2\,a + 1)\, (x_1\, y_2^3 - x_2\, y_1^3)$$
$$f_5 = x_1^3\, y_1 + x_2^3\, y_2 - 3\, (x_2\, y_1 + x_1\, y_2) + (4\,a + 2)\, x_1\, x_2\, (x_1\, y_1 - x_2\, y_2)$$

$$f_6 = x_1^6 + x_2^6 + (4\,a + 2)\, x_1\, x_2\, (x_1^4 - x_2^4) - 5\, x_1^2\, x_2^2\, (x_1^2 + x_2^2)$$
$$f_7 = y_1^6 + y_2^6 + (4\,a + 2)\, y_1\, y_2\, (y_1^4 - y_2^4) - 5\, y_1^2\, y_2^2\, (y_1^2 + y_2^2)$$

$$f_8 = x_1^3\, y_2^3 - x_2^3\, y_1^3 - (2\,a + 1)\, (x_1\, y_1 - x_2\, y_2)\, (x_1^2\, y_2^2 - x_2^2\, y_1^2) - (x_1\, y_2 - x_2\, y_1)\, (x_1^2\, y_1^2 + x_2^2\, y_2^2 - 3\, x_1\, x_2\, y_1\, y_2).$$

The degrees of the fundamental invariants are

$$(1, 1),\ (0, 4),\ (4, 0),\ (1, 3),\ (3, 1),\ (0, 6),\ (6, 0),\ (3, 3).$$

[3] docs.oscar-system.org

Computing diagonal invariants

Group	#inv	$\mathbb{N}$-degrees	$\mathbb{N}\times\mathbb{N}$-degrees	#$\mathbb{Z}$-deg=0
G4	8	2, 4, 4, 4, 4, 6, 6, 6	(1,1), (0,4), (1,3), (3,1), (4,0), (0,6), (3,3), (6,0)	2
G5	12	2, 6, 6, 6, 8, 8, 8, 12, 12, 12, 12, 12	(1,1), (0,6), (3,3), (6,0), (1,7), (4,4), (7,1), (0,12), (3,9), (6,6), (9,3), (12,0)	4
G6	10	2, 4, 4, 6, 8, 8, 10, 10, 12, 12	(1,1), (0,4), (4,0), (3,3), (2,6), (6,2), (1,9), (9,1), (0,12), (12,0)	2
G7	12	2, 6, 8, 12, 12, 12, 12, 12, 14, 14, 16, 16	(1,1), (3,3), (4,4), (0,12), (0,12), (6,6), (12,0), (12,0), (1,13), (13,1), (2,14), (14,2)	4
G8	10	2, 6, 6, 8, 8, 8, 12, 12, 12, 12	(1,1), (1,5), (5,1), (0,8), (4,4), (8,0), (0,12), (4,8), (8,4), (12,0)	2
			⋮	
G26	21	2, 6, 6, 6, 8, 8, 8, 10, 10, 10, 12, 12, 12, 12, 12, 14, 14, 14, 14, 18, 18	(1,1), (0,6), (3,3), (6,0), (1,7), (4,4), (7,1), (2,8), (5,5), (8,2), (0,12), (3,9), (6,6), (9,3), (12,0), (1,13), (4,10), (10,4), (13,1), (0,18), (18,0)	5
G27	29	2, 6, 6, 8, 10, 10, 10, 12, 12, 12, 12, 12, 14, 14, 14, 16, 16, 16, 16, 18, 18, 20, 20, 22, 22, 26, 26, 30, 30	(1,1), (0,6), (6,0), (4,4), (2,8), (5,5), (8,2), (0,12), (3,9), (6,6), (9,3), (12,0), (4,10), (7,7), (10,4), (2,14), (5,11), (11,5), (14,2), (3,15), (15,3), (1,19), (19,1), (2,20), (20,2), (1,25), (25,1), (0,30), (30,0)	5
G28	40	2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12	(0,2), (1,1), (2,0), (0,6), (1,5), (2,4), (3,3), (4,2), (5,1), (6,0), (0,8), (1,7), (2,6), (3,5), (4,4), (4,4), (5,3), (6,2), (7,1), (8,0), (0,12), (1,11), (2,10), (3,9), (3,9), (4,8), (4,8), (5,7), (5,7), (6,6), (6,6), (7,5), (7,5), (8,4), (8,4), (9,3), (9,3), (10,2), (11,1), (12,0)	6
G29	44	2, 4, 4, 6, 8, 8, 8, 8, 8, 8, 8, 10, 10, 10, 10, 10, 10, 12, 12, 12, 12, 12, 12, 12, 12, 12, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 16, 16, 16, 16, 18, 18, 20, 20	(1,1), (0,4), (4,0), (3,3), (0,8), (2,6), (2,6), (4,4), (6,2), (6,2), (8,0), (1,9), (3,7), (5,5), (5,5), (7,3), (9,1), (0,12), (2,10), (4,8), (4,8), (6,6), (8,4), (8,4), (10,2), (12,0), (1,13), (3,11), (3,11), (5,9), (7,7), (7,7), (9,5), (11,3), (11,3), (13,1), (2,14), (6,10), (10,6), (14,2), (1,17), (17,1), (0,20), (20,0)	8

Theorem (M, Thiel)

Let F be a set of fundamental invariants for $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathcal{W}}$. Then: $\forall f \in F$ with $\deg(f) = (d, e)$, $\exists \tilde{f} \in F$ with $\deg(\tilde{f}) = (e, d)$.

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Open questions

- For fixed $(d, e) \in \mathbb{N}^2$, how many fundamental invariants with bidegree (d, e) are there?
- What properties do the symplectic singularities of the complex reflection groups $G1, \ldots, G37$ have?
- How to compute fundamental invariants for $G30$, ..., $G37$?

"One can learn a lot about a mathematical object
by studying how it behaves under small deformations."

B. Mazur

Definition

Let A, R be graded $\mathbb{C}$–algebras and $\mathfrak{m}$ be a maximal ideal in R. An R–algebra $\tilde{A}$ is called a **deformation** of A over R if $A \cong \tilde{A}/\mathfrak{m}\tilde{A}$.

To learn more about $X := \mathfrak{h} \oplus \mathfrak{h}^*/\mathcal{W}$, define the **skew group ring** $A := \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathcal{W}$ with basis $\mathcal{W}$ and multiplication

$$(f_1\, s_1)\,(f_2\, s_2) = f_1\, f_2^{s_1} s_1\, s_2.$$

(This is an infinite–dimensional noncommutative algebra.)

Lemma

$$Z(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathcal{W}) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathcal{W}} =: R$$

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Deformations

PBW Theorem (Etingof–Ginzburg 2002)

$\tilde{A} := \mathbb{C}\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle \rtimes \mathcal{W} / \{[x_i, x_j] = [y_i, y_j] = 0,\ [y_i, x_j] = \sum_s \omega_s(y_i, x_j)\, s\}$

is a deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathcal{W}$ for $\omega_s(v, w) := \frac{\omega((v, \rho_s^\vee), (\rho_s, w))}{\rho_s^\vee(\rho_s)}$.

Idea: $\tilde{A}$ has a very nice basis, the **PBW basis**.
$\rightarrow$ Compute the center $Z(\tilde{A})$ to derive $R = Z(A) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathcal{W}}$.

Remark

Computing $Z(A) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\mathcal{W}}$ directly is inefficient with the general tools of invariant theory (e.g., King's algorithm).

Computing $Z(\tilde{A})$ on the other hand is possible thanks to work of Levandovskyy, Shepler, Bonnafé, Thiel and implementations in Singular (Letterplace, Plural) and Magma (Champ).

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Computations

```
julia> W = shephard_todd(5,5,2); W

Matrix Group over Cyclotomic Field of order 5 and degree 4
Generators:
[0 1]
[1 0]
[0 zeta_5]
[-zeta_5^3 - zeta_5^2 - zeta_5 - 1 0]

julia> symplectic_doubling_fundamental_invariants(W)
[
y1*y2,
y1*x1 + y2*x2,
x1*x2,
y1^5 + y2^5,
y1^4*x2 + y2^4*x1,
y1^3*x2^2 + y2^3*x1^2,
y1^2*x2^3 + y2^2*x1^3,
y1*x2^4 + y2*x1^4,
x1^5 + x2^5
]
```

Computations

```
julia> symplectic_doubling_invariant_ring_presentation(W)

Ideal of Polynomial ring of rank 9 over Cyclotomic Field of order 5 and degree 4
Order: Lexicographical
Variables: z1, z2, z3, z4, z5, z6, z7, z8, z9

Basis:
[
z1*z6 - z2*z5 + z3*z4,
z1*z7 - z2*z6 + z3*z5,
z1*z8 - z2*z7 + z3*z6,
z1*z9 - z2*z8 + z3*z7,
-4*z1^2*z2*z3^2 + 5*z1*z2^3*z3 - z2^5 + z4*z9 - z6*z7,
4*z1^4*z3 - z1^3*z2^2 + z4*z6 - z5^2,
4*z1^3*z2*z3 - z1^2*z2^3 + z4*z7 - z5*z6,
-4*z1^3*z3^2 + 5*z1^2*z2^2*z3 - z1*z2^4 + z4*z8 - z5*z7,
-4*z1^2*z2*z3^2 + z1*z2^3*z3 - z5*z8 + z6*z7,
4*z1^2*z2^2*z3 - z1*z2^4 + z4*z8 - z6^2,
-4*z1^2*z3^3 + 5*z1*z2^2*z3^2 - z2^4*z3 + z5*z9 - z6*z8,
4*z1*z2^2*z3^2 - z2^4*z3 + z5*z9 - z7^2,
4*z1*z2*z3^3 - z2^3*z3^2 + z6*z9 - z7*z8,
4*z1*z3^4 - z2^2*z3^3 + z7*z9 - z8^2
]
```

```
julia> A = rational_cherednik_algebra(W,0);

julia> center_presentation(A)

[
z1*z6 + -1*z2*z5 + z3*z4,
z1*z7 + -1*z2*z6 + z3*z5,
z1*z8 + -1*z2*z7 + z3*z6,
z1*z9 + -1*z2*z8 + z3*z7,
-4*z1^2*z2*z3^2 + 5*z1*z2^3*z3 + -100*K1_1^2*z1*z2*z3 + -1*z2^5 + 100*K1_1^2*z2^3 + z4*z9
+ -1*z6*z7,
4*z1^4*z3 + -1*z1^3*z2^2 + 100*K1_1^2*z1^3 + z4*z6 + -1*z5^2,
4*z1^3*z2*z3 + -1*z1^2*z2^3 + 100*K1_1^2*z1^2*z2 + z4*z7 + -1*z5*z6,
-4*z1^3*z3^2 + 5*z1^2*z2^2*z3 + -100*K1_1^2*z1^2*z3 + -1*z1*z2^4 + 100*K1_1^2*z1*z2^2
+ z4*z8 + -1*z5*z7,
-4*z1^2*z2*z3^2 + z1*z2^3*z3 + -100*K1_1^2*z1*z2*z3 + -1*z5*z8 + z6*z7,
4*z1^2*z2^2*z3 + -1*z1*z2^4 + 100*K1_1^2*z1*z2^2 + z4*z8 + -1*z6^2,
-4*z1^2*z3^3 + 5*z1*z2^2*z3^2 + -100*K1_1^2*z1*z3^2 + -1*z2^4*z3 + 100*K1_1^2*z2^2*z3
+ z5*z9 + -1*z6*z8,
4*z1*z2^2*z3^2 + -1*z2^4*z3 + 100*K1_1^2*z2^2*z3 + z5*z9 + -1*z7^2,
4*z1*z2*z3^3 + -1*z2^3*z3^2 + 100*K1_1^2*z2*z3^2 + z6*z9 + -1*z7*z8,
4*z1*z3^4 + -1*z2^2*z3^3 + 100*K1_1^2*z3^3 + z7*z9 + -1*z8^2
]
```

Computations

```
julia> X = calogero_moser_space(A);

julia> Xsing = singular_subscheme(X);

julia> minimal_basis(Xsing)

[
z9,
z8,
z7,
z6,
z5,
z4,
z3,
z2,
z1
]
```

This means: $X = \mathfrak{h} \oplus \mathfrak{h}^*/\mathcal{W}$ has an isolated singularity at $0 \in \mathbb{C}^9$.

Thank You.

Merci.